Partial Differential Equations
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Lecture – 57
IBVP for Heat Equation

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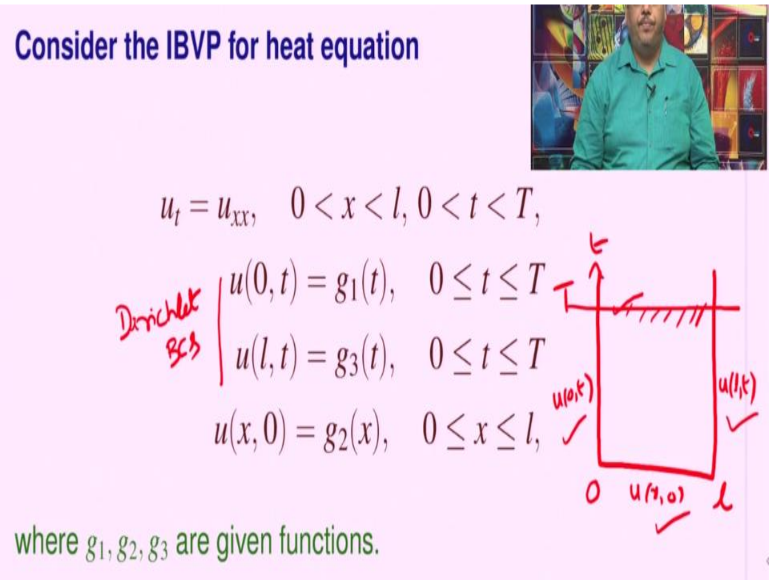

Welcome to this lecture on initial boundary value problems for heat equation. We are going to use methods of separation of variables to solve the initial boundary value problem. So, what is an initial boundary value problem we will define and we will define what the meaning of its solution is. So, the initial boundary value problem for heat equation consists of solving the heat equation u t = u xx posed on the domain x belonging to the interval 0, l and t belonging to the interval 0, capital T where capital T is a fixed number.

The u of 0, t = g 1 of t valid for t between 0 and capital T. So, this is one of the boundary conditions. The second boundary condition is u of l, t that is also prescribed to be g 3 of t and then the initial condition u x, 0 = g 2 of x where x is between zero and l. So, where g 1, g 2, g 3 are given functions. So, we are given these three functions g 1, g 2, g 3; we are supposed to find a function which solves this equation and also satisfies these three conditions.

So, this is x = 0, x = l. So, this is the boundary. So, we are prescribed u of 0, zero t here and here you have l, t and this is u of x, 0. Of course, we have put this as t axis. So, t = T here. We are not prescribing any condition on this. If we prescribe a condition on this line as well, then it will be a boundary value problem. We are going to see in the future lecture that boundary value problem for heat equation is not well posed.

So, that is why we consider only initial boundary value problem. So, initial condition you have 0 at time t = 0 and these are the boundary conditions which are given. Of course, we can consider other kinds of boundary conditions. Here we have considered what are called Dirichlet boundary conditions. We can consider other types of boundary conditions exactly like we did for the wave equation.

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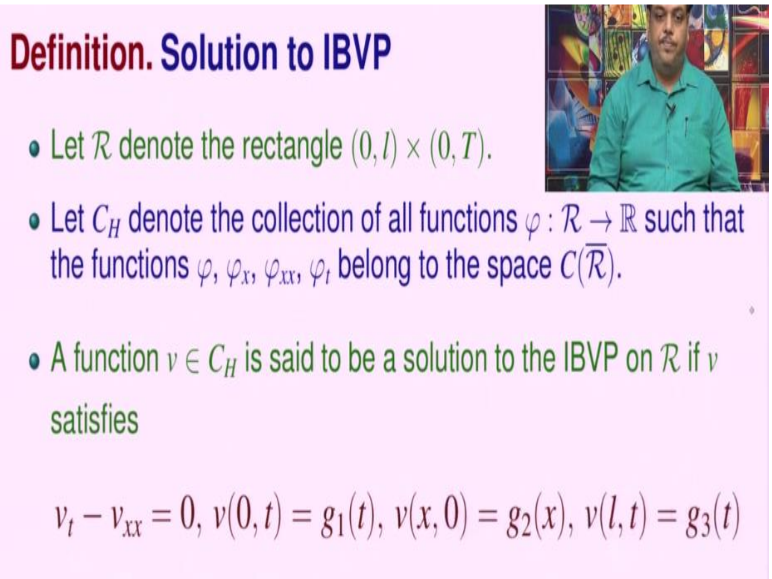

So, what is the meaning of a solution to the initial boundary value problem? Let R denote the rectangle 0, l cross 0, T. Let C H, H for heat, denote the collection of all functions phi defined on this rectangle taking values in real numbers such that the functions phi, the first order derivatives phi x and phi t and second order derivative with respect to x phi xx these are all continuous on R closure.

In fact, we do not require it to be continuous in R closure because the conditions are prescribed only on this, on this and on this we do not require it to be continuous here. A function v belongs to C H is said to be a solution to the initial boundary value problem on R if v satisfies the heat equation and satisfies the three conditions, the two boundary conditions; v 0, t = g 1 t and v l, t = g 3 t and the initial condition.

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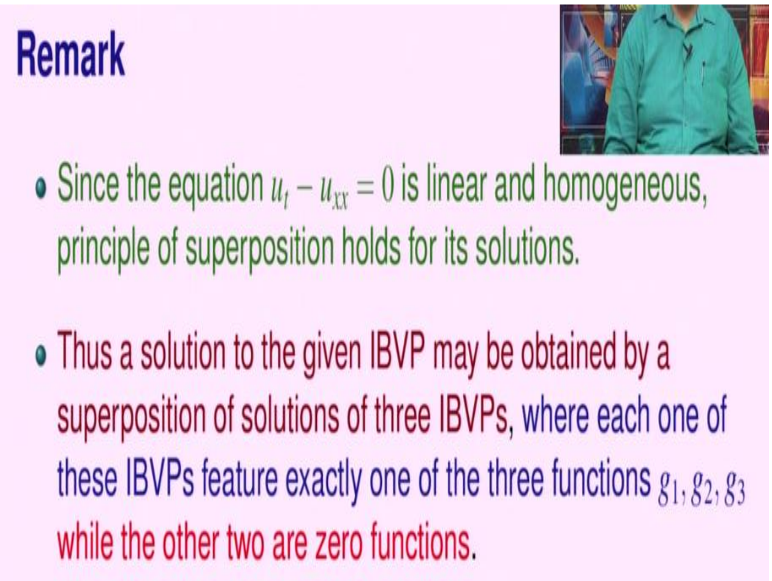

Remark: Since the equation u t = u xx is linear and homogeneous, the principle of superposition holds for its solutions. Thus solution to the given IBVP maybe have time by a superposition of solutions of three IBVPs where each one of these IBVPs feature exactly one of the three functions g 1, g 2, g 3 and other two functions are zero functions.

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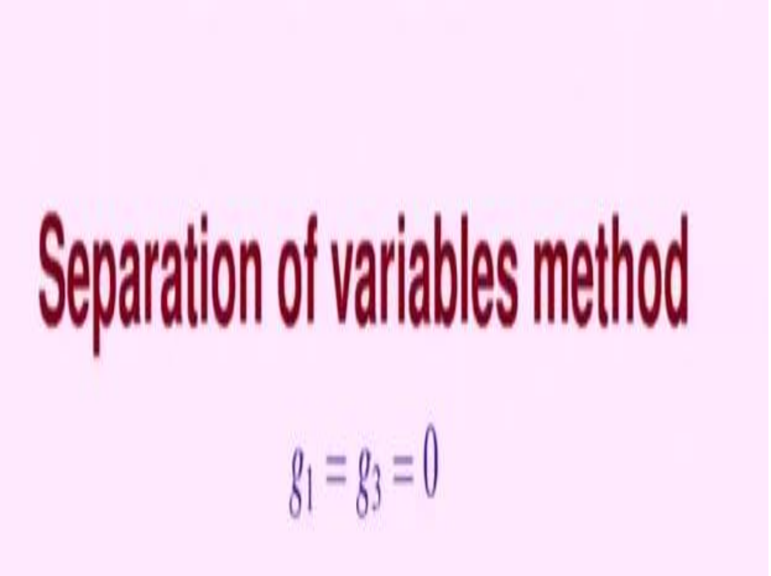

We are going to describe separation of variables methods when g 1 and g 3 are equal to 0.

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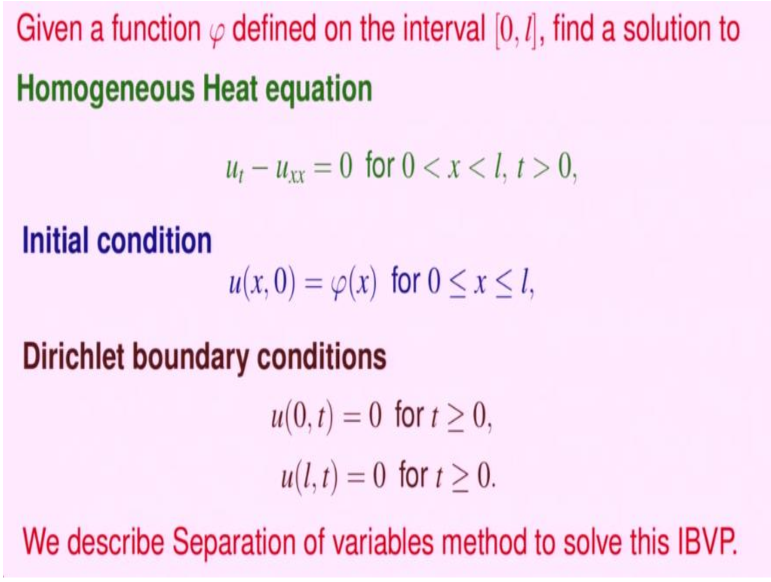

So, given a function phi defined on the interval 0, l find a solution to the homogeneous equation u t = u xx for x in 0, l and t positive. Initial condition u x, 0 = phi x for x between 0 and l and the boundary conditions which are Dirichlet boundary conditions we are considering u of 0, t is 0; u of l, t is 0. In other words, what we have done is we have set u = 0 as a boundary condition here and we are considering only this to be general function.

So, this is just one of those three IBVPs that we mentioned earlier. If you notice here, we have t positive, of course we can consider t in the interval 0, capital T for some capital T positive that also we can consider. So, as a consequence if you want to define what is the solution of this, it must be in C H for each capital T positive. We are going to discuss more about the solution of this in the future lecture. So, we describe a separation of variables method to solve this IBVP.

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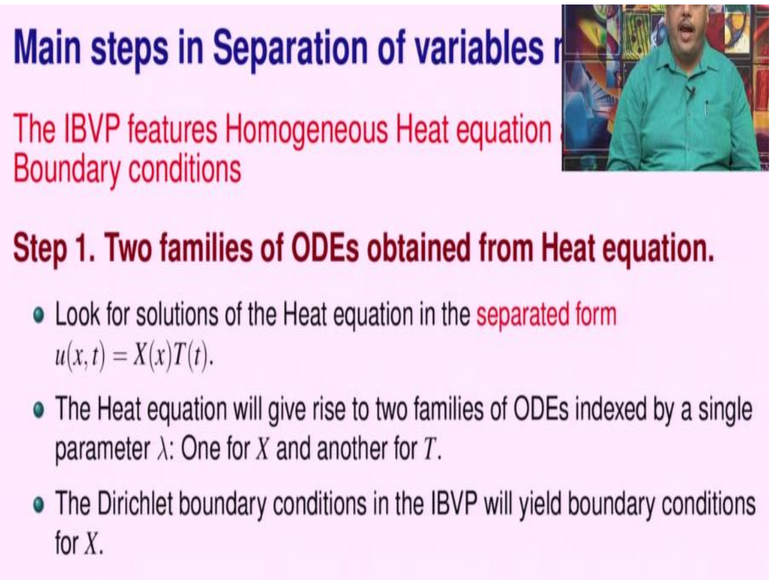

So, what are the main steps involved in separation of variables method? The IBVP features homogeneous heat equation and zero Dirichlet boundary conditions. So step 1, Two families of ODEs obtained from heat equation, how do we get that? We look for solutions to heat equation in the separated form u x, t = X x into T of t. The heat equation will give rise to two families of ODEs indexed by a single parameter lambda; one for X and another for T that is one family of ODE is for X and one family of ODE is for T. The Dirichlet boundary conditions in the IBVP will yield the boundary conditions for X.

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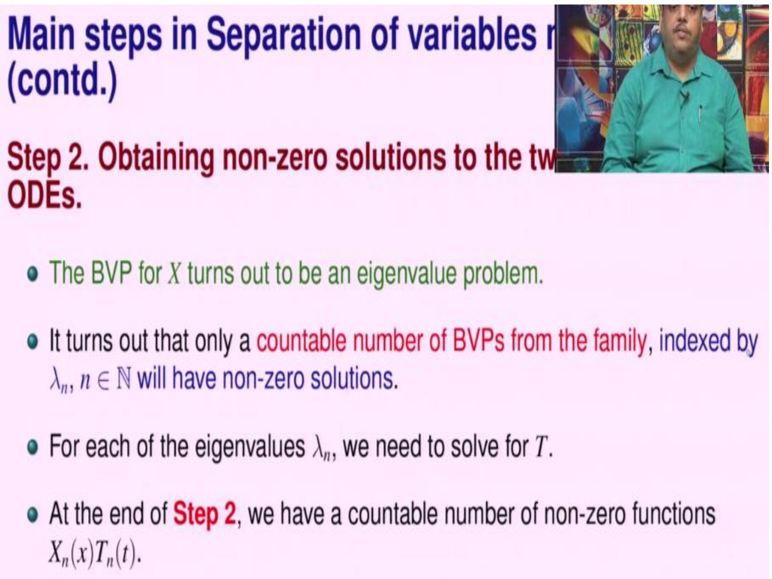

The step 2 is obtaining non-zero solutions to the two families of ODEs. The BVP for X turns out to be what is known as eigenvalue problem. So, it turns out that only a countable number of BVPs from the family indexed by lambda n, n and belongs to N we have non-zero solutions. For each of these eigenvalues, we need to then solve for T. At the end of step 2, we have a countable number of non-zero functions X n x into T n t.

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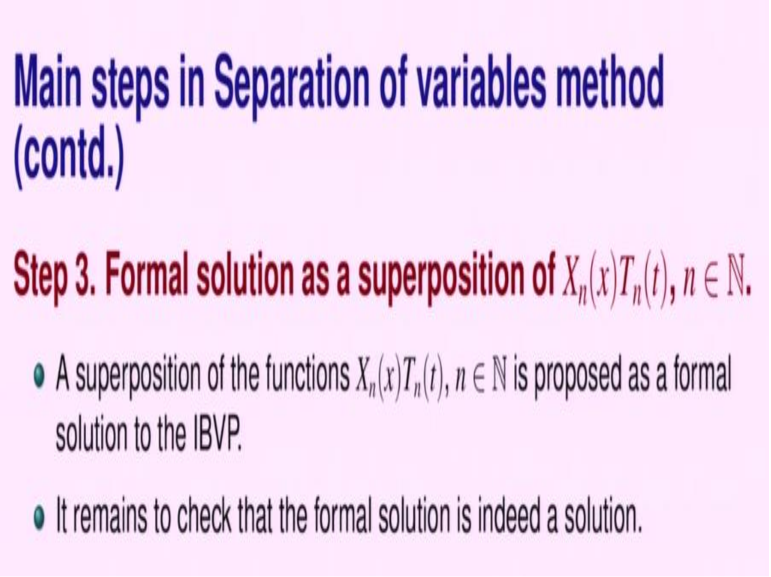

Step 3, proposing a formal solution as a superposition of these nonzero functions that we have obtained. So a superposition of the functions X n x T n t is proposed as a formal solution to the IBVP. And then it remains to check whether this formal solution is indeed a solution? What are the conditions that are needed on the initial data file?

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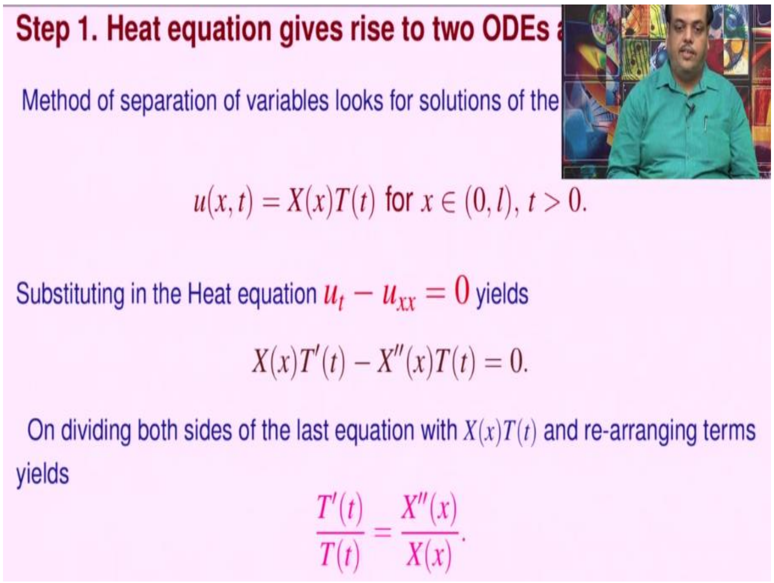

So step 1 heat equation gives rise to two ODEs and one of them there is a boundary value problem. So, method of separation of variables looks for solutions of the form u x, t = X x into T t, of course x in this finite interval 0, l and t positive. So, substituting this ansatz in the heat equation gives us this equation. On dividing both sides of this equation with X x and T t and rearranging terms will give us T prime by T = X double dash by X.

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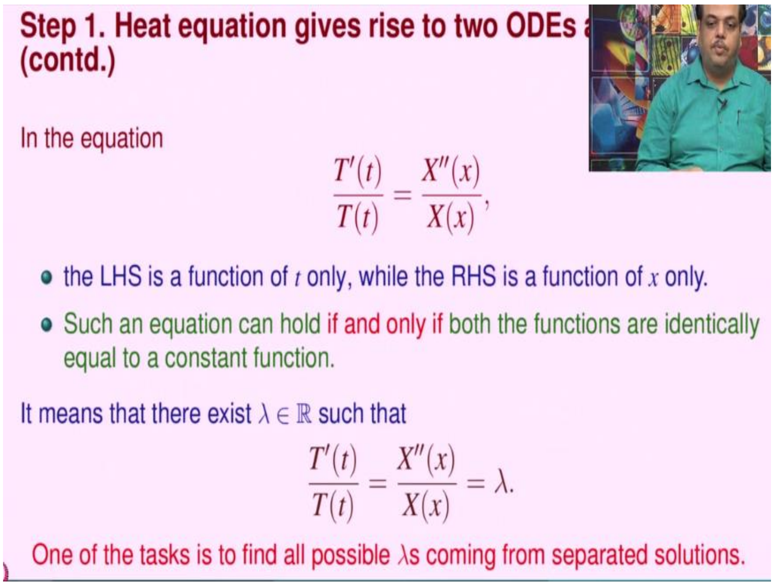

So, in this equation T prime by T = X double dash by X, the LHS is a function of t only and RHS is a function of x only. Such an equation can hold if and only if both the functions are identically equal to a constant function. It means that there exists lambda in R, a real number lambda such that T prime by T = X double dash by X = the lambda. One of the tasks is to find all possible lambdas which are coming from the non-zero separated solutions.

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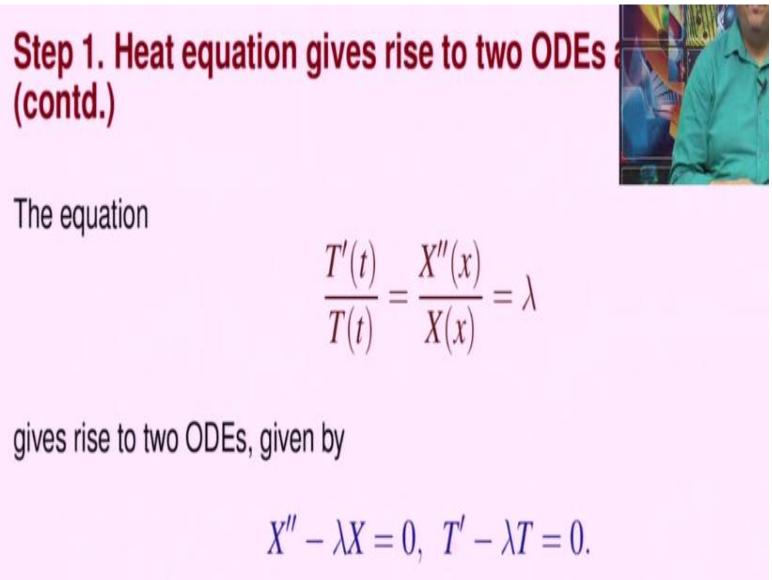

So, this equation which we have got on the last slide gives rise to two ODEs, one for T and one for X, what are they? X double dash – lambda X = 0 and T prime – lambda T = 0. It is not surprising the heat equation has two derivatives with respect to X variable. So, therefore the equation satisfied by capital X is second order and the equation satisfied by capital T is first order because the heat equation features only the first order derivative with respect to T.

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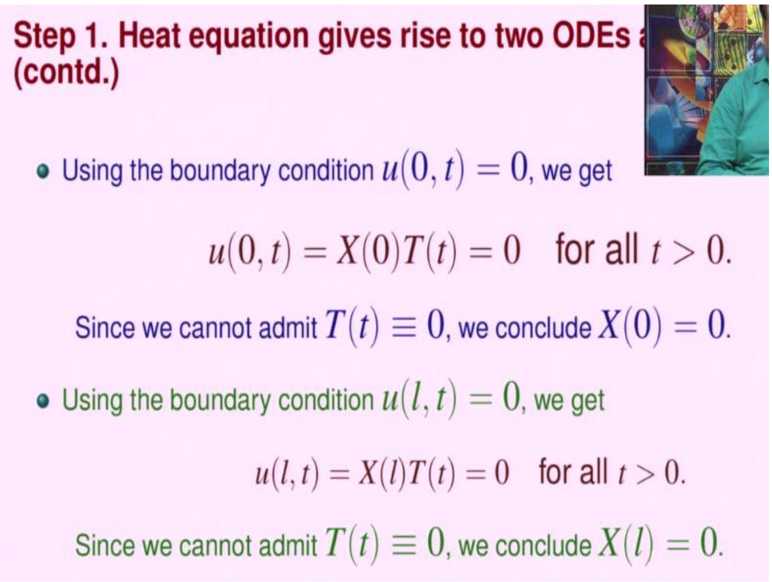

Now, we have two boundary conditions. Using them we will get a boundary value problem for X. So, using the boundary condition u of 0, t = 0 what we get is X of 0 into T of t = 0 for every t. So, this can mean that either X of 0 is 0 or T is identically equal to 0 function. We are not interested in T of t identically equal to 0, therefore X of 0 is 0. If T of t identically equal to 0, we get nothing because as you saw in step 3 we are going to propose as superposition of X on x and T on t as solution.

If T on t is 0 for some n, it does not make sense because it is 0, there is no term like that. Therefore, we cannot admit it to be 0 and hence we conclude X of 0 is 0. Similarly, using the boundary condition, u of l, t = 0 we get X of l = 0. So we got two conditions for X; X of 0 is 0, x of l is 0.

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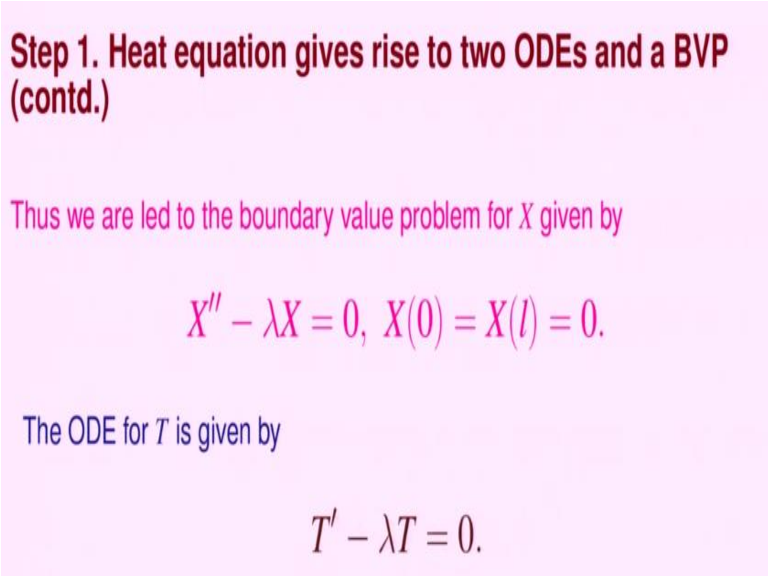

So we have the boundary value problem for X given by X double dash – lambda X = 0 and the values of X at X = 0 and X = l are 0. So the ODE for T is T prime – lambda T = 0, we do not get any condition for T unlike what we have got in the wave equation case.

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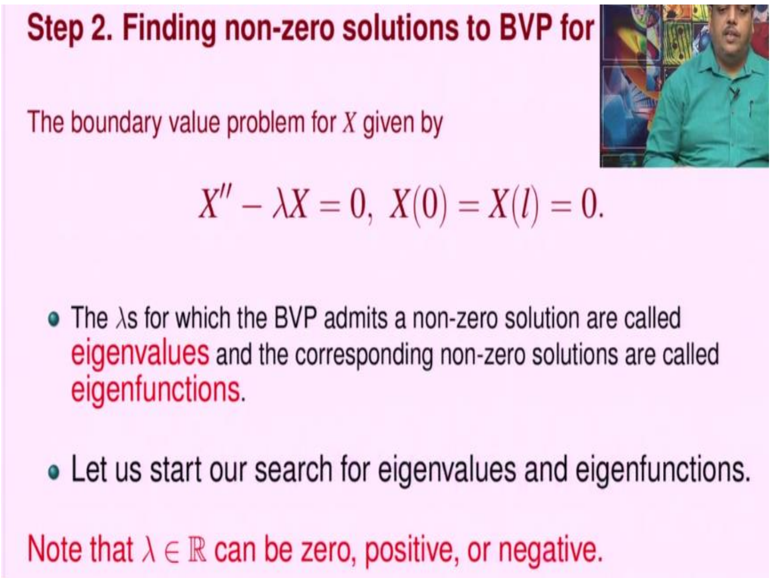

So, finding non-zero solutions to BVP for X that is step 2. So, this is the boundary value problem we are interested in solving and we are looking for non-zero solutions of this boundary value problem. Of course, as you see X identically equal to 0 is a solution for every lambda, but we are not interested in that. So the lambdas for which the BVP admits a non-zero solution are called eigenvalues and the corresponding non-zero solutions are called eigenfunctions.

So, let us start our search for eigenvalues and eigenfunctions. Note that the lambda which is a real number it can be 0, positive or negative. Therefore, our search we will conduct in three steps lambda equals 0, lambda positive lambda negative. Why is that? It is because of this equation that we have X double dash – lambda X = 0. If lambda is positive, we can write the solution in terms of exponentials, a general solution.

If lambda is negative, the general solution is in terms of the sine and cosine functions. And of course, if lambda equals 0, X is ax + b, X of x = ax + b. So the nature or the form of the solution changes depending on whether lambda is 0, positive or negative that is the reason why we are going to divide our search for eigenvalues and eigenfunctions into three steps.

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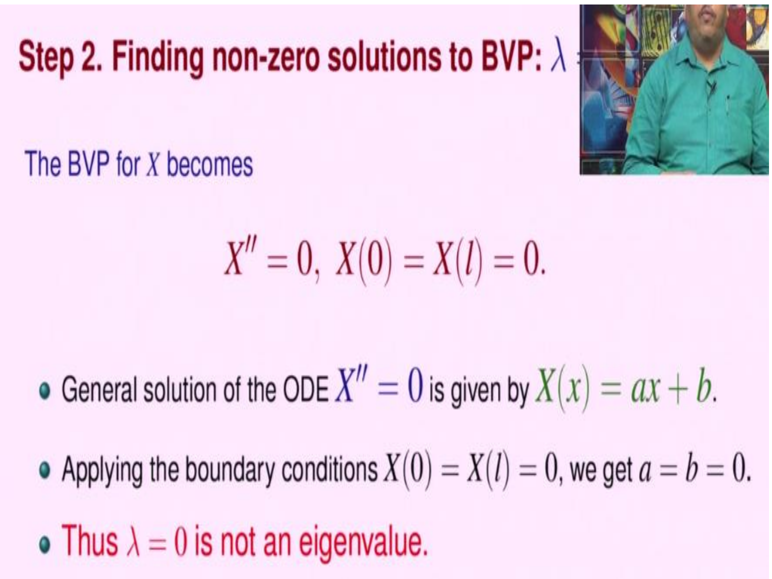

So let us take lambda = 0. Then the boundary value problem for X becomes X double dash = 0 and X of 0 = X of l = 0. General solution of X double dash = 0 is ax + b for some constants a and b, real numbers. But now we are going to look for those solutions which satisfy these boundary conditions. That means what we get is a = b = 0 because if you see, look at this the graph of this is a straight line and that is supposed to pass through 0, 0 as well as l, 0 and the only straight line which passes through 0, 0 as well as l, 0 is X of x = 0, y = 0. So a = b = 0, the line is the x axis itself. So this lambda = 0 is not an eigenvalue.

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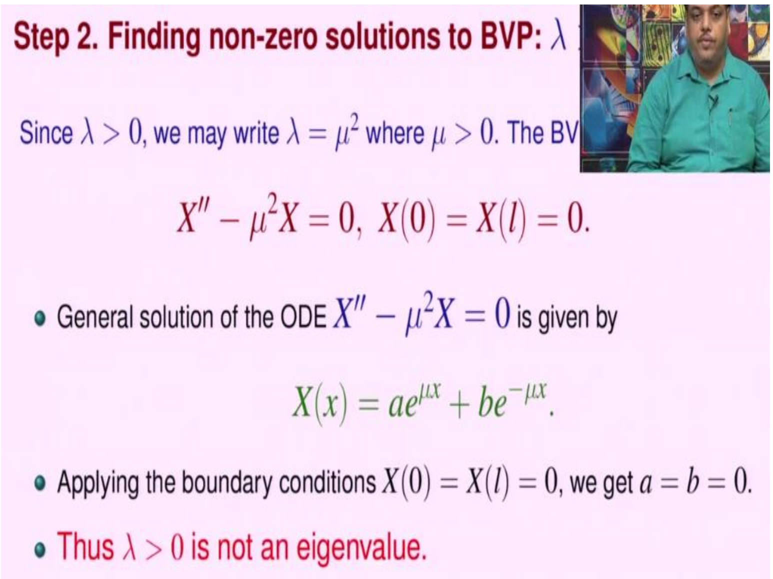

What about lambda positive? Are there positive eigenvalues? So because lambda is positive, we can always write down as lambda equal to mu square, it is in the interest of noncomplicated notations where mu is positive. Equation is X double dash – mu square X = 0, boundary condition remains the same. Now general solution of the ODE X double dash – mu square X = 0 is a combination of exponentials.

So, ae power mu x + be power –mu x. Imagine if we are not set lambda equal to mu square, we would have had here square root of lambda instead of a simple looking mu that is the reason we always do like this. The moment something is positive, we write it as mu square for mu positive. So, now we have to check if there are solutions which satisfy these boundary conditions and non-zero functions.

So, applying the boundary conditions X of 0 = X of l = 0, it is easy to see that we get a = b = 0. It means no non-zero solutions exist for this boundary value problem. So, no positive number is an eigenvalue.

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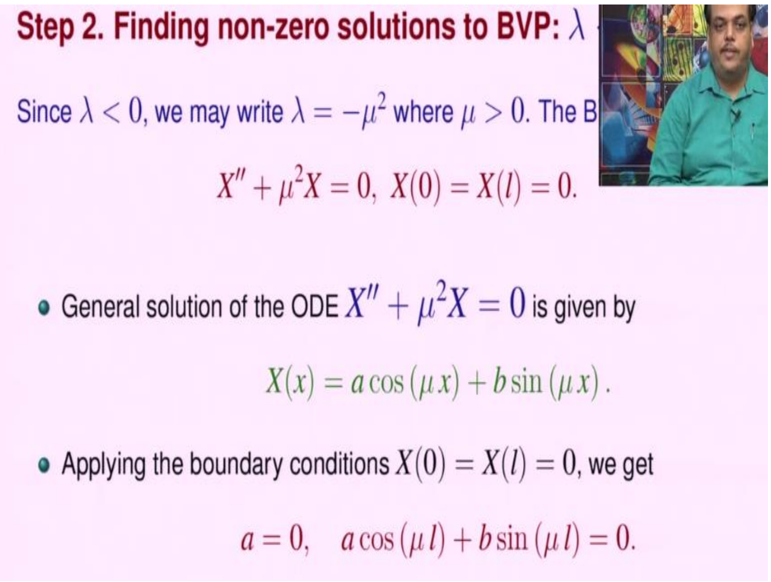

So, now what remains is to look for eigenvalues which are negative. So, since lambda is negative, we can write down lambda to be –mu square where mu is positive. So the BVP for X becomes X double dash + mu square X = 0 and the boundary condition is X 0 = X l = 0. So general solutions of the ODE X double dash + mu square X = 0 is X of x = a cos mu x + b sin mu x.

Once again, if you have not set lambda equal to –mu square what we would have had here is square root of – lambda X which is confusing that is why we choose to write lambda equal to –mu square. Now applying the boundary conditions, X of 0 = X of l = 0, what do we get? Suppose we use X of 0 = 0, then what we get when I set X = 0 a + 0 = 0 that means a is 0. When I put X of l = 0, I get this condition.

But note a is already 0, so essentially the condition is b sin mu l = 0. I do not want b to be 0, so the condition is sin mu l = 0. So mu must satisfy sin mu l = 0 that is what we are going to see on the next slide.

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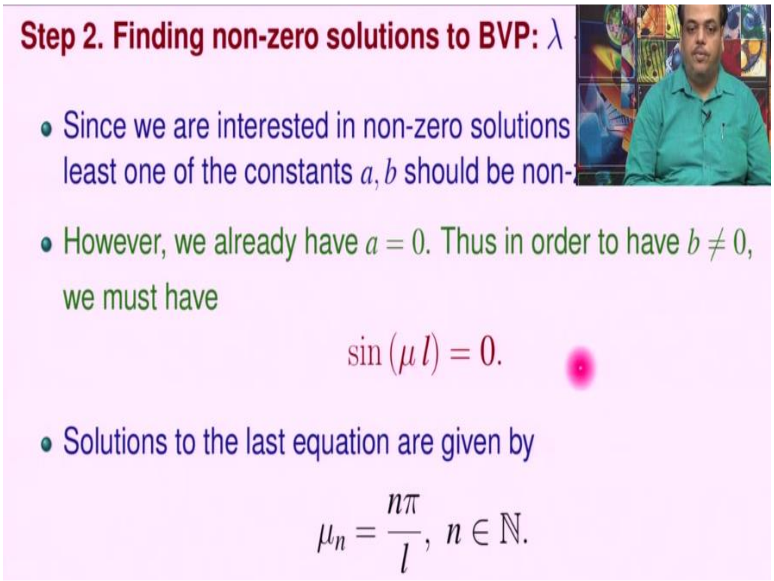

So, since we are interested in non-zero solutions to the boundary value problem, at least one of the constants a, b should be nonzero, but we already have a = 0. Therefore, to have b nonzero, we must have sin mu l = 0 which means mu n = n pi by l where n is a natural number. So, whenever mu n is of the form n pi by l for some natural number n, sin of mu n l is 0.

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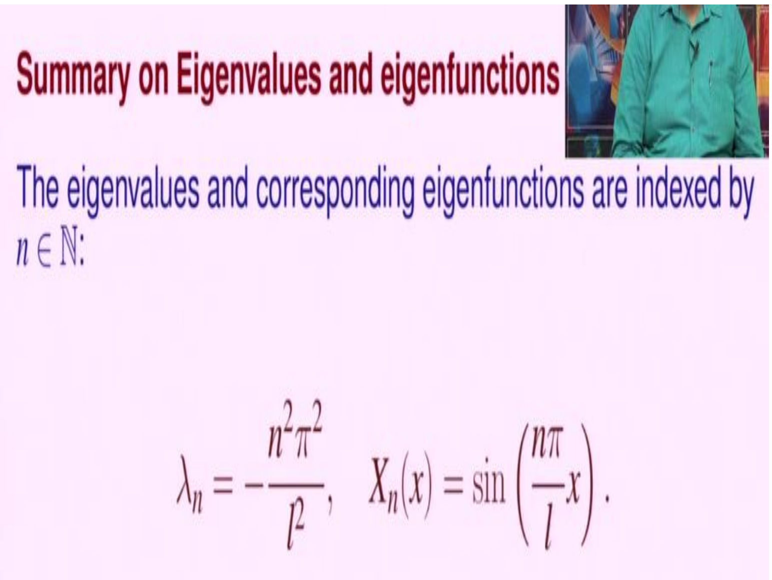

So, let us summarize what we have got on eigenvalues and eigenfunctions for the boundary value problem for X. The eigenvalues and corresponding eigenfunctions are indexed by n belongs N; lambda n is – n square pi square by l square, remember lambda = – n square, lambda is eigenvalue, so lambda = –mu square and X n x is sin mu x. So, X n x is, sin mu is n pi by l x.

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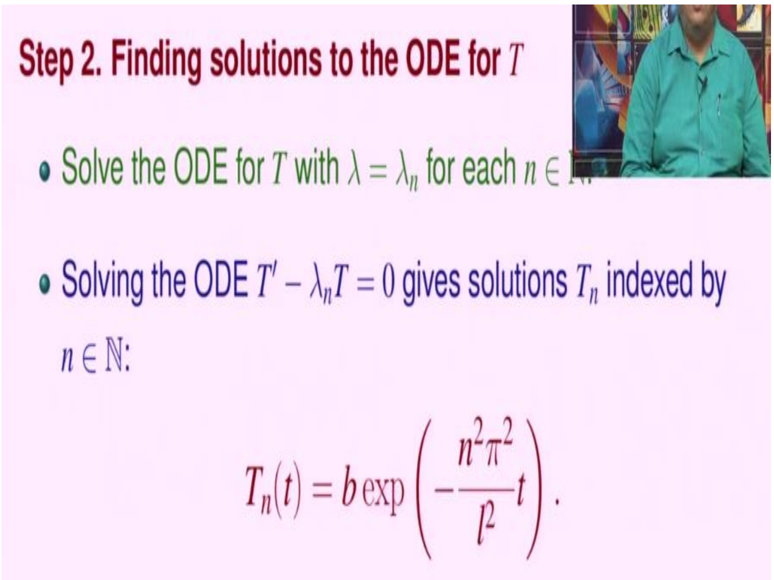

Now, we need to solve the ODE for T that is a step 2 that is very simple. So, solve the ODE for T with a lambda = lambda n for each n in N and call it T n. What is the equation? T dash – lambda n T = 0 that is T dash = lambda n T. So answers will be in terms of exponential. So, it is a constant times exponential of –n square pi square by l square into t are the solutions.

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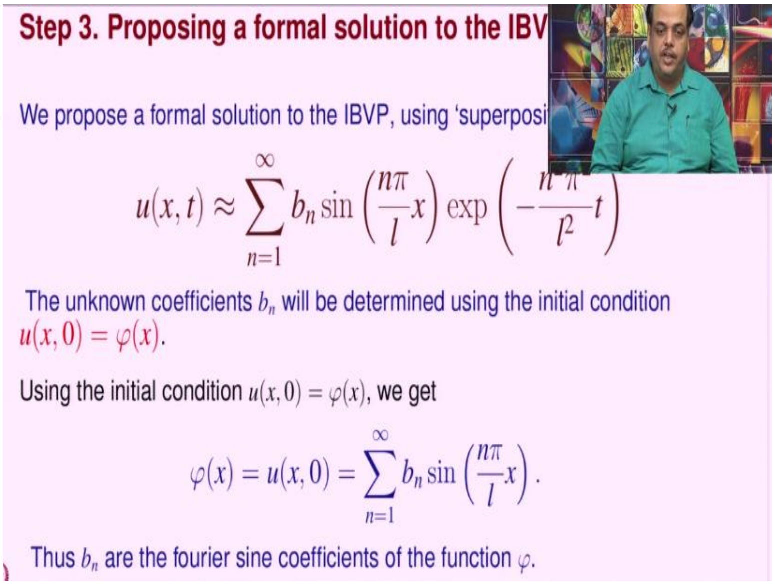

Now, let us propose a formal solution to the initial boundary value problem. So, we propose a formal solution using superposition principle as this. This is not really superposition principle, this is infinite superposition that is why I have put them in quotes. So, u x, t is proposed to be this. This is X n x, this is T n t and we take a combination of them and propose that as a solution to u x, t.

Of course, each one of these terms solves heat equation that is why we believe that the sum will also solve. But we have infinite sum, so some justification is required. Since we are not doing the justification currently, we just put this symbol. Now, what are the coefficients b n, they need to be determined, but we have one condition that we are not yet used which is u x, 0 = phi x. So, put t = 0, you get u x, 0 = phi x.

When you put t = 0 this term will not be there because this will be 1. So, what we have is phi x = summation n = 1 to infinity b n sin n pi by l into x. So, thus b n are the Fourier sine coefficients of the function phi. See phi of x is given in terms of only pure sine series that is why this is called Fourier sine series for the function phi and b n are the coefficients. So, we need to determine them.

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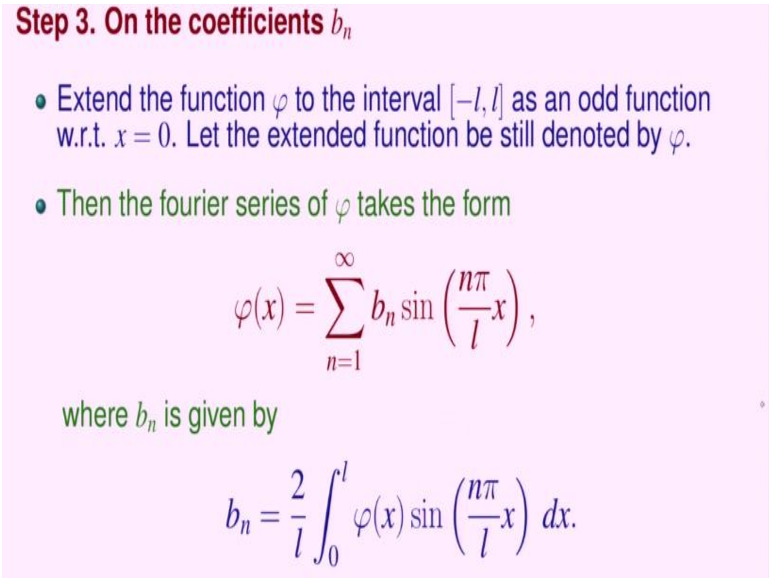

So, extend the function phi to the interval –l, l as an odd function with respect to x = 0 because phi is given on the interval 0, l. So, let the extend function be still denoted by phi. Then the Fourier series of phi takes this form because now the extended function is odd function, it will not feature cosine terms. The Fourier series will not feature cosine terms, it features only the sine terms. And the coefficient b n is given by 2 by l integral 0 to l phi x sine of n pi by l x dx. We already discussed this in the context of wave equation.

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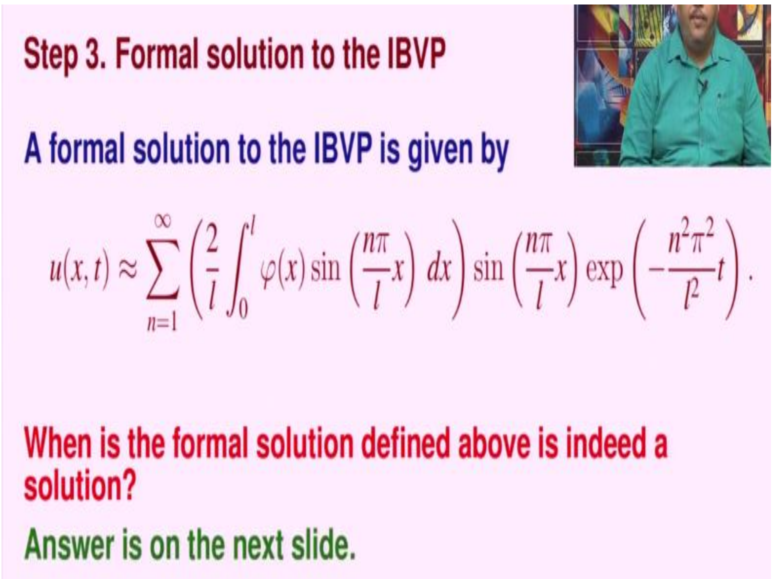

So, a formal solution to the IBVP is given by this. The only difference is we have substituted what are the coefficients b n, the coefficient b n are the Fourier sine coefficients for the function phi. So when is this formal solution defined above is indeed a classical solution? So, the answer is on the next slide.

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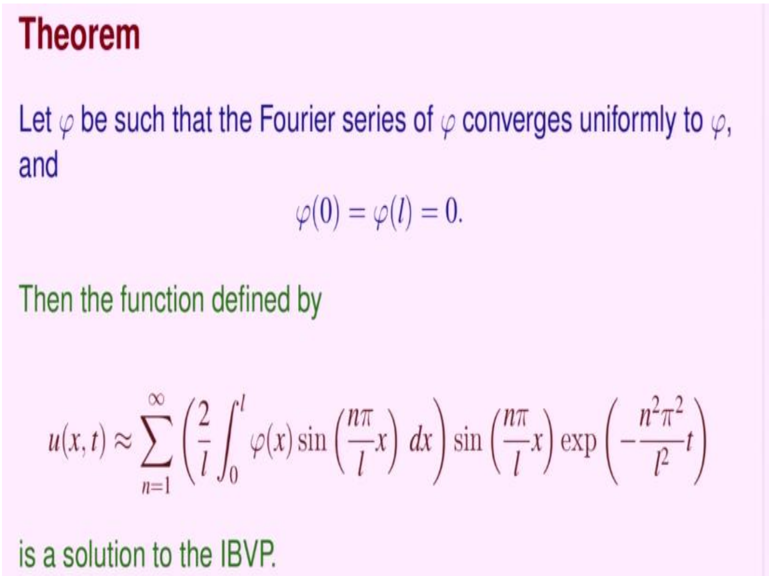

So theorem. Let phi be such that the Fourier series of phi converges uniformly to phi. In fact, the Fourier sine series of phi converges uniformly to phi and phi of 0 = phi of l = 0. Then the function defined by this is a solution to IBVP. If you recall, we have defined the solution to IBVP only when t is restricted to finite interval 0, t. But here the IBVP we have posed these for all t positive. So what it means is that this function belongs to C H for every capital T fixed that is the meaning of the solution.

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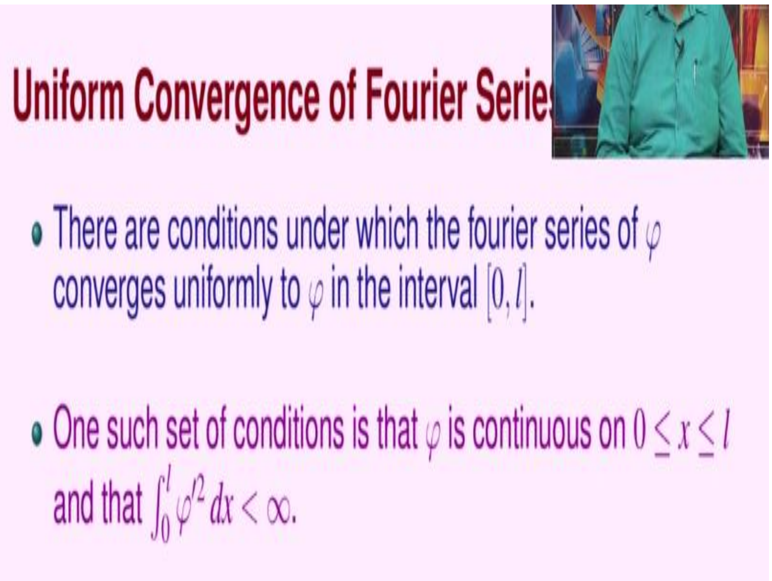

A comment on the uniform convergence of Fourier series because it is what we have assumed in the theorem. There are conditions under which the Fourier sine series of phi converges uniformly to phi in the interval –l to l or equivalent interval 0, l. One such set of conditions is that phi is continuous and the integral 0 to l phi dash square dx is finite. And of course phi of 0 = phi of l = 0 that is also required.

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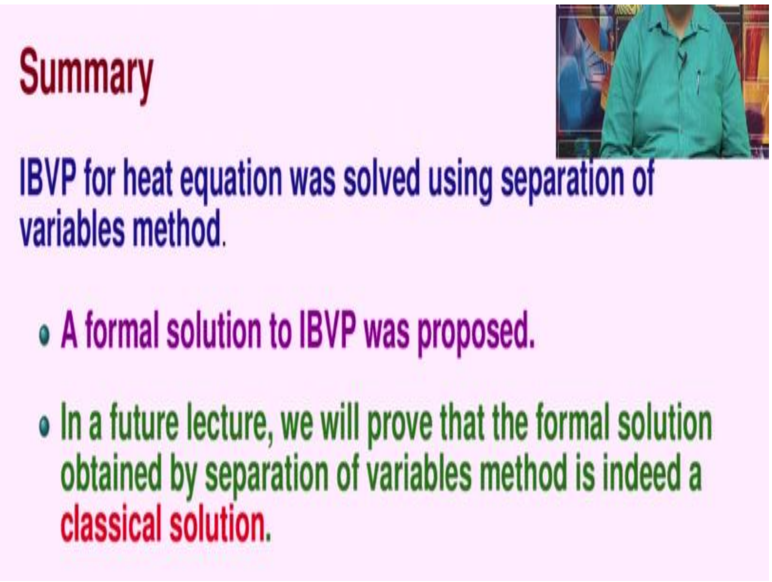

So, let us summarize what we did in this lecture. IBVP for heat equation was solved using separation of variables method. A formal solution to IBVP was proposed. In a future lecture, we will prove that the formal solution obtained by separation of variables method is indeed a classical solution. Thank you.