Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology-Bombay

Lecture-53
Laplace Equation

Strong Maximum Principle and Dirichlet Principle

Welcome, in this lecture we are going to study a stronger version of the weak maximum principle that we considered in the last lectures, it is called strong maximum principle. We will also be discussing what is known as Dirichlet principle. Dirichlet principle roughly speaking, it says that solving a Dirichlet boundary value problem is same as solving a minimization problem for a functional; let us get into the lecture. The outline is consisting of 2 points.

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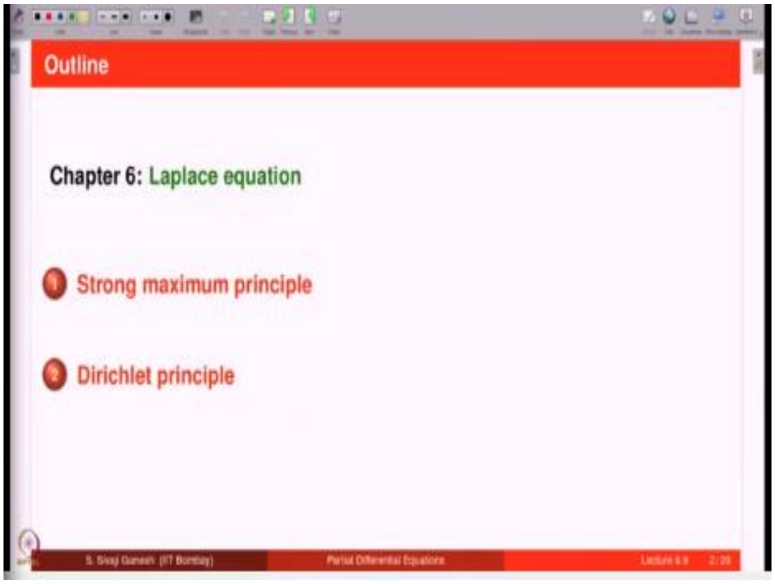

Strong maximum principle and then we move on to discuss Dirichlet principle.

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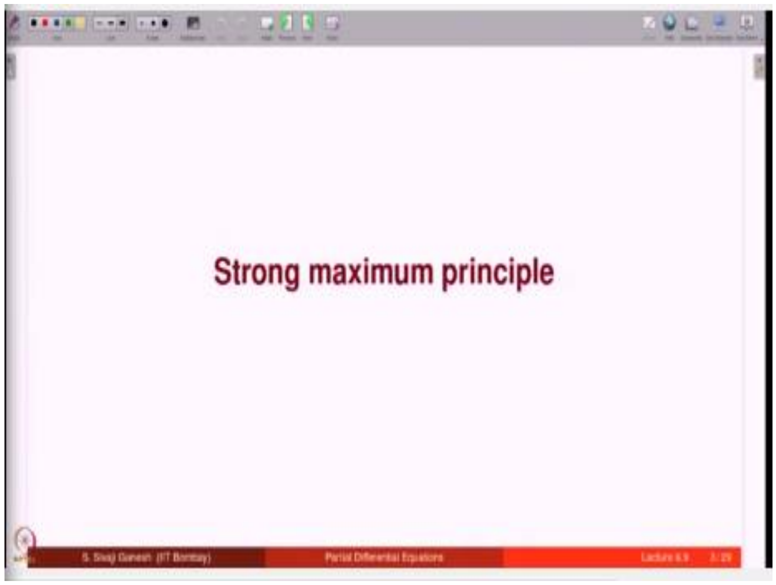

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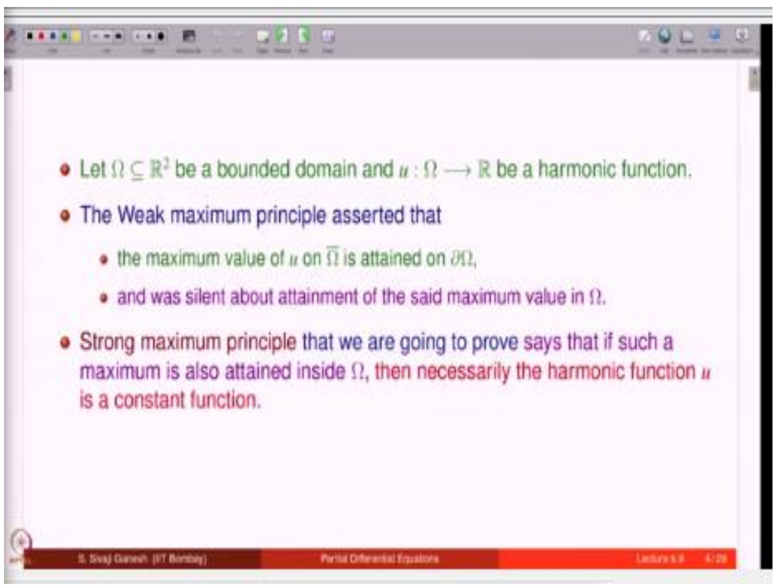

So, let omega be a bounded domain in R 2 and u be a harmonic function in omega. The weak maximum principle asserted that the maximum value of u on omega closure is attained on the boundary of omega. Of course, it never told where else the maximum is attained. In particular, whether maximum is attained in the domain omega or not, it did not say. Strong maximum principle that we are going to prove says that if such a maximum is also attained inside omega, then the harmonic function must be a constant function.

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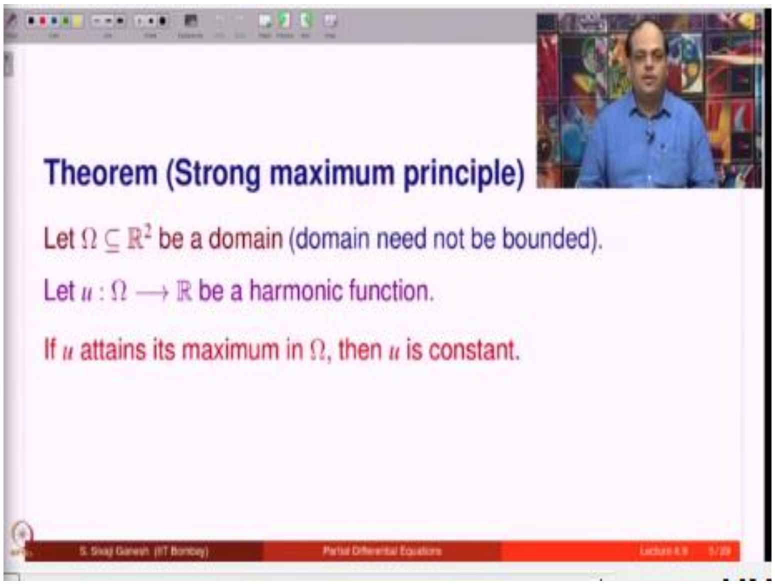

So, let us take strong maximum principle as a theorem. Let omega inside R 2 be a domain, domain need not be bounded let u from omega to R be a harmonic function. If u attains its maximum in omega, then u is constant. Notice in the hypothesis of strong maximum principle, we are not assuming that omega is a bounded domain. Therefore, even if you have a continuous function, which is even continuous on the closure of omega the maximum or minimum may not make sense.

Therefore, strong maximum principle does not talk about that. On the other hand, what does it talk about is if there is a maximum and that maximum is attained in omega, then the harmonic function has to be a constant function, that is what the strong maximum principle assert.

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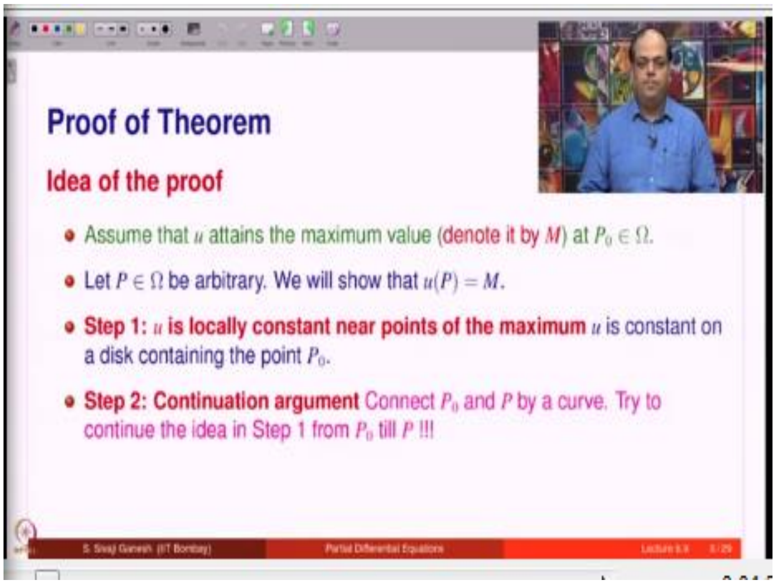

So, what is the idea of the proof? Assume that u attains the maximum value denoted by M. In other words, there is a hidden assumption in the background that is supremum is indeed meaningful and that supremum is attained at some point in omega. That when it is attained the supremum is called maximum and let us denote it by M at some point P 0 in omega. So, P 0 is a point in omega where u attains the maximum value namely the M.

Let P be any other point in omega, we will show that u of P = M, so what does it mean? u of P = M for every P in omega, that means u is a constant function, how do we show this? Step 1, u is locally constant near points of the maximum. We are assuming that the maximum is attained at P 0, therefore what is the meaning of the sentence u is locally constant near points of maximum. There is a disk around P 0 on which u is constant and that constant is M.

Step 2, continuation argument. We have shown in step 1 that u is constant in a disk around P 0. But I want to show that you have P is also M, so natural idea is to go from P 0 to P using a curve and show all along the curve u is the constant M. Then it follows that u of P = M, we will see how this idea is implemented in the step 2. So, connect P 0 and P by a curve, try to continue the idea in step 1 from P 0 till P.

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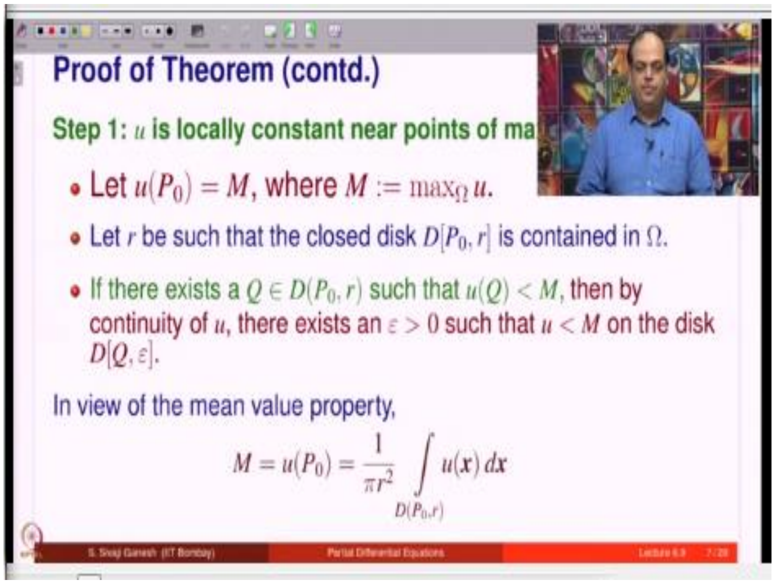

So, let us move to step 1. We are going to show that u is locally constant near points of maximum. I already mentioned how we should read this kind of words locally. So, this u is locally constant near points of maximum. It means take any point of maximum then there is a disk around that on which u is constant that is the meaning of locally constant. Let u of P 0 be M as we assumed, where M is the maximum of u over omega.

Let r be such that the closed disk D P 0, r is contained in omega. Recall this notation, we use is closed brackets here these square brackets to denote the closed balls or closed disks if there is a point Q in this disk P 0, r where u is not M. In other words if u is not M, M being the maximum u will be strictly less than M. So, suppose this happens then by continuity of u, there is a closed disk around Q on which the function remains less than M, this follows by continuity.

So, in view of the mean value property, we know that M is given by u of P 0. Now u of P 0 is given an average or the mean over this disk D of P 0, r this is the area of the disk pi r square integral of u over the disk divided by the area of the disk. So, therefore, this is the average on the disk that is equal to u of P 0, we know this because of the mean value property.

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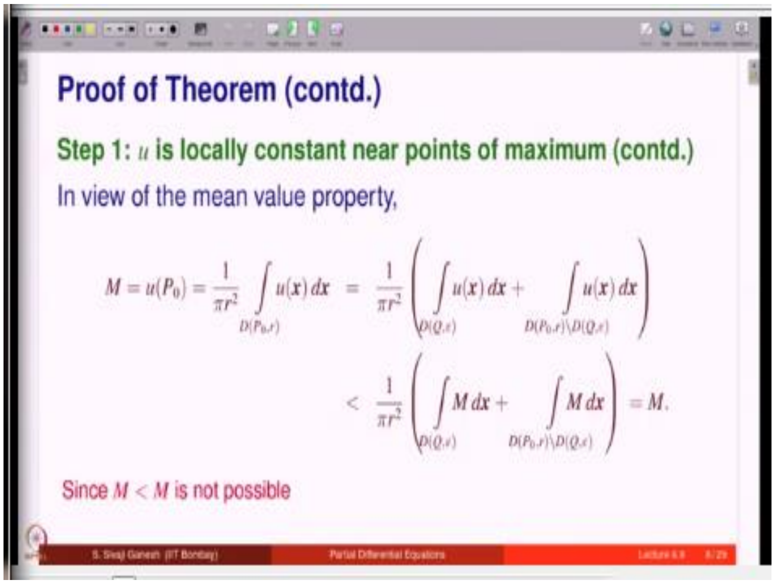

So, we have stated this already on the last slide. Now, we write D of P 0, r as union of 2 things 1 is D P 0, r - D Q epsilon union D Q epsilon. Therefore the integral becomes sum of these 2 integrals. Now, I know that on this it is strictly less than M u is strictly less than M, on this u is less than or equal to M. Therefore, we have a strict less than now, because the first term is strictly less than and is a non zero quantity.

So, if u multiply with a non zero non negative quantity with strict inequality is respective. So, we have strict inequality here, I have replaced u with the bond for u, which is strictly as an M on the disk, and u is less than or equal to M, anyway in omega. Now, if you see these 2 integrals add up to an integral D P 0, r therefore this evaluates to M, because M is a constant, it comes out, what you have is integral of D P 0, r, which will give you the area. And anyway you have a area here, so both get cancelled and you get M. So, what we have got is M less than M, it is not possible.

We conclude that u is constant on the disk D P 0, r, where did we get this contradiction of M less than M? That is because we assumed that there is a point Q at which u is strictly less than M. That is the reason why we got M less than M here. Given any real number, it has to be equal to itself; it cannot be strictly less than or strictly greater than because of the law of trichotomy in real numbers. So, this concludes the proof of step 1, where we have established that there is an r such that u is constant on disk of radius r with centered P 0.

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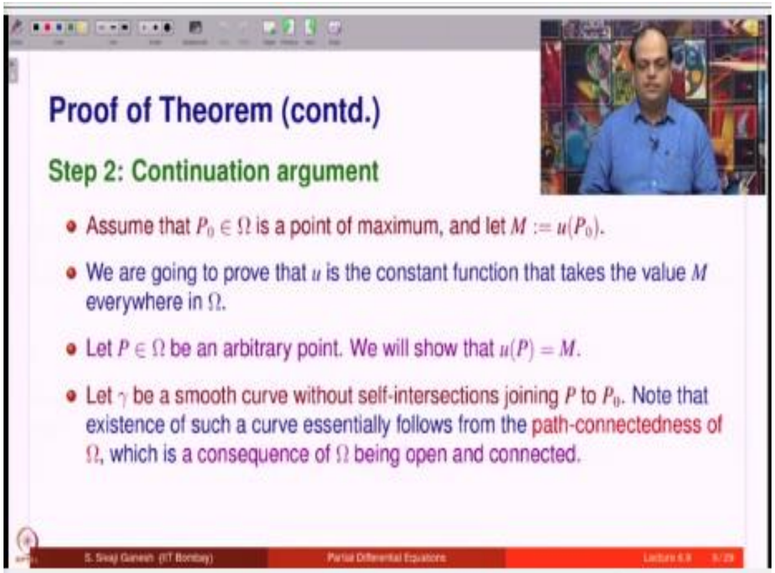

Let us move on to step 2, the continuation argument. Assume P 0 is a point of maximum let M be = u of P 0. We are going to prove that u is a constant function that takes the value M everywhere in omega. So, take a arbitrary point P in omega, we will show that u of P = M. So, let gamma be a smooth curve without self intersections, these kinds of things we need for the technical things that will follow.

Otherwise, simply speaking takes a curve which connects P and P 0. Existence of such a curve essentially follows from the path connectedness of omega, which in turn follows from omega being open and connected. So, once you have a curve, you can always have a smooth curve. So, take it as a fact or try to do the exercise if you have enough background, try to do this as an exercise.

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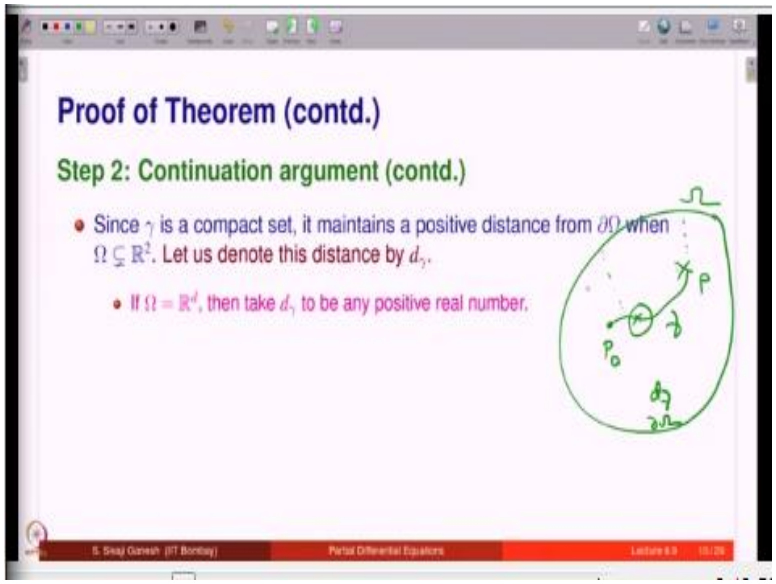

So, since gamma is a compact set, it maintains a positive distance from boundary of omega when omega is a proper subset of R 2. So, let us denote this distance by d gamma. If omega is R d then take d gamma to be any positive real number. So, in other words, we have P 0 here, we take any point P and take a curve gamma. Now, if imagine this is a bounded domain omega, or domain is omega is like this, it has boundaries. Then what is the distance of this curve to this boundary of omega? That is a positive number.

So, we have to see which is a closest point perhaps this is the closest point in this picture or maybe this. So, if you call d gamma as a distance, what does that would mean is that? If you take at any point on this curve, take a circle of radius strictly less than d gamma, it will not intersect the boundary of omega and that is the reason why we are taking this. In other words, this ball of radius which is strictly less than d gamma is properly contained in omega. Of course, if omega is R d you can take any d gamma you do not have to be careful at all.

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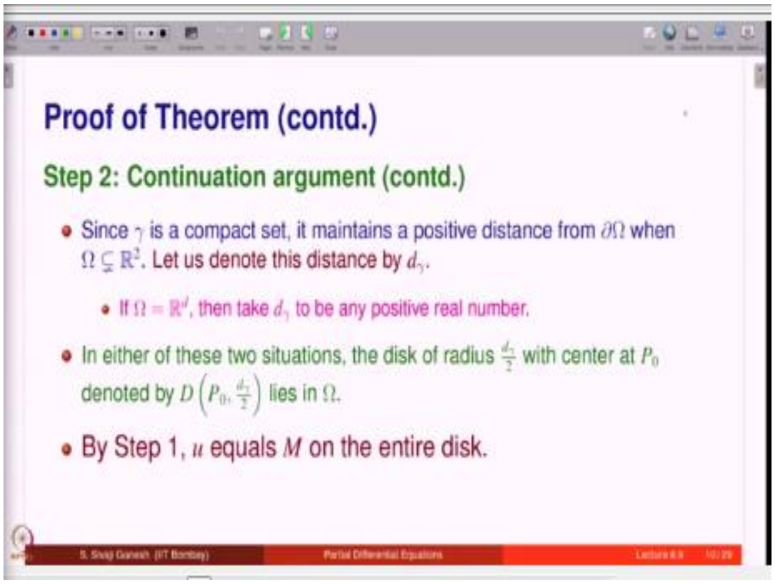

So, in either of these 2 situations the disk of radius d gamma by 2 which is strictly less than d gamma with centered at P 0 denoted by D P 0 d gamma by 2 lies in omega. Not only this even the closed disk D closed bracket P 0 d over 2 close bracket that also lies in omega. So, by step 1, we know that u is constant on this disk, because P 0 is a point of maximum therefore u is locally constant, we proved in step 1.

Therefore, u is identically equal to M on this ball, on this disk. Actually, step 1 says that u is locally constant around points of maximum of u. But if u carefully observe step 1, what we have actually proved is that whenever you find a disk with centre at P 0, where P 0 is a point of maximum of u such that the closed disk is contained in omega then u is identically equal to M on this disk.

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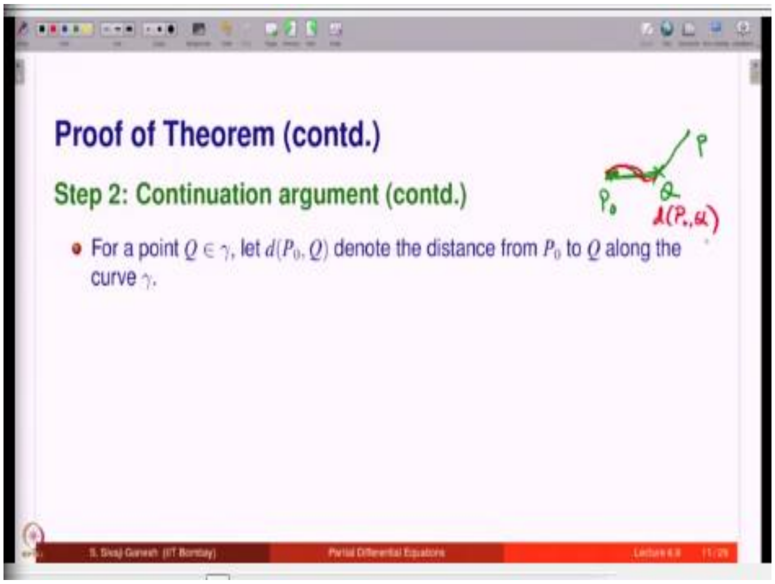

Now, for a point Q in gamma let d P 0, Q denote the distance from P 0 to Q along the curve gamma, so let me just illustrate. Suppose, you have find point P 0 P here and you take any point Q, the usual distance is the Euclidean distance which is the length of this line. But what I am asking you to do is take the length along this curve. You know that if you move along curves and not along straight line the distance will be more. So, go along this that distance is called d of P 0, Q.

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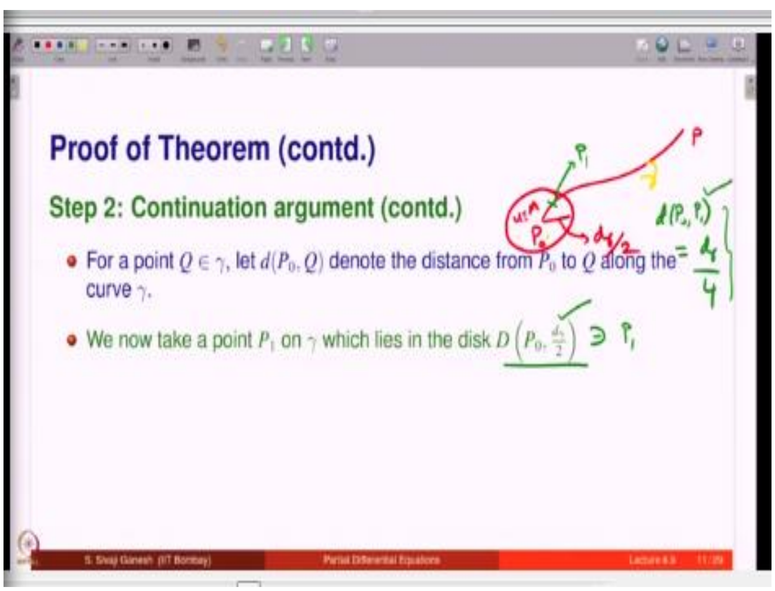

So, we now take a point P 1 on gamma which lies in this disk P 0 d gamma by 2. That means, so this is P 0, this is p and by step 1 we have proved that on this disk whose radius is d gamma by 2 u is constant. Now we plan to take a point P 1 which is on this curve lies in this disk, but where will we take? Will we take here, will we take here? Let us see more prescription, we are going to take on this curve, I want P 1 to be on the curve gamma also. So, I will take P 1 inside this disk here, this is my P 1, so P 1 is inside this ball, P 1 is here.

Not only that, I want to maintain some distance D of P 0, P 1 I will take it to be d gamma by 4. Recall, what is this d P 0, P 1? It is a distance along the curve d gamma by 4, therefore P 1 will lie inside because along a curve I am doing a smaller distance d gamma by 4, where the disk is d gamma by 2. So, this point P 1 cannot be outside, if a P 1 satisfies this criterion, that the distance along the curve is d gamma by 4, it is not outside this disk, therefore full d be inside. So, it is possible to find such a P 1 which is in this disk as well as this criterion.

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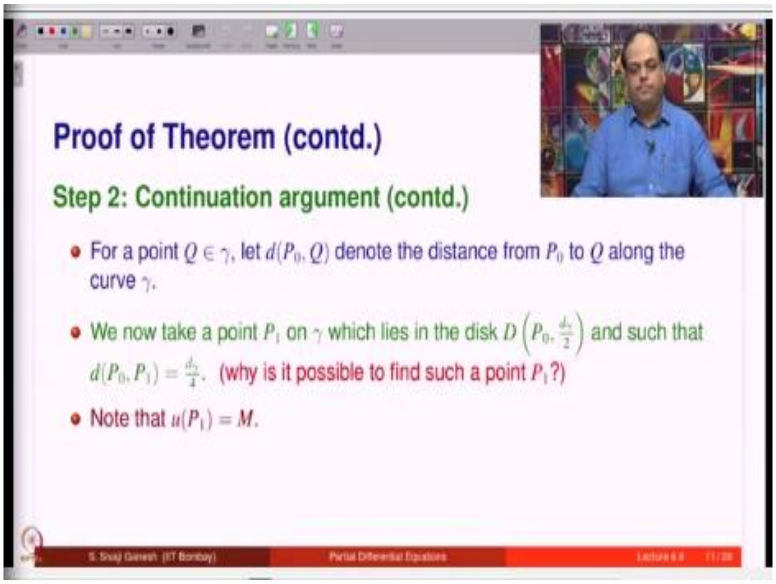

I have already explained why is it possible to find such a point. And u of P 1 = M, because P 1 is in the disk P 0 d gamma by 2, therefore u of P 1 is M.

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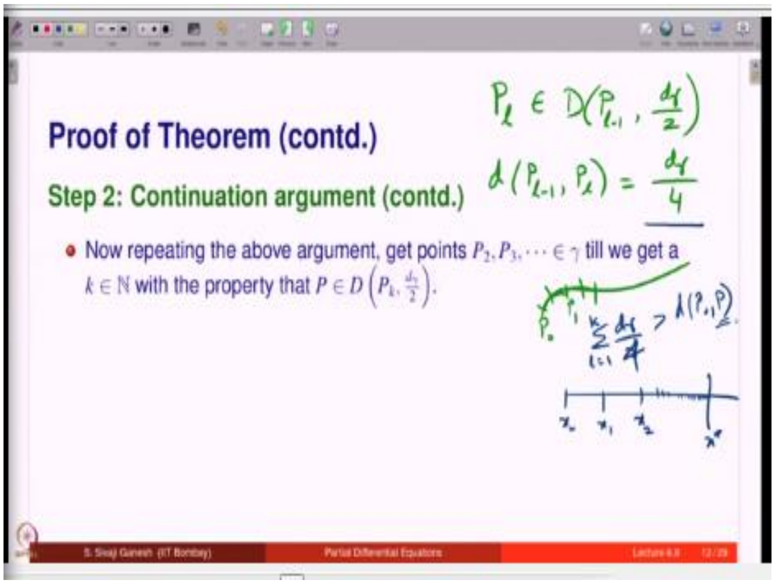

Now repeating the above argument get points P 2, P 3 and so on till you get a k in N such that P belongs to the disk with centre P k and radius d gamma by 2. So, in each of these steps when we try to find P 2, P 3 etcetera, we are insisting on this. That P l of course belongs to the previous one P l - 1 radius d gamma by 2, we are not compromising the radius, radius is always the same. And importantly, the distance between the centre and the point we are choosing is d gamma by 4, the distance is fixed.

That means, we are definitely moving along this curve, P 1 is here, so this distance along the curve is d gamma by 4 and it is here, so these 2 distances are same. See sometimes it might happen that you are moving from some point to another point in the step 1 let us say x 0, x 1, x 2. But then you may be never crossing this some point x star, it can happen that the steps are becoming smaller and smaller and you are getting accumulated somewhere, you are not crossing this x star.

But every time if you move a fixed step like this, you will definitely exhaust the distance which you need to do is simply d of P 0, P that is a distance if you cross using these tiny steps d gamma by 4. How many number of times that is the thing really here. So, l = 1 to k perhaps then definitely you will exceed this. So, somewhere before the P will fall into one of these disks, this is extremely important.

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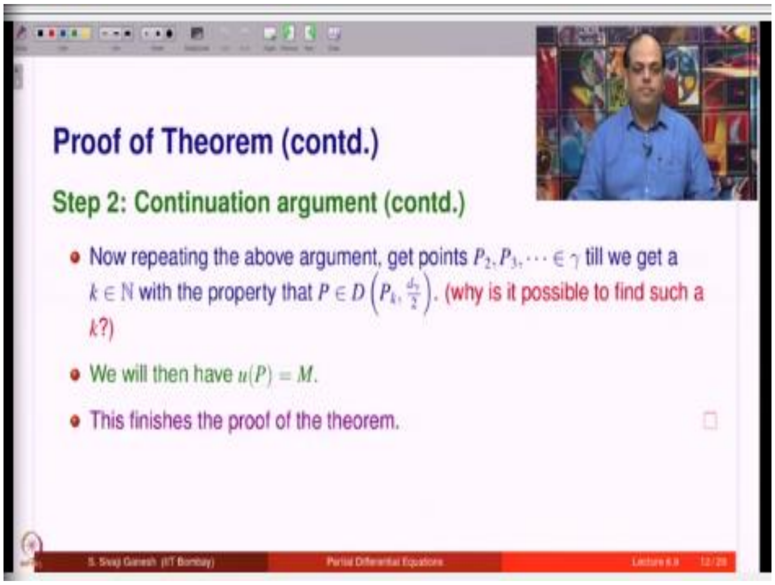

We will then u of P = M because u of P k = M and u is constant on this disk. Therefore at any point in this in particular at P which is in this desk, u will be M. This finishes the proof of the theorem.

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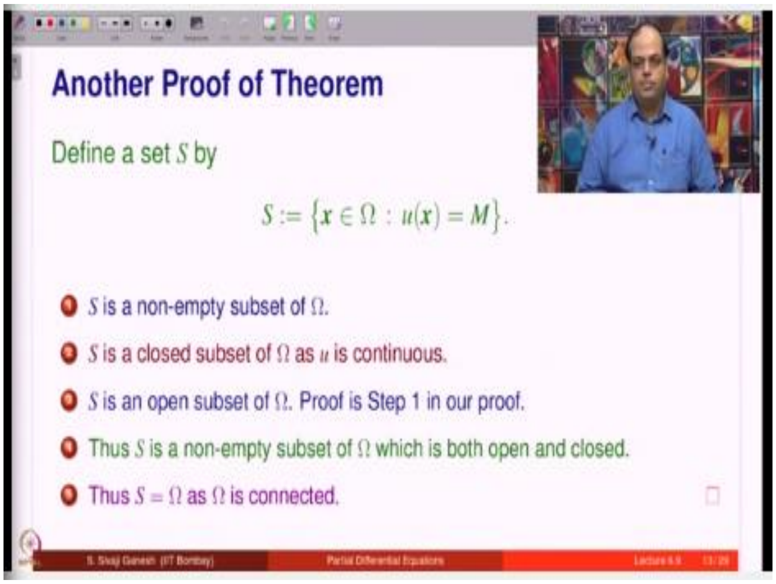

Another proof let us look at. Define a set S by set of all x in omega such that u x = M, S is a non empty subset of omega because we are assuming that u takes the value M. What is the M? M is at maximum value of u, u takes the value M at some point in omega that is the assumption, therefore the set is non empty. And it is a close subset of omega, because u is a C 2 function in particular continuous function, continuous function equaling a constant will be a closed set.

You can also think like this, this is a continuous function, constant function is also continuous, so 2 functions are continuous. That set where they are equal is a closer or you can also look at u x - M = 0 and set of all x where a continuous function takes a value 0 here close it. Now, it is an open subset of omega, it is not clear just from this definition, using continuity you can only prove it is closed, we have to u is something extra that we know about u namely u is harmonic.

Open set is exactly step 1 in our proof, where we have proved that if u take the maximum value at some point, then there is a disk around that point where u take constant value M. That is precisely the meaning of showing the set is open set. So, we have got a set which is non empty, open and closed, it is a subset of omega, what is omega? It is a open and connected set, there you have a subset which is non empty and both open and closed. By connectedness of omega such set has to be the full set that is S = omega, this is the another poof because we are using here omega is connected.

The proof looks so simple, because we are using the fact that if you are working in a connected set or a connected topological space, a subset which is both open and closed has to have only 2 choices either it is empty set or the whole set, we are using that result here, that is why it is simple. And step 1 proof anyway, we have to supply here. So, essentially step 2 is removed and we are appealing to the connectedness. Of course we have used the connectedness in another format in the other proof also.

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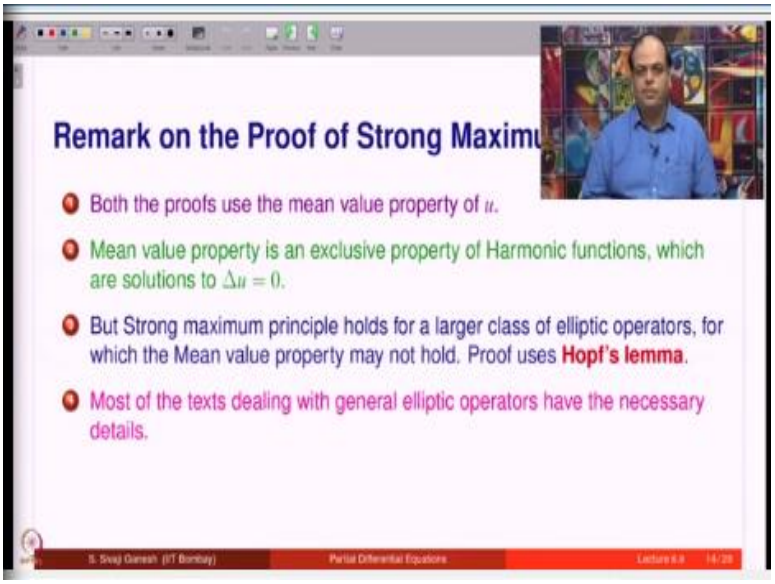

Remark on the proof of strong maximal principle. Both the proofs u is the mean value property of u, namely in step 1. Mean value properties in exclusive property of harmonic functions, recall we have not only shown that every harmonic function has mean value property, but also every continuous function which has mean value property is harmonic. So, almost mean value property is exclusively a property for the harmonic functions. Harmonic functions means solutions are Laplacian u = 0.

But, strong maximum principle holds for a larger class of elliptic operators for which mean value property may not hold, because general elliptic operator need not be just Laplacian all the time. So, there are operators which are more general than Laplacian for which also the strong maximum principle holds. And the proof uses what is known as Hopf's lemma, lemma of Hopf. Most of the text dealing with general elliptic operators has the necessary details, you may cancel them if you are interested. But in this course, we will not go beyond Laplace equation as I have pointed out at the beginning.

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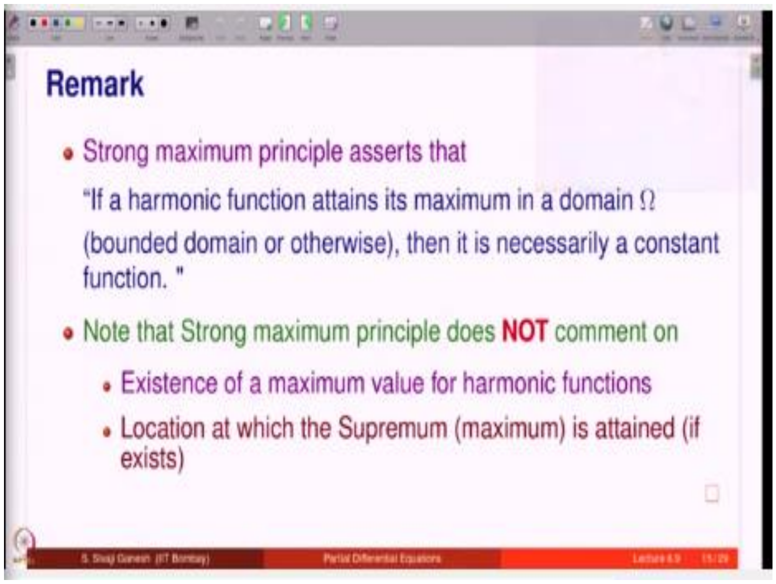

Strong maximum principle asserts that, if a harmonic function attains it is maximum in a domain omega bounded or otherwise, then it is necessarily a constant function. Note the strong maximum principle does not comment on existence of a maximum value for harmonic functions, does not comment about location at which supremum is attained if it exist.

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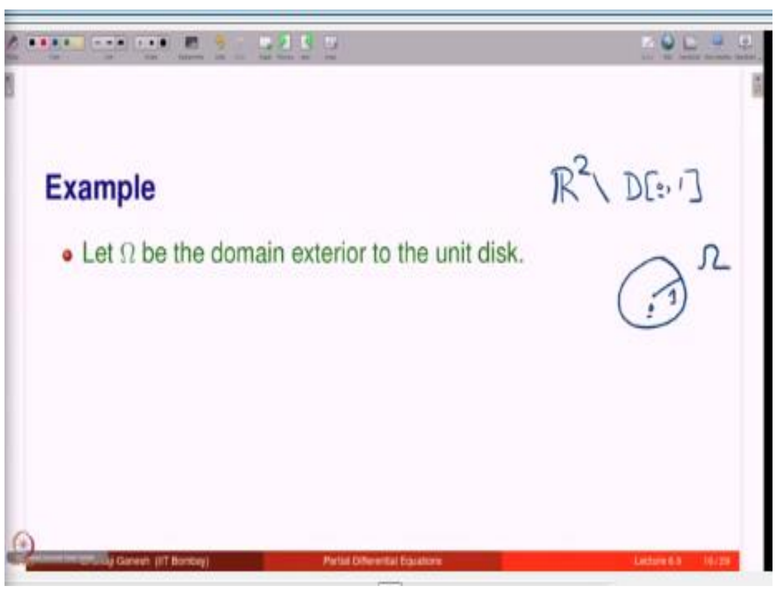

Let us look at an example. Let omega be the domain exterior to the unit disk, it means my domain is R 2 - the unit disk that is origin radius 1, my omega is this.

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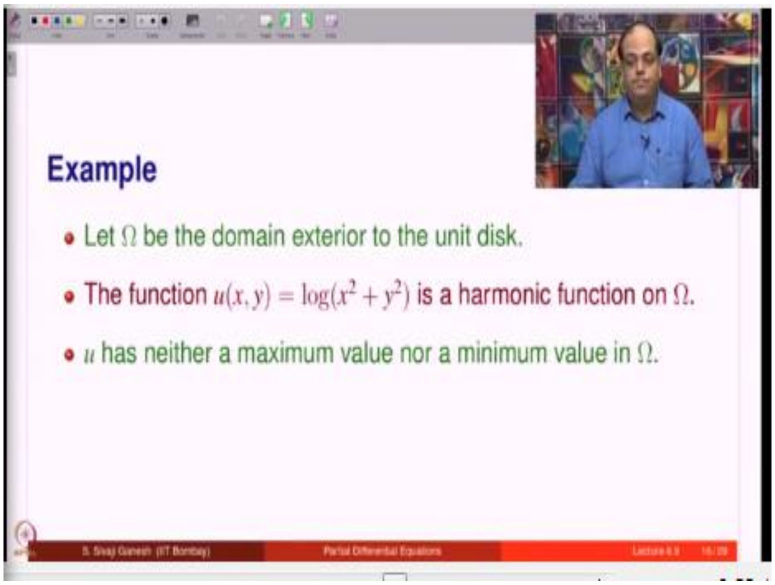

This function u x, y = log x square + y square is a harmonic function and omega, this can be easily checked, u have nice neither a maximum value nor a minimum value in omega.

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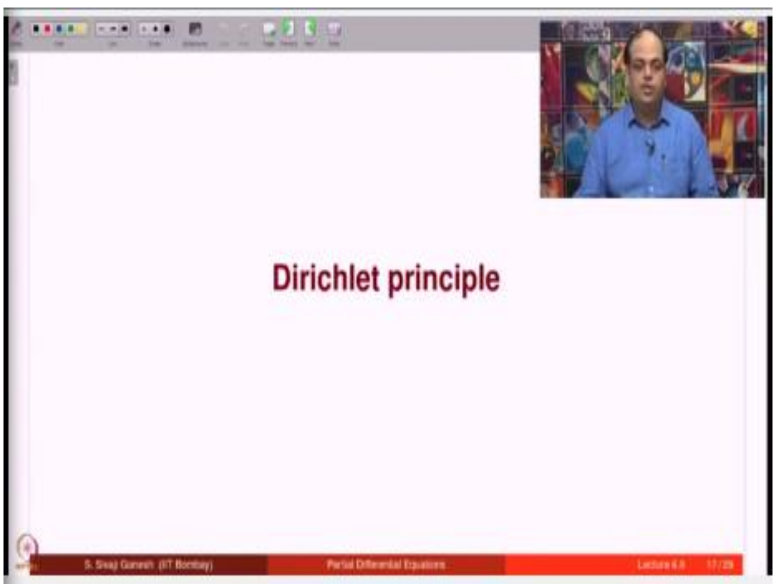

Let us look at Dirichlet principle.

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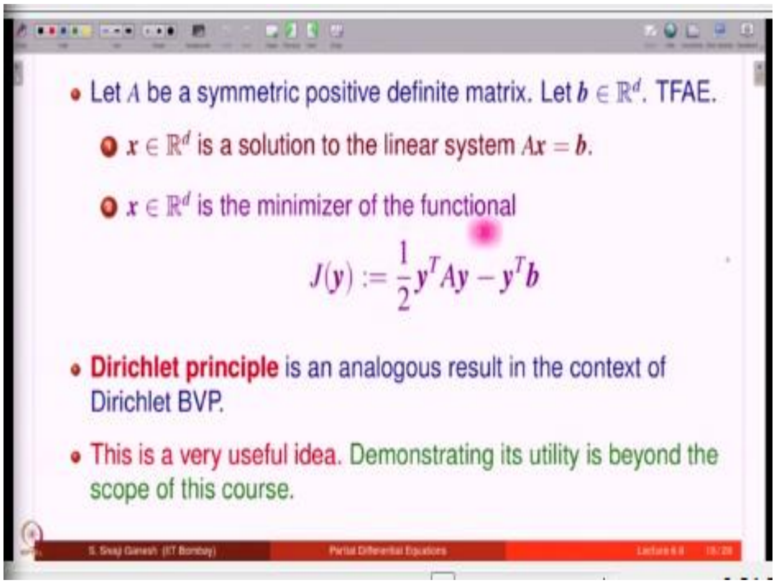

Before that, let us recall some facts about a system of linear equations. So, let A be symmetric positive definite matrix, let b be a vector in R d then the following statements are equivalent. x belongs to R d is a solution to the linear system Ax = b that is same as saying x is the minimizer of this functional J of y half y transpose Ay - y transpose b. Dirichlet principle is an analogous result in the context of Dirichlet boundary value problems, this is a very useful idea.

Demonstrating it is utility is beyond the scope of this course, nevertheless let me mention a couple of points. We have many methods to solve the system of linear equations Ax = b. For example, we have what is known as direct methods, which will give us exact solutions like Gaussian elimination method and some modifications of that. Exact methods are good, but when this A is a big size matrix, matrix of big size that is d is very big then it is not profitable.

In fact, a lot of errors might get enhanced in the method of solution. And people have found that conjugate trading method is one of the very useful methods which is based on minimizing this functional. So, Dirichlet principle is analogous to this result and if you understand the utility of this result, it is easier to understand how this will also be useful. In fact, these kinds of ideas are used in establishing solutions to elliptic equations and that method is also called calculus of variations.

And method itself is called the first method is called the direct principle of calculus of variations where they will look at a minimizing sequence. That is a sequence, for example in this context sequence of vector y n such that J of y n converges to infimum of this functional and show that y n converges to some y and J of y is actually the minimum of the functional. So, that is the general idea in the direct method in calculus of variations.

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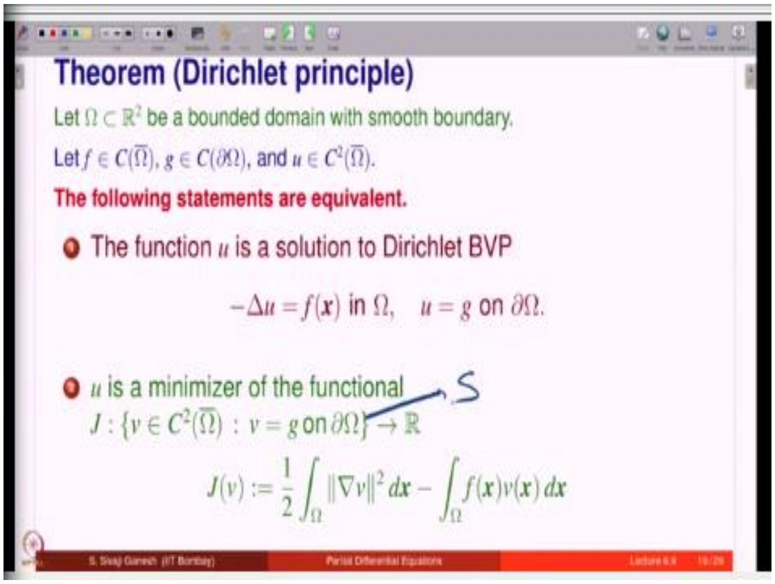

What is Dirichlet principle? Let omega be a bounded domain in R 2 with smooth boundary. Let f be continuous on omega bar, g belongs to C of boundary of omega and u is C 2 of omega bar. Then the following statements are equivalent. The function nu is a solution to the Dirichlet boundary value problem, -Laplacian u = f in omega and u = g on the boundary of omega. And that is same as saying u is a minimizer or the functional J defined by this formula below.

And defined for v, which is in C 2 omega bar such that v = g on boundary of omega. So, minimizer in this set, because if u is a solution to Dirichlet boundary value problem using this set, u = g on boundary of omega u is C 2 of omega bar by our assumption. So, for a C 2 omega bar function, these 2 statements are equivalent. If you know that u is a solution to the Dirichlet boundary value problem, you can prove that it is a minimizer of this functional. And so, conversely if u is a minimizer of this functional then it actually solves the Dirichlet boundary value problem. So, let us denote the set by S.

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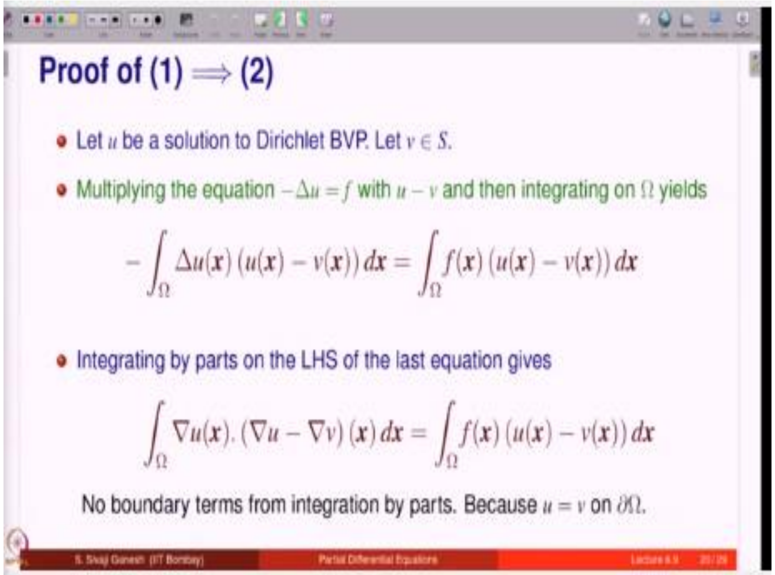

So, let us move to proof of 1 implies 2. Let u be a solution to the Dirichlet BVP (()) (26:07) let v be an element of S. Multiply the equation -Laplacian u = f with u - v and then integrate on omega which by this we get this. Then integrate by parts on the LHS that means the Laplacian u becomes grade u and you will get a gradient here which is here, gradient of u - v grad u - grad v. No boundary terms from integration by parts because u = v on the boundary.

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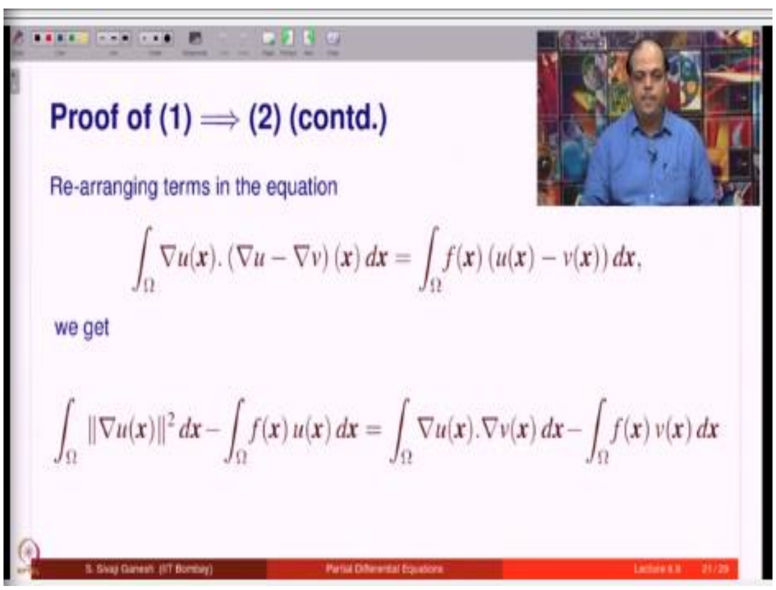

So, rearranging the terms in this equation we get this. So, bring this term to this side take this time here to that side.

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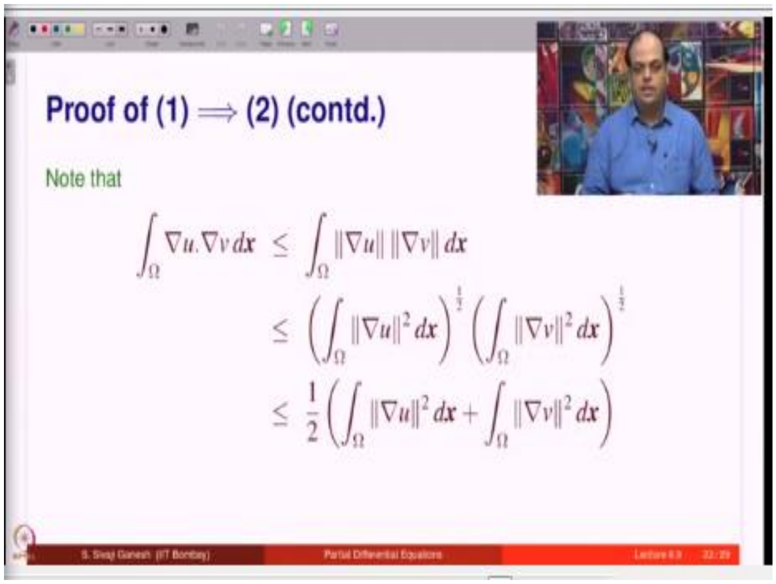

Now, look at this inequality, integral grad u dot grad v is less than or equal to integral non grad u into non grad v. Simply because a dot b is less than or equal to norm a into norm b, this is a Euclidean norm after vector. So, it is grad u of x grad v of x is less than or equal to non grad u of x into non grad v of x. Now, here this integral is less than or equal to this integral. After that, this is called the first time as a, second term as b this is a into b. That is less than or equal to a square + b square by 2, I have used that, so we get this.

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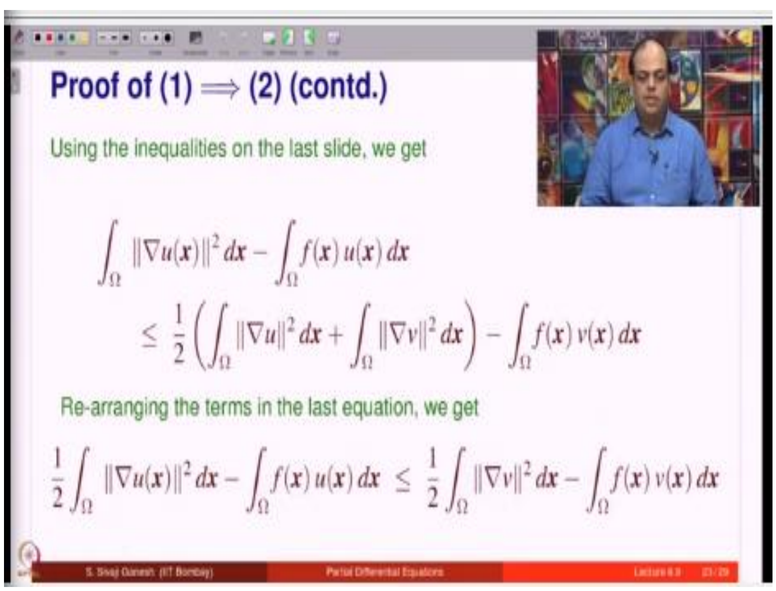

So, using the inequalities on the last slide, we get that integral over omega of non grad u of x square dx - integral over omega of f u is less than or equal to this - integral over omega of f v.

So, rearranging in terms we get this, all the terms featuring u on one side and v on the other side, but what is this?

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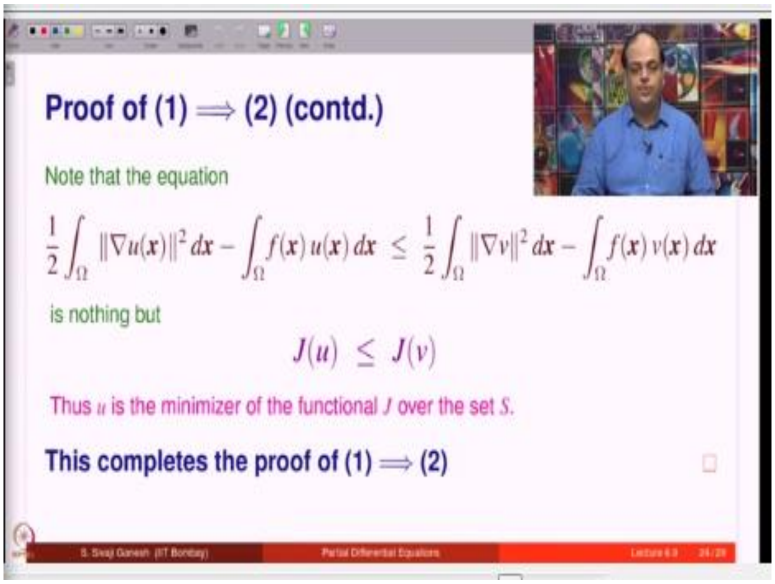

This is nothing but J u is less than or equal to J v, this is the definition of J u; this is the definition of J v. Thus u is a minimizer of the functional J over the set S, this completes the proof of 1 implies 2.

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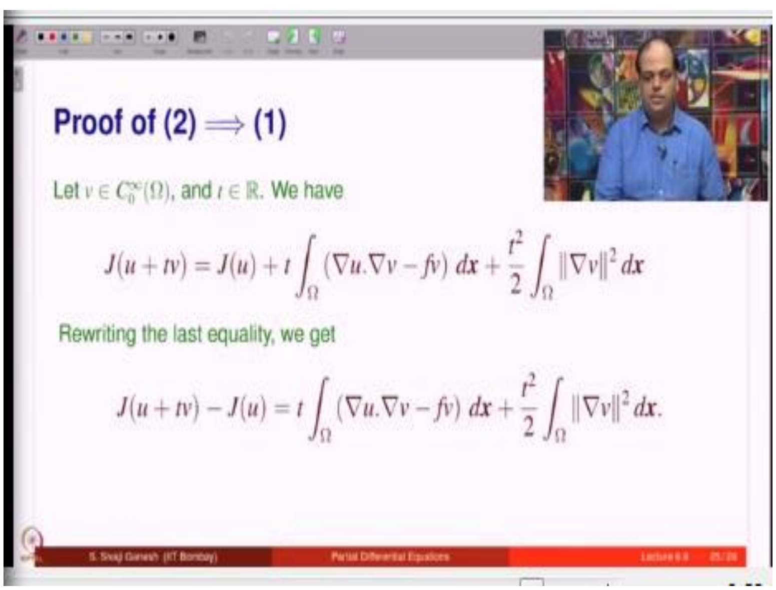

Let us look at proof of 2 implies 1. So, let v be a C 0 infinity function defined an omega and t be a real number. Then we have J of u + tv, u plug into the formula of J you get this expression.

And rewriting this what we get is J of u + tv - J if u take this to the left hand side this remains as it is, RHS.

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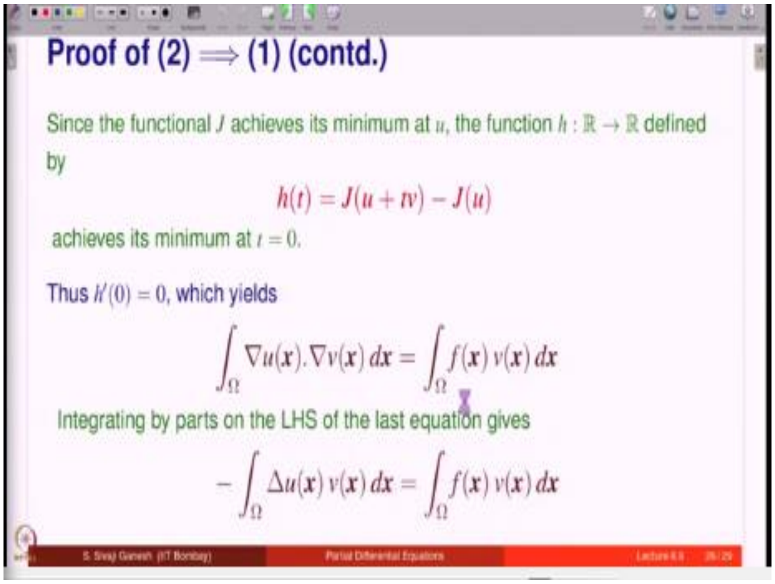

So, since the functional J achieves it is minimum at u that is the hypothesis in 2. The function h here from R to R defined by h of t = J of u + tv - J of u that achieves minimum at t = 0. Thus, h prime of 0 is 0 which will give us this relation. Integrating by parts on the LHS will give us - Laplacian into v. So, integral -Laplacian u into v = integral f v and this is true for every v which is C 0 infinity of omega.

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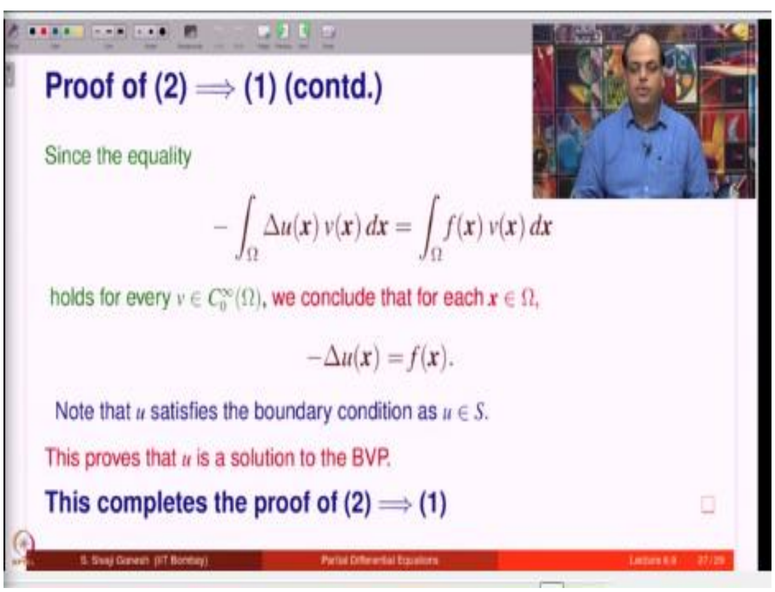

Therefore, -Laplacian u of x = f of x. Note that u satisfies the boundary condition as u belongs to the set S, the set S in the definition itself includes that u = g on the boundary of omega. So, here we need not work with C 0 infinity of omega, we can as well work with C 0 2 of omega, that is C 2 functions with compact support in omega. If this equality holds for every v which is C 2 0 of omega, then you have that -Laplacian u = f holds at every point of omega. So, this proves that u is a solution to the BVP. Thus completing the proof of 2 implies 1.

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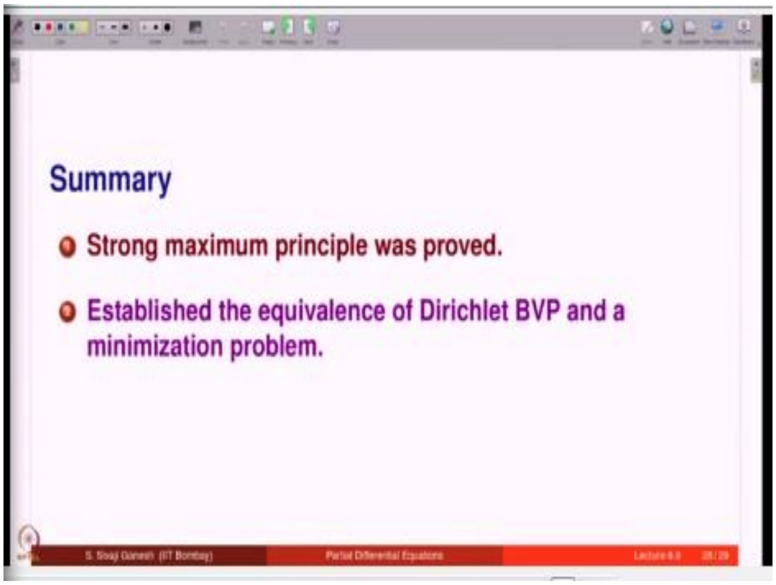

So, let us summarize what is done in this lecture. We have proved a strong maximum principle and we have established the equivalence of Dirichlet boundary value problem and a minimization problem. Thank you.