Partial Differential Equations
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Lecture – 55
Cauchy Problem for Heat Equation - 1

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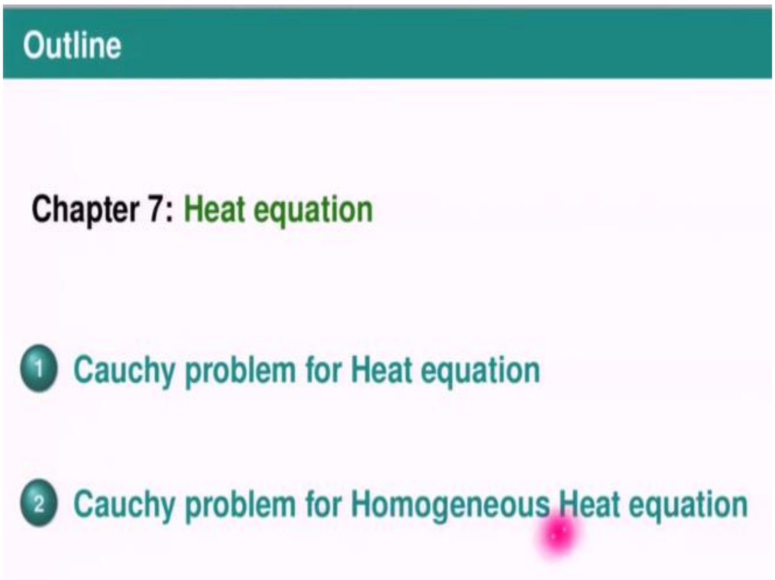

Welcome, we begin the study of heat equation starting from this lecture. In this lecture, we are going to take a few preliminary steps into the solution of Cauchy problem for the heat equation. The outline of this lecture is as follows. First, we introduce what is Cauchy problem for heat equation and take the first steps in solving the Cauchy problem for homogeneous heat equation. In the next lecture, we are going to complete the study of the Cauchy problem for nonhomogeneous heat equation.

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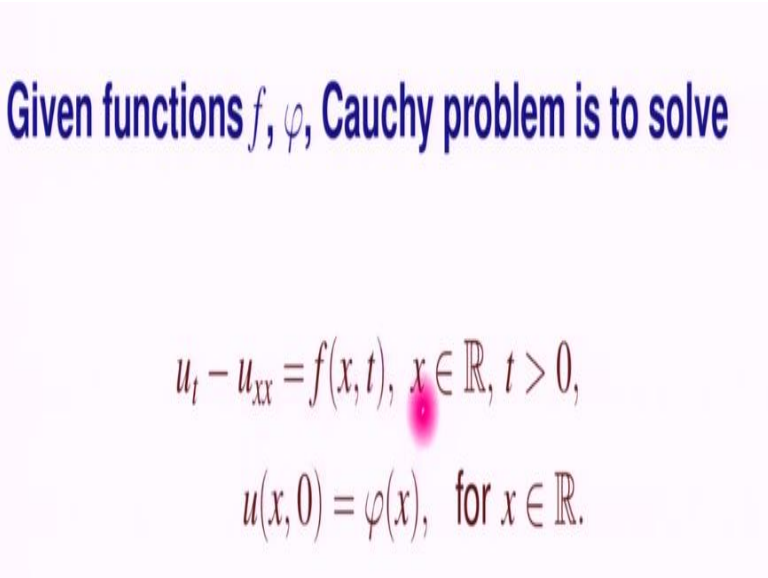

So, what is Cauchy problem for heat equation? Given functions f, phi on appropriate domains Cauchy problem is to solve u t – u xx = f of x, t; x belonging to R and t positive and u of x, 0 = phi of x for x in R.

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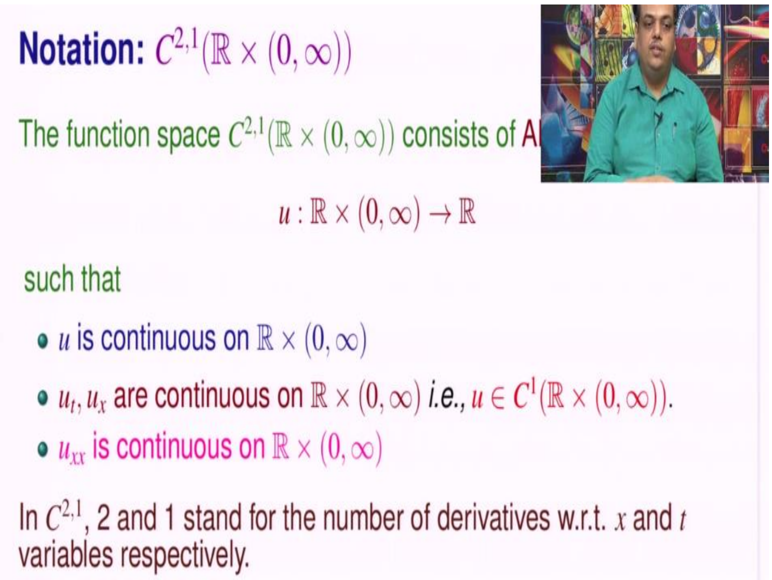

Notation: C 2,1 of R cross 0, infinity. In case you have seen this notation C 2,1 somewhere else, you forget about that. It is not about that notation; it is a notation restricted to the heat equation study. So, the functions space C 2,1 of R cross 0, infinity consists of all functions defined on the domain R cross 0, infinity taking values in real numbers such that u is continuous on R cross 0, infinity; u t, u x are continuous on R cross 0, infinity that is u is C 1 of R cross 0, infinity; u xx is continuous on R cross 0, infinity.

In this notation C 2,1 2 and 1 stands for the number of derivatives with respect to x and t variables respectively. As you see with respect to x, we have two derivatives that is why the 2 here and this 1 is for one derivative of u with respect to t.

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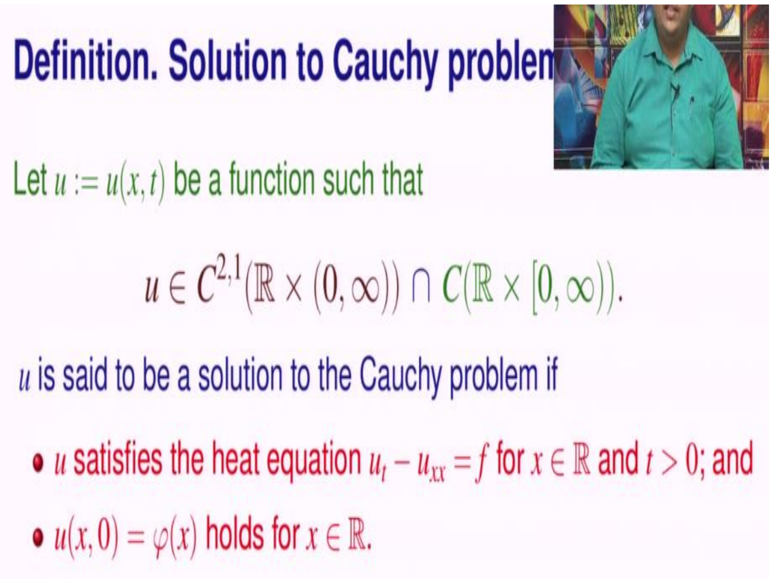

Let us define what is the meaning of solution to the Cauchy problem. Let u be a function such that u is C 2,1 of R cross 0, infinity and that is intersection C of R cross closed 0, infinity. Observe here it is open 0, infinity, this is closed 0, infinity. The u is said to be a solution to the Cauchy problem if u satisfies the heat equation u t – u xx at every point x, t = f of x, t for x in R and t positive that holds and u of x, 0 is phi of x that also holds for every x in R.

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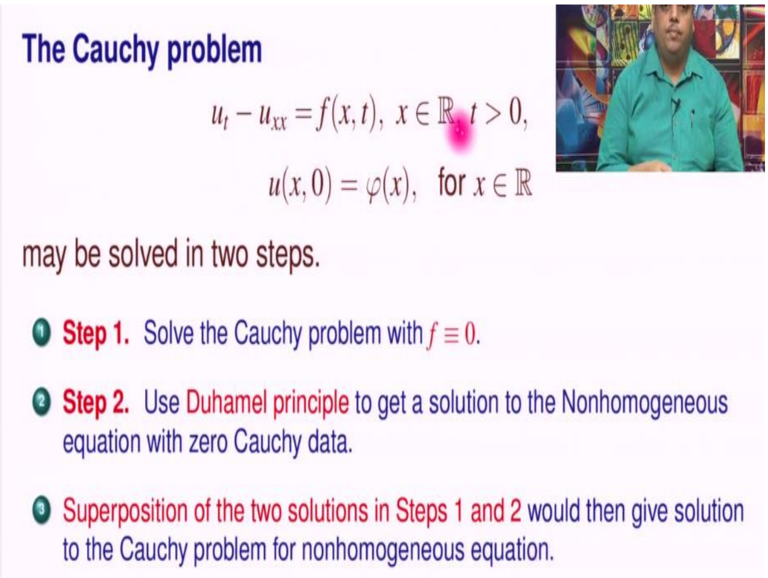

So, the Cauchy problem u t – u xx, you may call this as he operator h of u equals to f of x, t and u x, 0 = phi x on appropriate domains x belongs to R, t positive here and x belongs to R here may be solved in two steps. Step 1, solve the Cauchy problem with f identically equal to 0. Step 2, use Duhamel principle to get a solution to the nonhomogeneous equation with zero Cauchy data. And superposition of the two solutions which are obtained in step 1 and 2 would then give solution to the Cauchy problem for a nonhomogeneous equation that is solution to this problem.

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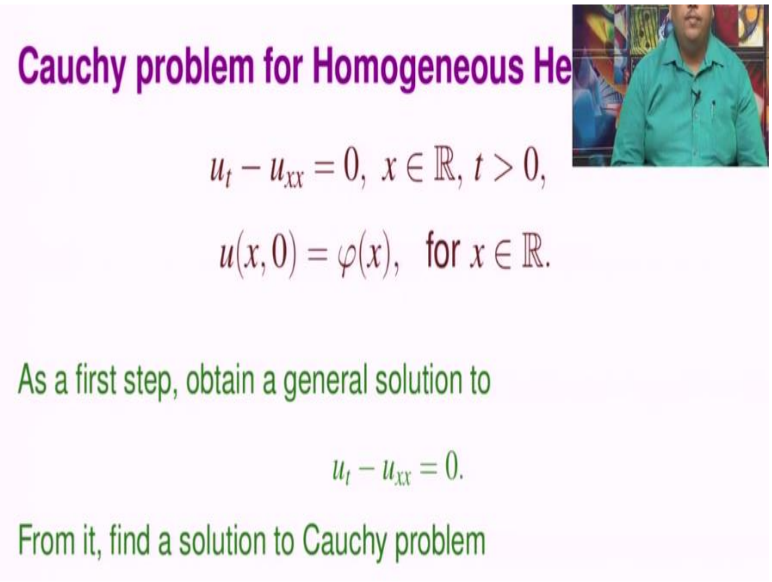

So, Cauchy problem for homogeneous heat equation u t – u xx = 0 and u x, 0 = phi of x. So, the first step; obtain a general solution to this equation, a solution which satisfies u x, 0 = phi of x that is the idea, let us try it out.

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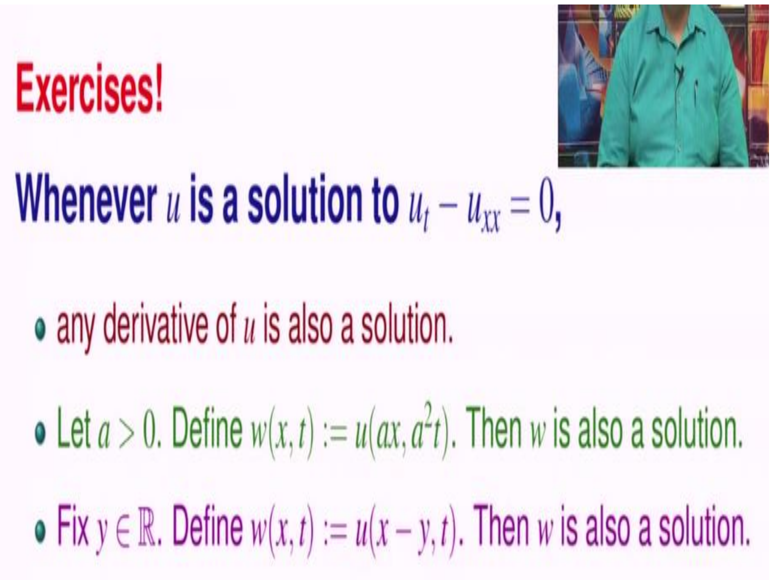

Before that some exercises. Whenever u is a solution to u t – u xx = 0 any derivative of u is also solution assuming that such derivatives exist okay. And let a be positive defined w of x, t

= u of ax, a square t. Then w is also a solution. Fix y in R and look at a translate of u, u of x – y, t call it w of x, t that w is also a solution.

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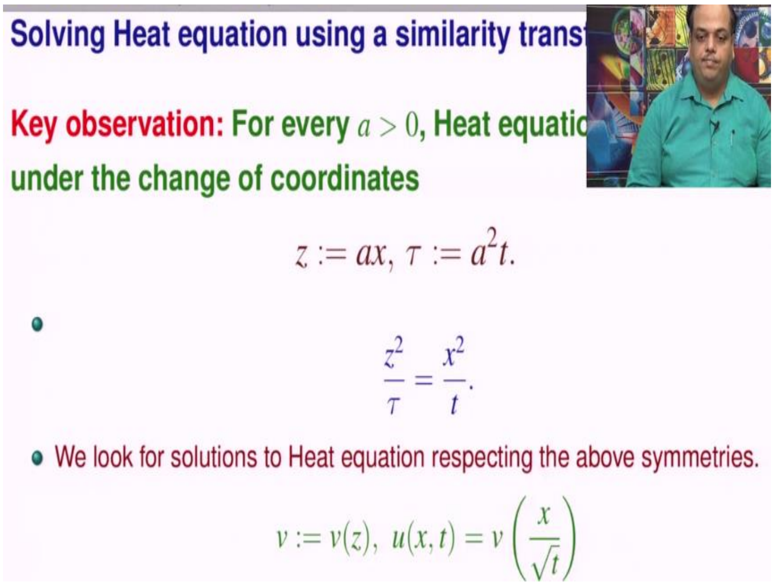

Let us solve heat equation using a similarity transformation. We will see what that means. A key observation is as follows. For every a positive, heat equation is invariant under the change of coordinates. That means if you shift from x, t to z, tau, the heat equation now will become w tau – w zz = 0. Observe this z square by tau is x square by t. So, we look for solutions to heat equation respecting the symmetries.

So, let v be a function of one variable v of z, okay this z is different from this z, I should have called it v of some other variable s. So, u of x, t = v of x by root t. So take a function of one variable and look for u of x, t as v of x by root t. So, x by root t is coming from this equation x square by t equivalently x square root t.

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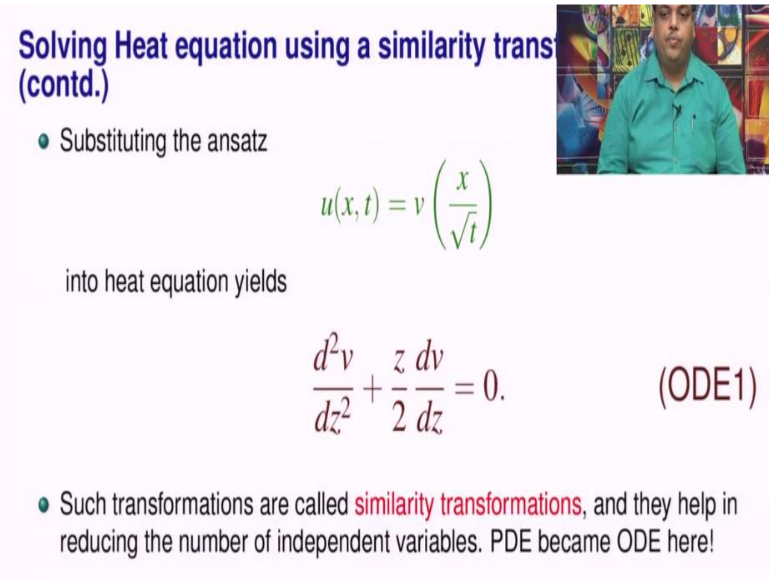

So, substituting this ansatz u of x, t = v of x by root t into heat equation gives us this equation very easy to check, I am not going to prove this. Now, such transformations are called similarity transformation, which transformation this instead of looking for x, t look for x square root t and they help in reducing the number of independent variables. PDE became ODE in our example. The heat equation became this ODE.

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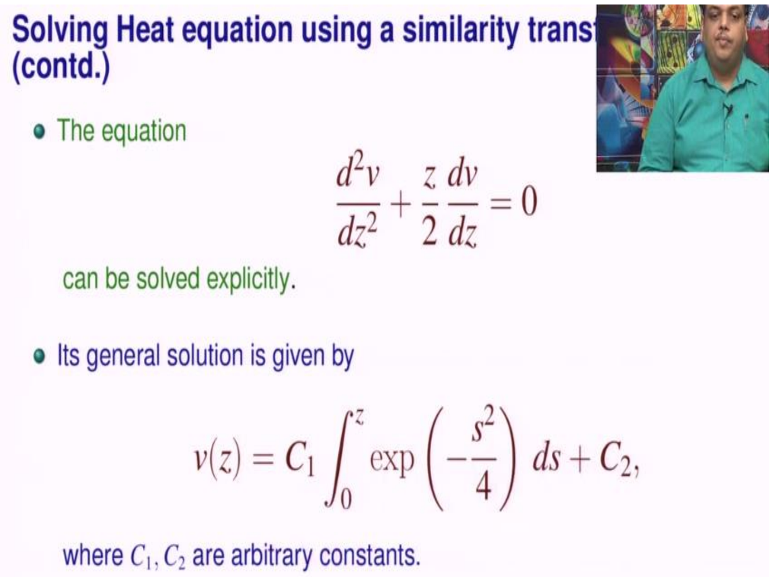

So, the equation which we have obtained on the earlier slide it can be solved explicitly. So, its general solution is given by this v z = C 1 times 0 to z exponential – s square by 4 ds + C 2 where C 1 and C 2 are arbitrary constants.

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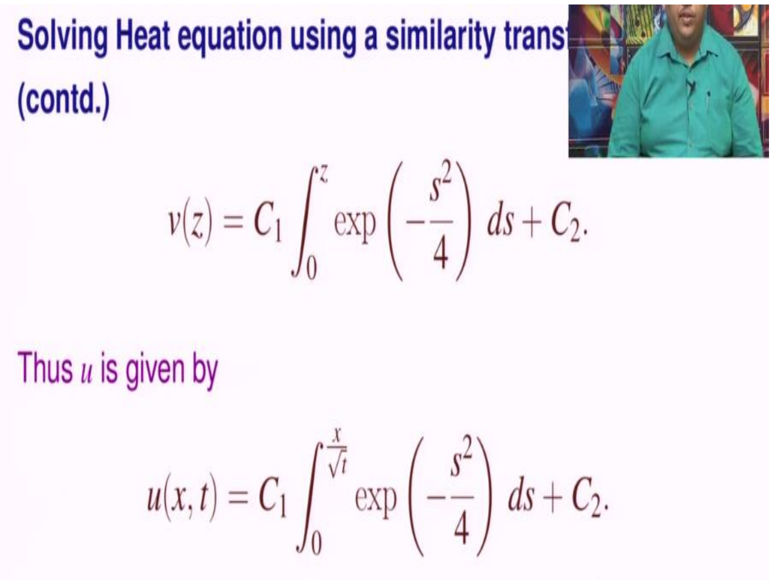

So, therefore, u of x, t is v of x by root t. So put z = x by root t. So, that is only here, z is only here, so it is only here. So, this is an expression for u of x, t. Now, we want to make sure u of x, 0 is phi x, then we are done, we have solved the Cauchy problem for the homogeneous heat equation.

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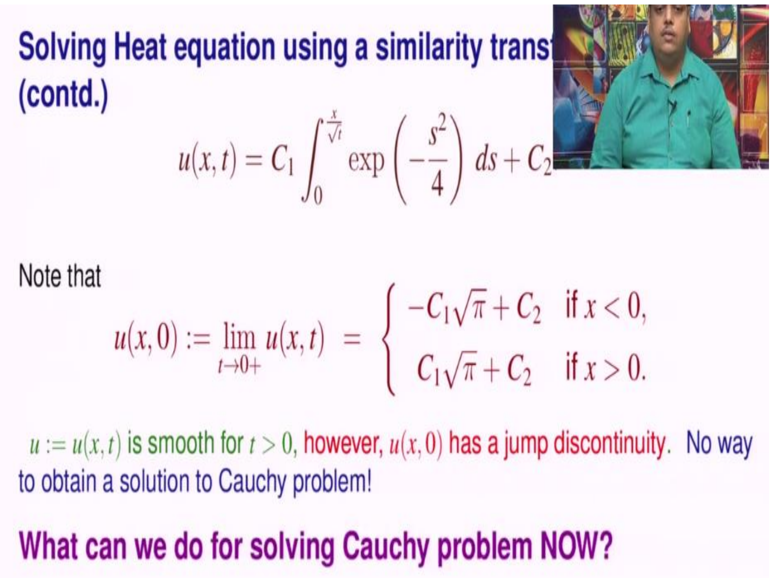

Let us see what is u of x, 0? u of x, 0 is limit of u of x, t as t goes to 0+ which is actually – C 1 root pi + C 2 if x is negative and if x is positive it is C 1 root pi + C 2, please check this. Now, u which is given here, u of x, t is smooth function for every t positive, only at t = 0 there is some issue. If t is positive it is an indefinite integral of course with this variable here of some nice function, smooth function C infinity function. So, therefore, u is smooth, there is no problem for u.

However, u of x, 0 has a jump discontinuity. But in any case, there is no way to obtain a solution to our Cauchy problem, we want u of x, 0 = phi x. How will I get phi x from this? This is just piecewise constant function with some jump discontinuity if C 1 is nonzero. If C 1 is 0, of course it is just a constant function. So, we need to exploit this of varying C 1 if possible and try to get solution, but there is no way u of x, 0 is not going to be phi of x. So, what can we do for solving Cauchy problem now?

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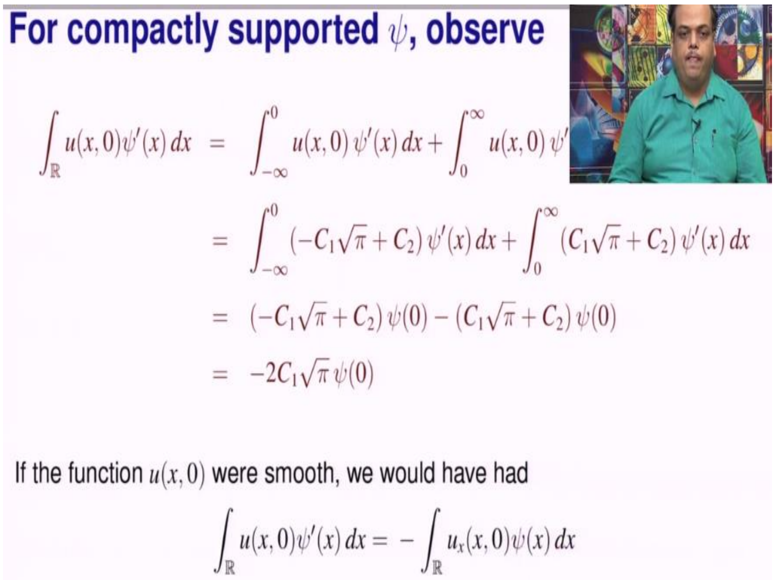

Let us observe this for compactly supported function psi, look at u of x, 0 into psi dash of x. Of course, this integral can be split into minus infinity to 0 + 0 to infinity. I have not done anything, the integration domain is split into two domains. I am going to substitute the value of u of x, 0. For x negative we have u of x, 0 this. For x positive u of x, 0 is this. Now we can do integration, right this is a constant.

So minus infinity to 0 of psi dash of x is psi of 0. And here this is constant, so 0 to infinity of psi dash of x is minus psi of 0, psi is having compact support, so the infinity does not matter, only 0 matters. If you simplify you get this –2 C 1 root pi psi of 0. If the function u of x, 0 was smooth and not like this, what we would have had is integral u of x, 0 psi hash of x = – u x of x, 0 psi of x dx.

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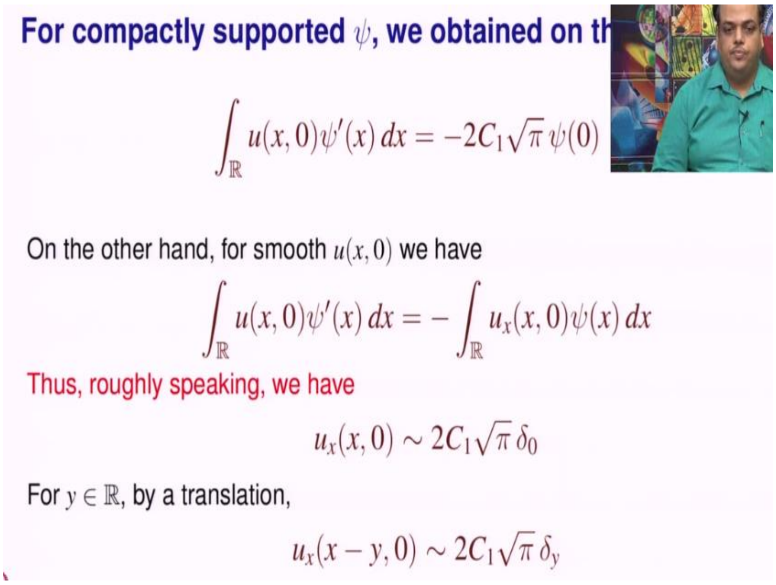

So, we obtained on the last slide this expression. For smooth functions, u of x, 0 we would get this. So, in some sense this and this are related loosely, roughly speaking. Roughly speaking u x of x, 0 is like 2 C 1 root pi delta 0, this is the Dirac delta. Dirac delta, when you integrate, loosely speaking when you integrate against some function will give you the value of the function at the point 0 that is why you have psi of 0. By translation, we can make this happen at any y. So, u x of x – y, 0 will be delta y instead of delta 0.

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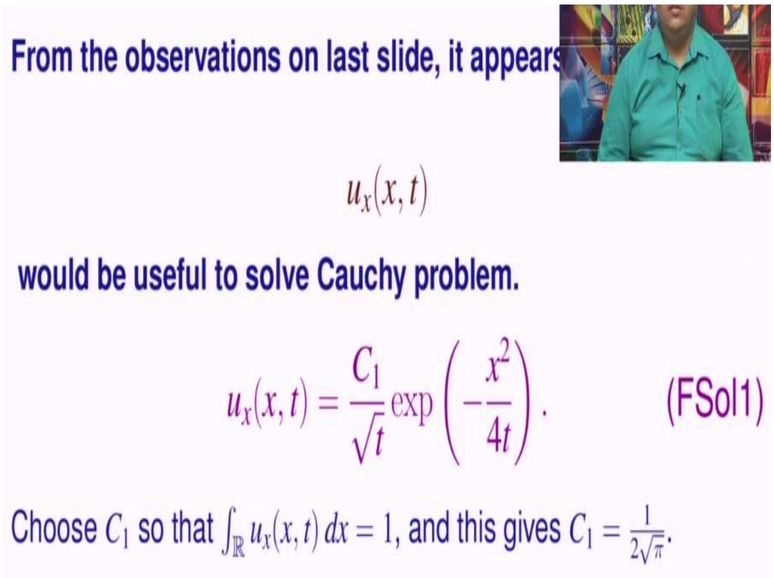

So, from the observations on the last slide, it appears that u x of x, t would be useful to solve Cauchy problem, u x of x, t of course is a smooth function, there is no problem for t positive. What happens at t = 0? It looks like the Dirac delta function relative of that. So, choose C 1 so that this integral is 1 that will give us C 1 to be 1 by 2 root pi. So, therefore, u x of x, t is 1 by 2 root pi t into exponential – x square by 4 t. This is going to play a crucial role in the solution of the Cauchy problem for homogeneous heat equation.

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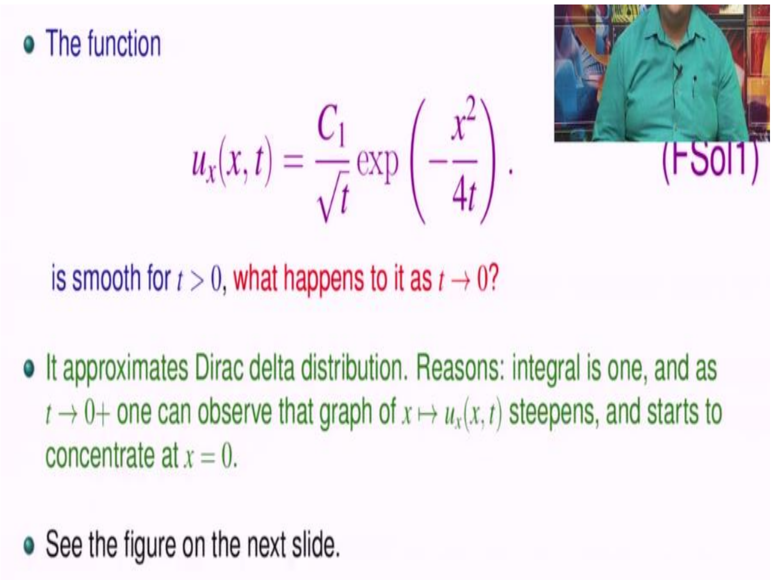

So, as observed this function is smooth for t positive. And what happens to it as t goes to 0? We will see some picture on the next slide. It approximates this Dirac delta distribution, in case you do not know what this is, simply ignore what I have said. The reason is that it has integral 1, for every t integral 1 with respect to x and as t goes to 0+ one can observe that the graph steepens and start to concentrate at x = 0. Look at the picture on the next slide.

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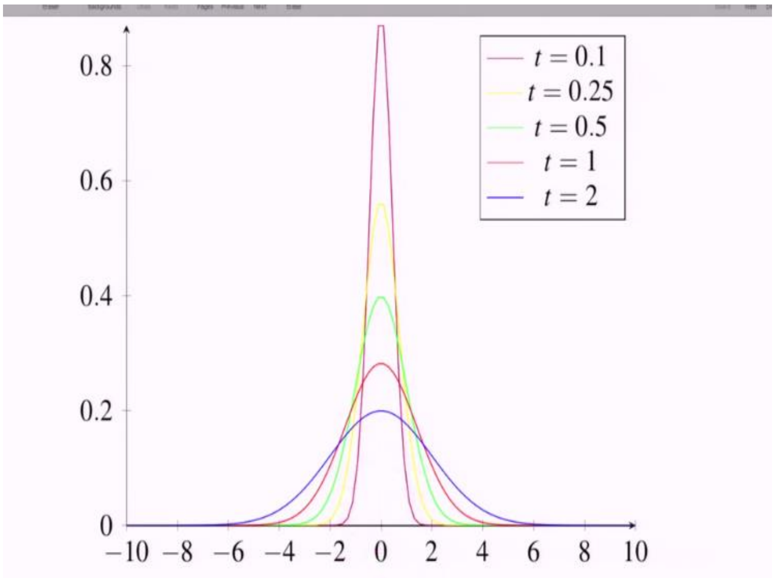

So, her u x of x, t graphs are drawn for various times; t = 2 is in blue, see here, these are one. The t= 1 is in this red colour and yellow is t = 0.25. You see it is getting steep and it steepens around 0, at the same time the values are tending to 0 and other places, not becoming 0, but then getting closer and closer to 0 as t goes to 0. So, t = 0.1 is in this here in magenta colour, see this steepens because the area it has to maintain to be 1, right, area under this curve.

And this is shrinking where u has big value, it is becoming smaller and smaller and therefore it is becoming bigger and bigger near x = 0. So the u x of x, t is concentrating near x = 0 as t is going to 0.

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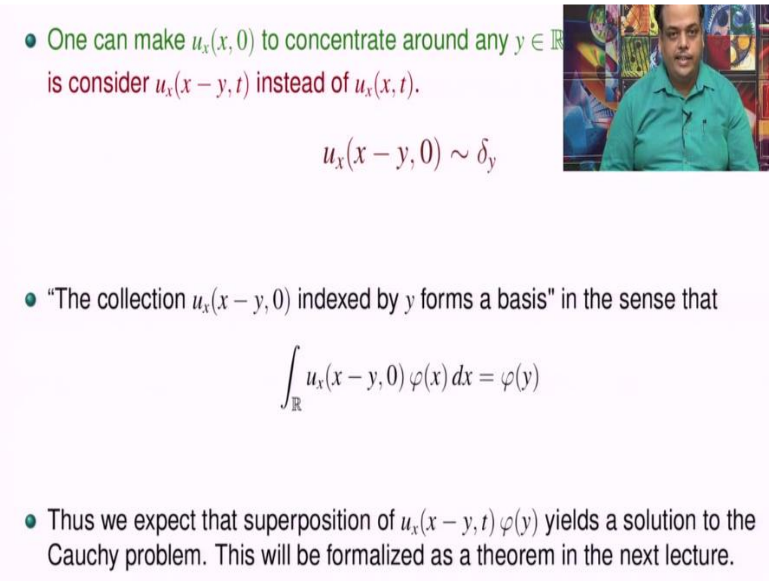

So one can make u x of x, 0 to concentrate around any y by a translation, we already observed that. So you consider u x of x – y, t instead of u x of x, t; u x of x – y, 0 will be like delta y. The collection u x of x – y, 0 indexed by y forms a basis roughly speaking in the sense that this when you integrate you will get phi of y. Thus, we expect that the superposition of u x of x – y, t into phi of y yields a solution to the Cauchy problem.

If I can get my phi of y using u x of x – y, 0 I hope that I can get my u of x, t with phi as the initial data using u x of x – y, t that is a hope we have. This will be formulated as a theorem in the next lecture.

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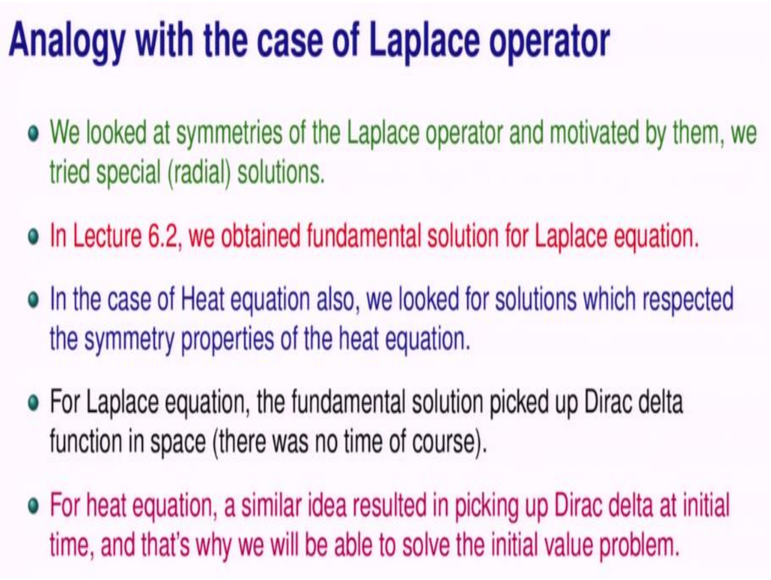

I would like to give an analogy with the case of the Laplace operator. We looked at symmetries of the Laplace operator and motivated by them. We tried special solutions that is radial solutions. And that gave us fundamental solution for the Laplace equation that is done in lecture 6.2. In the case of heat equation also, we looked for solutions which respected symmetry properties that heat equation enjoys.

For Laplace equation the fundamental solution picked up Dirac delta function in the space. Of course, there is no time in Laplace case. For heat equation, a similar idea resulted in picking up Dirac delta at initial time and that is why we will be able to solve the initial value problem. We will continue this in the next lecture.

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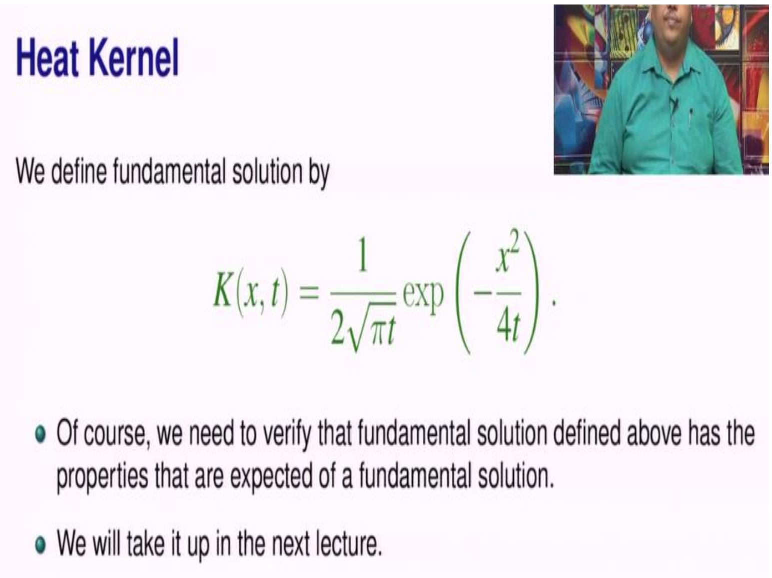

Let us define what is called heat Kernel that is 1 by 2 root pi t exponential – x square by 4 t, K of x, t. This is also known as heat Kernel, our fundamental solution, we can call fundamental solution. Of course, we need to verify that the fundamental solution defined above has the properties that are expected of a fundamental solution. We will take it up in the next lecture. Thank you.