Partial Differential Equations
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Lecture-52
Laplace Equation

More Qualitative Properties

Welcome, in this lecture, we are going to look at few more consequences of mean value property. Outline of lecture is as follows. First we prove that uniform limit of harmonic functions is harmonic.

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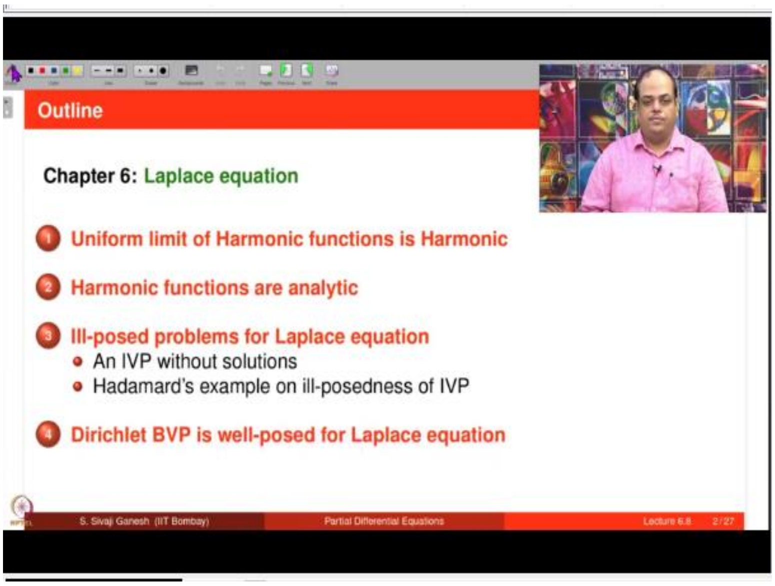

This is sounding like uniform limit of continuous functions is continuous, exactly like that uniform limit of harmonic functions is harmonic and then harmonic functions are analytic, then we are going to look at II ill posed problems for Laplace equation, we will see one problem without solutions and we see one problem where there is existence uniqueness, but the third requirement of Hadamard's phase and thereby we get a ill-posedness. It is an example of Hadamard. Then we show that Dirichlet boundary value problem is well posed for Laplace equation.

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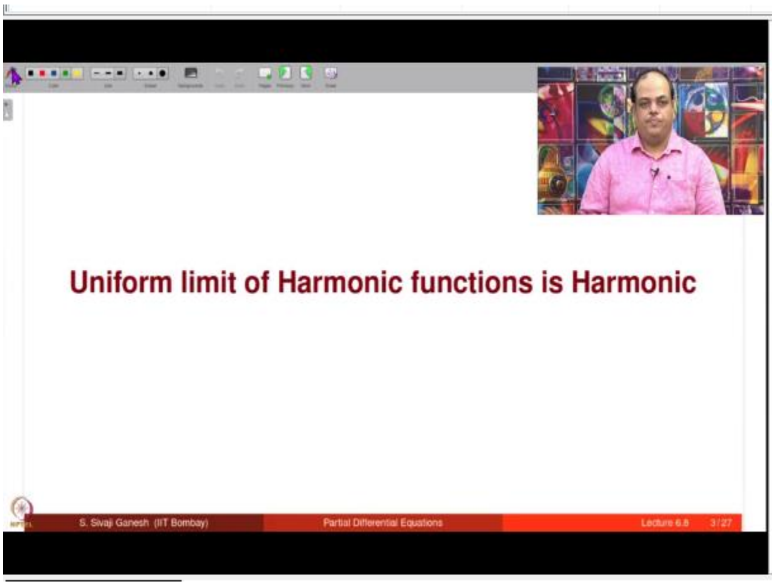

So, uniform limit of harmonic functions is harmonic.

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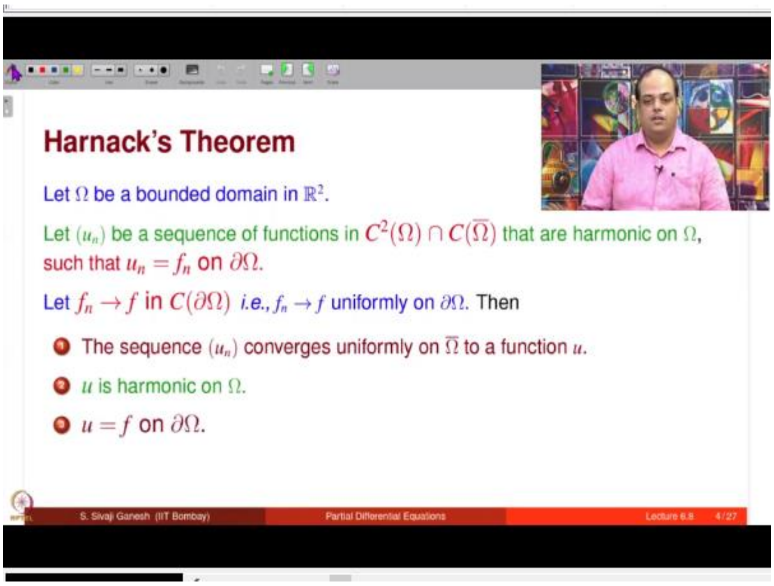

It is also called Harnack's theorem, there are too many theorems going after this name or not. So, you have to be careful when referring to these theorems. So, this theorem says take a bounded domain omega in R 2 and take a sequence of functions, which are C 2 of omega and C of omega bar which are harmonic on omega. That means Laplacian u n is 0 for every n. And to make sense of that we need C 2.

And C of omega bar, it means we are going to talk about the values of u n on the boundary of omega. So, that is going to come now. Such that u n = f n on boundary of omega, which means f n's are prescribed. And then u n's are solutions to Laplacian equals to 0 satisfying this Dirichlet boundary condition. Suppose, f n goes to f in C of boundary of omega, it means, modulus of f n - f goes to 0 uniformly as experiencing boundary of omega.

So, it is a uniform convergence, say omega is a bounded domain, boundary of omega is a compact set, continuous functions and compact set, you can define maximum norm of such functions, supremum norm it is called sometimes, it is actually the maximum. So, a maximum of mod f n - f as x varies in boundary for omega goes to 0 that is the meaning of uniform convergence. Then, the sequence u n converges uniformly on omega closure.

That means, u n also converges uniformly if the boundary values converges uniformly to a function u that means, there is a such a function u to which u n converges uniformly and u is harmonic. That means Laplacian u is 0. If you look at uniform convergence, if you have a sequence of whatever smoothness you have, if it converges uniformly, you can expect the limit to be only continuous, but now, we are saying use harmonic. That means Laplacian u = 0. That means, something needs to be proved that u is twice differentiable. So, that Laplacian u make sense and then further show that Laplacian = 0 and u = f on the boundary of omega.

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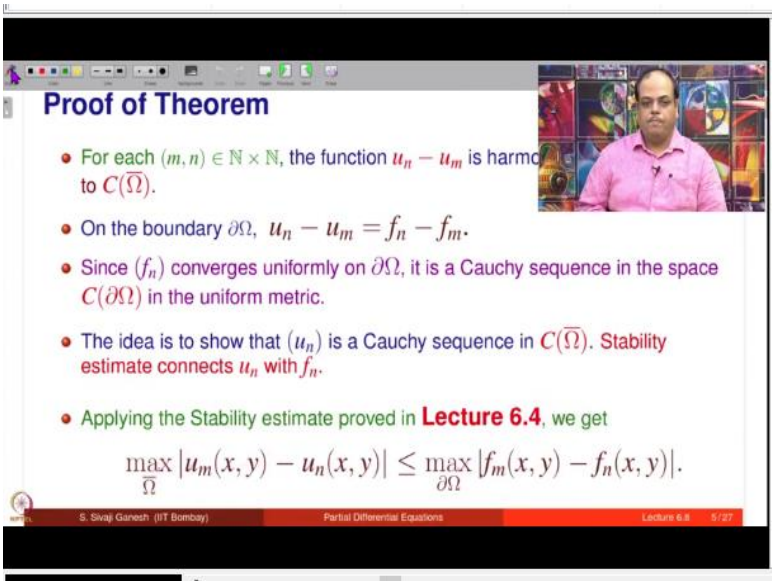

Let us turn to the proof of the theorem, fix m and n, then u n - u m is harmonic on omega, because it is a difference of 2 harmonic functions and belongs to C of omega bar because both u n and u m belongs to C omega bar. On the boundary of omega u n - u m is nothing but f n – f n. Because u n is f n on the boundary, u m is f m on the boundary. Since f n converges uniformly on boundary of omega it is a Cauchy sequence in C of boundary omega space that is a uniform metric.

Uniform metric means distance between 2 functions, let us say f and g in C of boundary of omega is defined as maximum as x varies in boundary for omega of modulus f of x - g of x. That is what is called the uniform metric. The idea is to show that u n is a Cauchy sequence in C of omega bar and stability estimate connects u n with f m. Applying the stability estimate which we have proved in lecture 6.4 we get this.

This is the stability estimate, because u m – u n is f m – f n on the boundary. So, maximum of u – u n on omega closure is less than or equal to maximum of u m – u n which is f n – f n on the boundary of omega. Now, we know that this is a Cauchy sequence. Therefore, this will be a Cauchy sequence.

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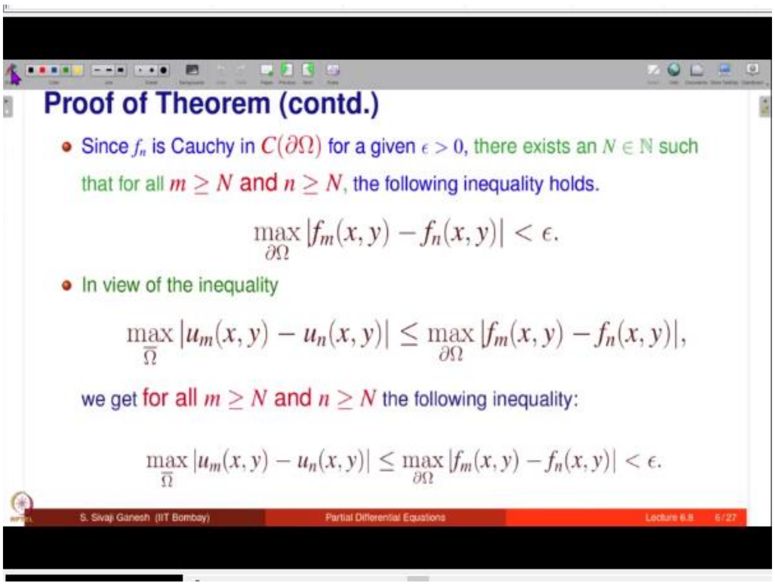

Since f n is Cauchy in C of boundary omega for given epsilon you can find a N such that this can be made at less than epsilon, whenever m and n are bigger than or equal to this N, this is a definition of a Cauchy sequence. In view of the inequality that we have written down on the last slide, which is coming from the stability estimate, we have this. Now, in view of this, this will tell us that maximum omega closure mod u m – u n is also less than or equal to epsilon in fact less than epsilon. For every m and n bigger than or equal to n.

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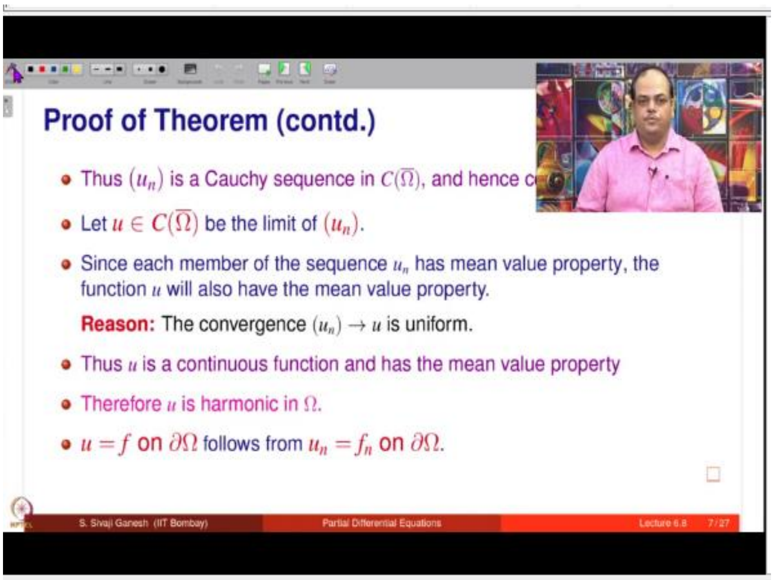

So, therefore, u n is a Cauchy sequence in C of omega bar and hence converges uniformly. Why a Cauchy sequence converges, because the space is a complete space, it is a complete metric space or it is a Banach space therefore every Cauchy sequence converges in particular u n converges. Call the limit as u. So, u will be an element in this space C of mega bar. So, let u belongs to C of omega bar be the limit of u n.

But now, we want to show that u is a harmonic function that means, we have to somehow show that u has 2 derivatives to start with. Since each member of the sequence u n has mean value property because u n's are harmonic functions. The function u will also have the mean value property. Reason the convergence u n going to u is uniform and uniform convergence tells us that we can swap integrals and limits.

This u is a continuous function and has the mean value property. Therefore, u is harmonic, we are shown this earlier a continuous function which has mean value property is harmonic function, we did this in lecture 6.7 and u = f on the boundary of omega follows from u n = f n on boundary of omega.

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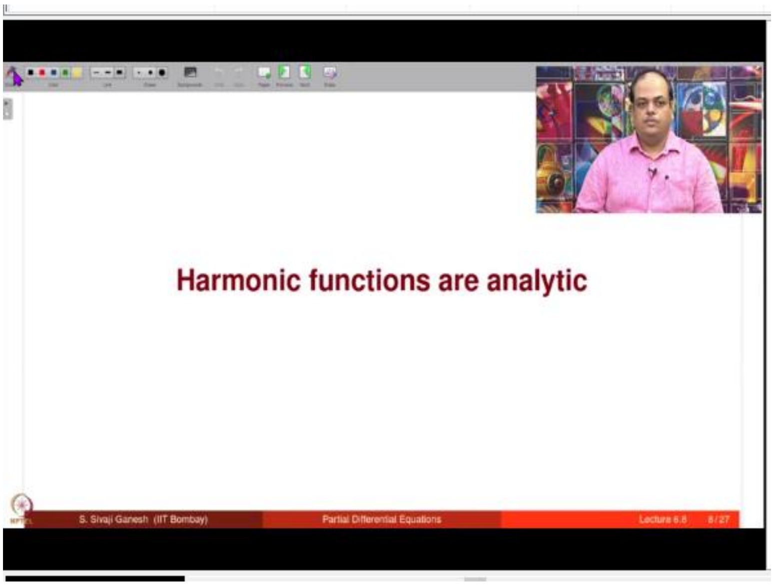

Now, let us show harmonic functions are analytic.

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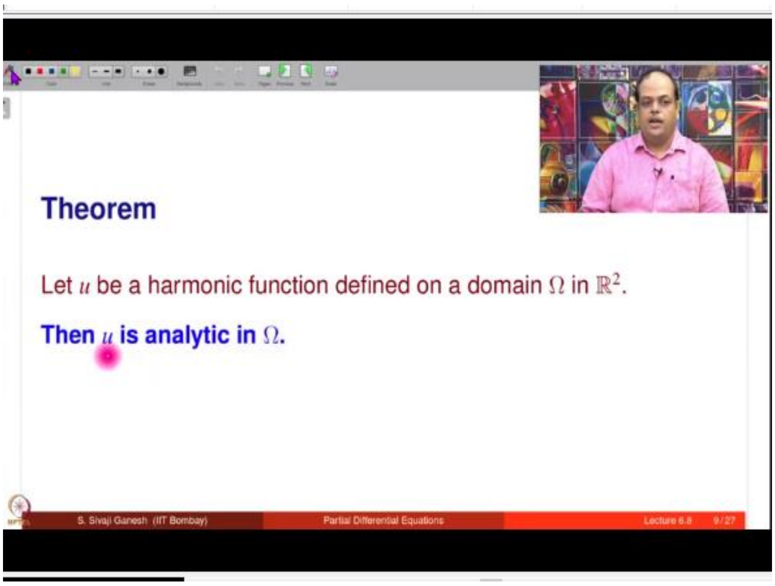

Let u be a harmonic function defined on a domain omega in R 2. Then u is analytic, what is the meaning of analytic? Given any point in omega, there is a disk around that point on which u has a Taylor series representation.

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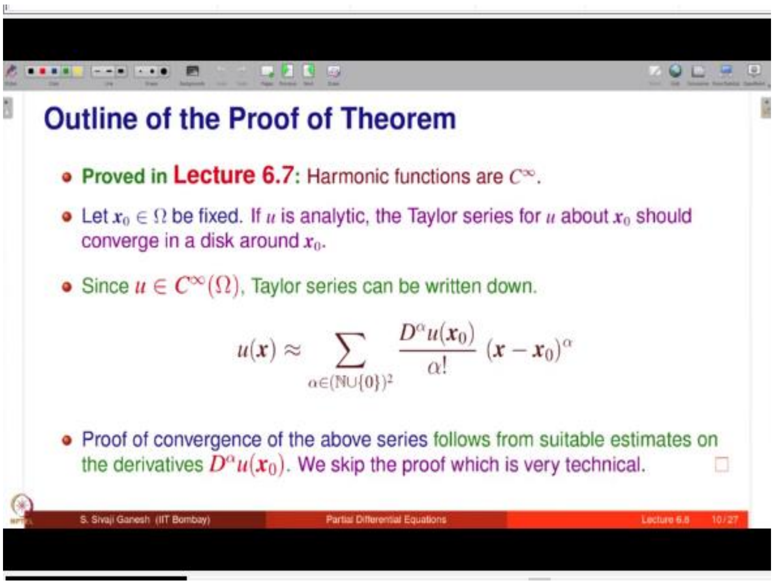

Outline of the proof of the theorem, we are not going to prove the theorem itself. So, we proved in lecture 6.7 harmonic functions are C infinity. Let x 0 belongs to omega be fixed, if u is analytic the Taylor series for u about the point x 0 should converge in a disk around x 0. That is the definition. Since u is C infinity of omega Taylor series can be written down. So, this is a formal Taylor series, we need to show that this series actually converges to u x at every point x in a disk around the point x 0.

Since the function is C infinity, this can be written down because each other term can be written down. Essentially what we know is d alpha u x 0 is meaningful because u is infinity. Proof of convergence of the above series follows from suitable estimates on the derivatives, because these are very generic terms, they are not going to help you much. This is what is required D alpha u, it should have some decay estimates, we skip the proof as very technical. Thus, harmonic functions are analytic.

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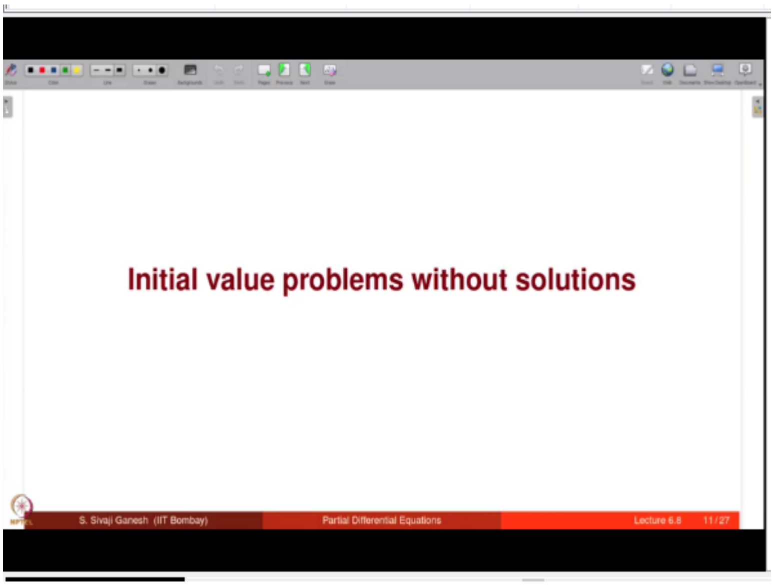

Initial value problems without solutions. So, example of an ill posed problem we are going to see.

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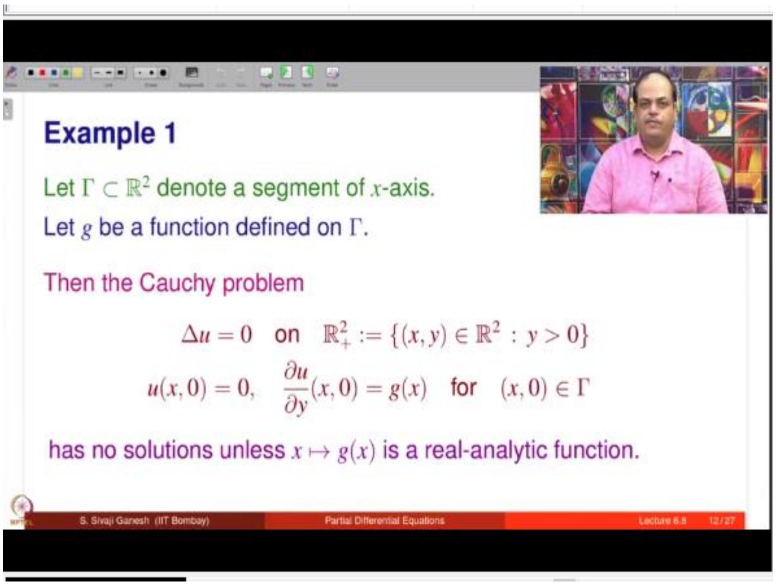

Example 1, let gamma denote a segment of x axis, let g be a function defined on gamma. Then this Cauchy problem Laplacian u = 0 in the upper half plane x, y in R 2 sided y positive. So, these upper half plane and on the x axis we have given the Cauchy conditions u of x 0, 0 and dou u by dou y at x, 0 is g x, it is given for all x, 0 belonging to gamma. That is points of gamma. It has no solutions. Unless this function g of x itself is an analytic function, sometimes we call it real analytic function just to distinguish it from the analytic function or the complex analysis.

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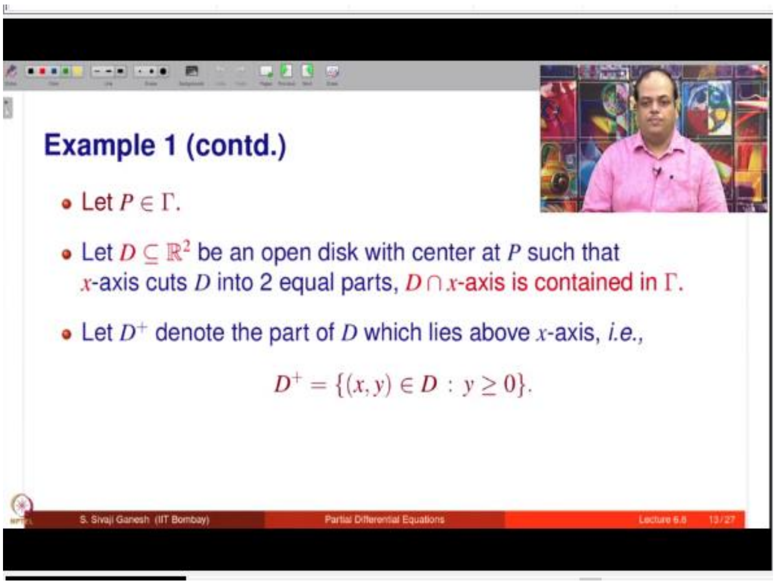

Let P belongs to gamma. Let D be an open disk with center at P such that x axis cuts D into 2 equal parts, it means that disk is symmetric about the x axis and D intersection x axis is contained in gamma. Recall gamma is a subset of x axis on which the function g is defined. Let D + denote the part of D which lies about x axis. That means set of all x, y in D such that y is greater than or equal to 0.

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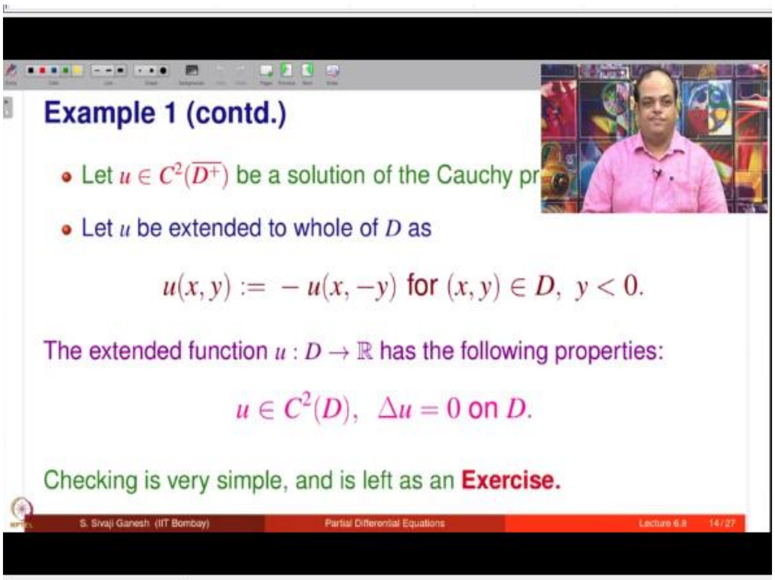

So, let u be C 2 function D + closure, which is a solution to the Cauchy problem, let u be extended to the whole of D as u of x, y = - u of x f – y. For x, y belongs to D and y negative, points x, y belonging to D have to have y positive or negative. If y is greater than or equal to 0, u is already defined. So, for y less than 0, we will define, what we do is that if y is less than 0 take - y that is positive. Therefore, u of x - y meaningful and put a minus sign in the front. So, this is the definition of u.

The extended function has the following properties, u is a C 2 function of D and Laplacian u = 0 on D. Because of these 2 requirements, we have put a minus sign here, if you do not put minus sign you will not get this. Checking is very simple and is left as an exercise.

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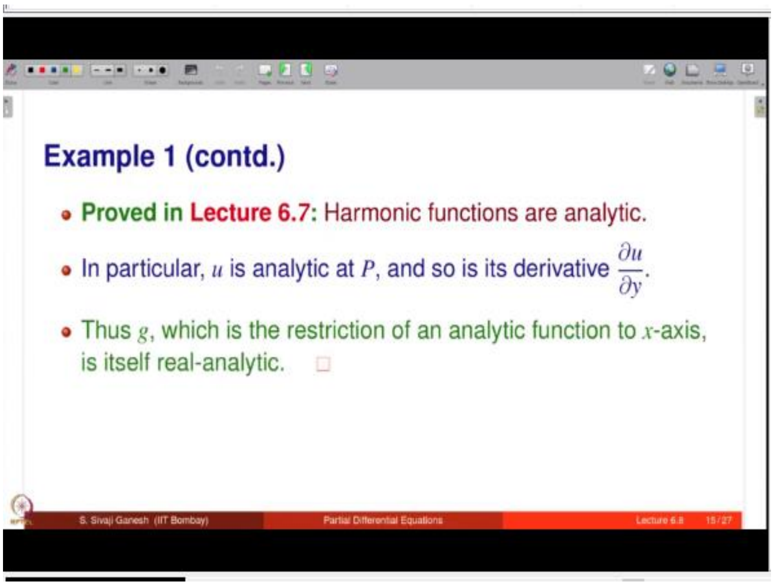

So, proved in lecture 6.7 harmonic functions are analytic. In particular, u is analytic at P and so, is its derivative dou u by dou y. Thus g which is the restriction of an analytic function to x axis is itself analytic or real analytic.

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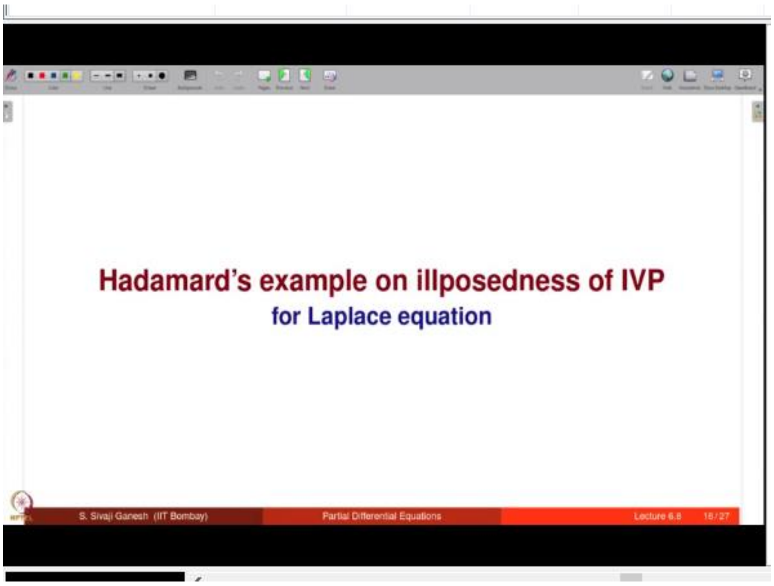

Now, let us look at the Hadamard's example, on illposedness of initial value problems for Laplace equation.

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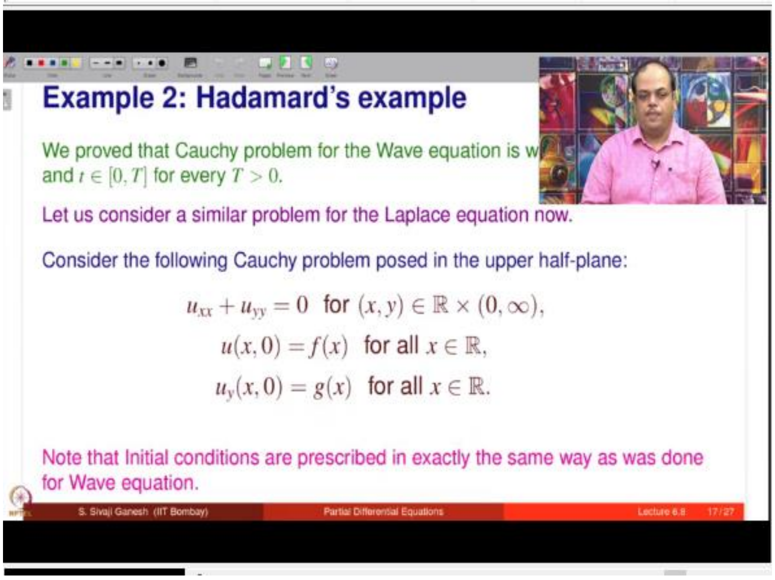

We proved that Cauchy problem for wave equation is well-posed on domains like this x belongs to R and t belongs to 0 T and this happens for every T positive. Let us consider a similar problem for the Laplace equation now, Consider the following Cauchy problem posed in the upper half plane, u xx + u yy = 0, for x, y belonging to R cross 0 infinity.

That is x belongs to R y positive, u of x, 0 = f x for all x in R, u y of x, 0 = g x for all x in R. This f and g are the Cauchy data. Note initial conditions are prescribed in exactly the same way as it was done for wave equation. In case of wave equation, the Laplace equation here is replaced with wave equation, the Cauchy conditions remain the same.

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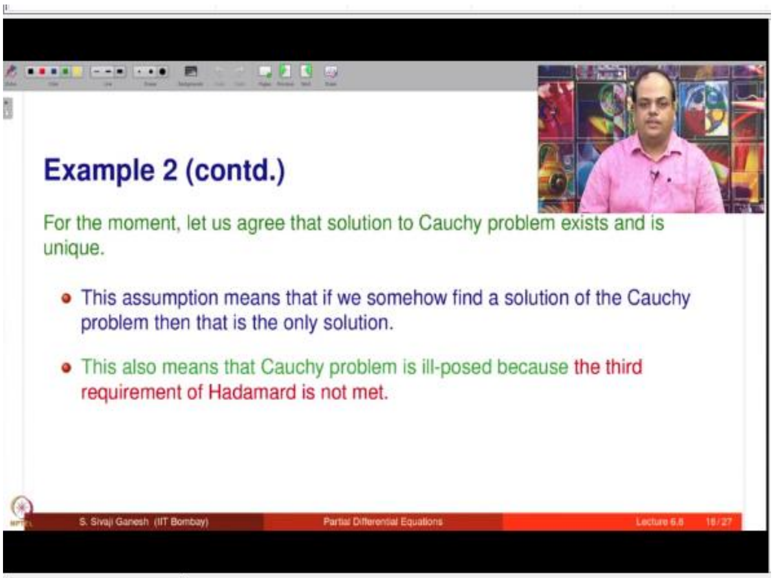

For the moment, let us agree that solution to Cauchy problem exists and is unique. In fact, we are going to consider a specific f and g where we explicitly know the solutions, but for discussion sake, let us assume that the solution exists and is unique. This assumption means that if you somehow find a solution of the Cauchy problem then that is the only solution. This also means that Cauchy problem is ill-posed because the third requirement of Hadamard is not much.

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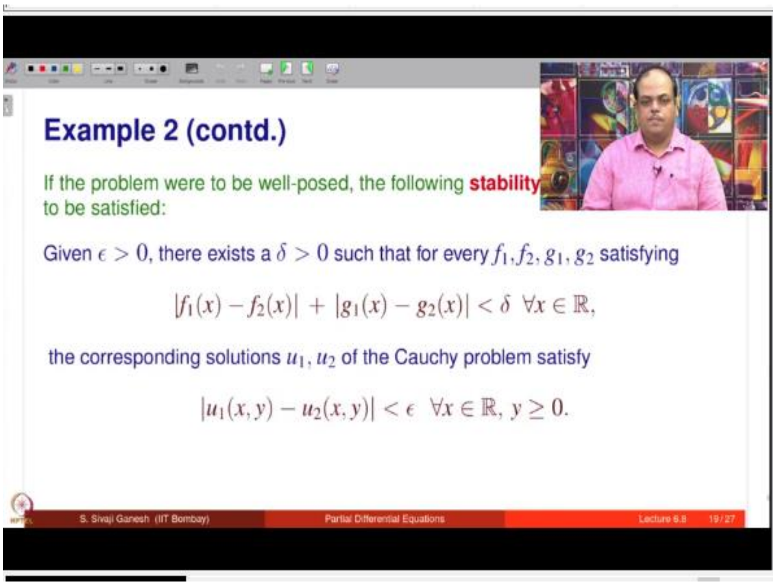

If the problem were to be well-posed the following stability estimate is expected to be satisfied. This is exactly the same way we have written for the wave equation I am writing here. So, given epsilon positive there is a delta positive such that whenever the delta close in some uniform sense, then solutions remain epsilon close in some uniform sense.

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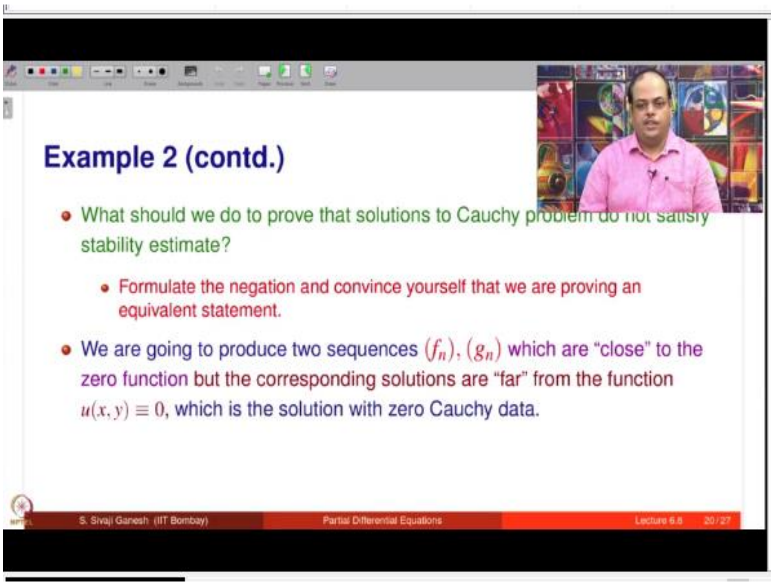

What should we do to prove that solutions to Cauchy problem do not satisfy stability estimate? Formulate the negation of the stability estimate which is there in the previous slide and convince yourself that we are proving an equivalent statement. We are going to produce 2 sequences f n and g n. The Cauchy data is f and g. So, we are going to take in place of f and g f n and g n respectively.

So, we are going to produce a sequence of Cauchy data f n, g n which are close to the 0 function, but the corresponding solutions are far from the solution u identically = 0, when Cauchy data is 0, the Cauchy problem for the Laplace equation has 0 as a solution. So, this is that solution which is the solution with zero Cauchy data.

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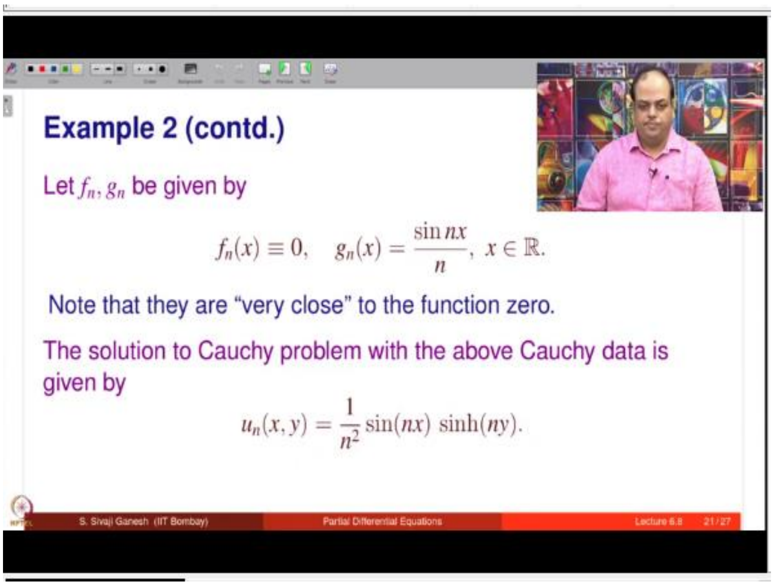

Let f n, g n be given by f n is identically = 0, g n is sine nx by n for x in R. Why are we taking f n = 0, because if something fails, it fails magnificently. Therefore we consider only g n. Note that they are very close to the function 0. It is very obvious one is anyway 0, this uniformly goes to 0 or g and x is mod sine nx by n which is less than or equal to 1 by n that goes to 0.

So, g n's for large n are very, very close to the function 0. The solution to Cauchy problem with the above Cauchy data is given by this formula one can easily check there is a solution to the Cauchy problem for the Laplace equation.

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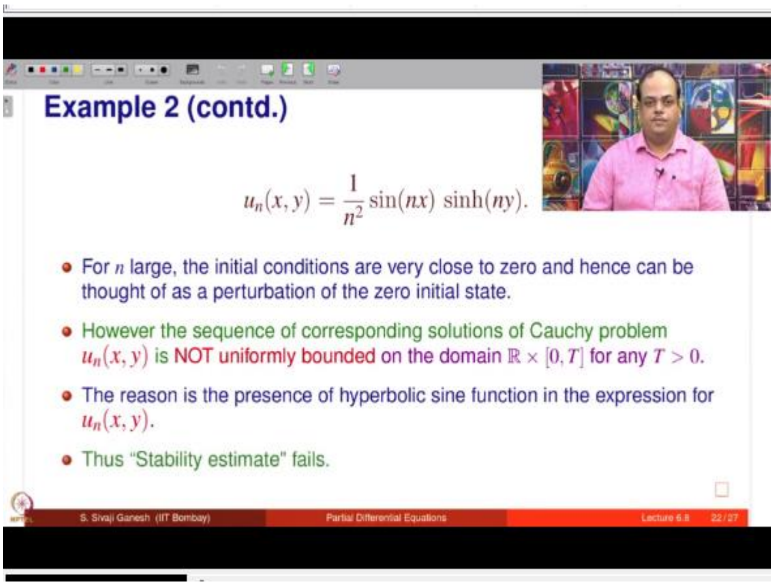

Now for n large the initial conditions are very close to 0 we have already observed and hence can be thought of as a perturbation of the 0 initial state. However, the sequence of corresponding solutions u n x, y given here is not uniformly bounded on the domain R cross 0 T whatever may be the T that you fix. The reason is the presence of the hyperbolic sine term here, sine hyperbolic function in the expression for u n x, y. Thus stability estimate fails.

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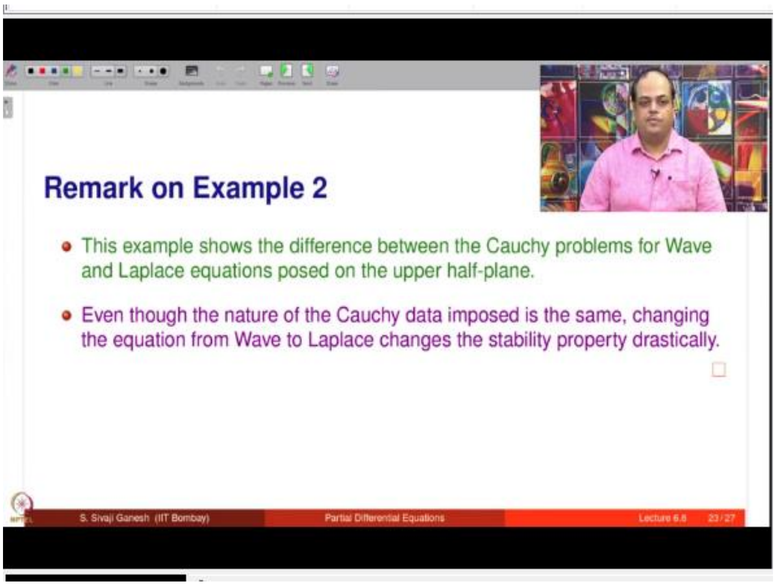

Remark on example 2. This example shows the difference between the Cauchy problems for wave equation and Laplace equations posed on the upper half plane. Even though the nature of the Cauchy data imposed is the same changing the equation from wave to Laplace changes the stability property drastically.

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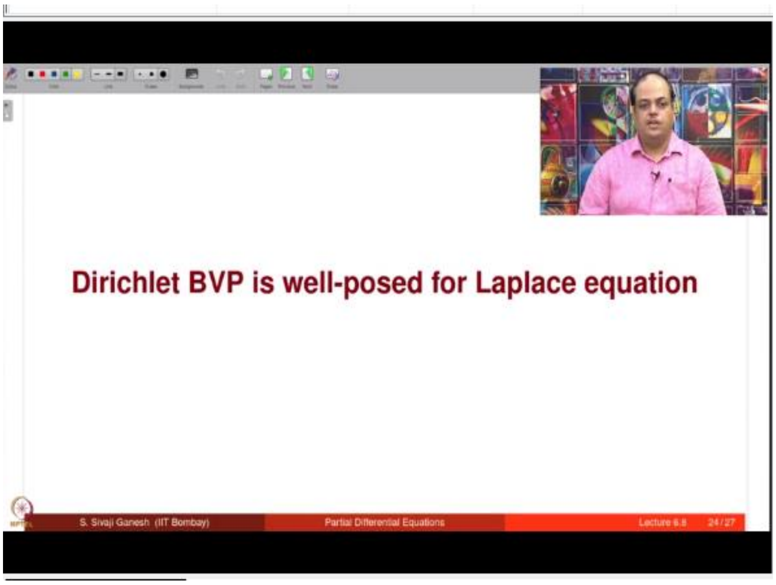

Dirichlet a problem is well-posed for Laplace equation.

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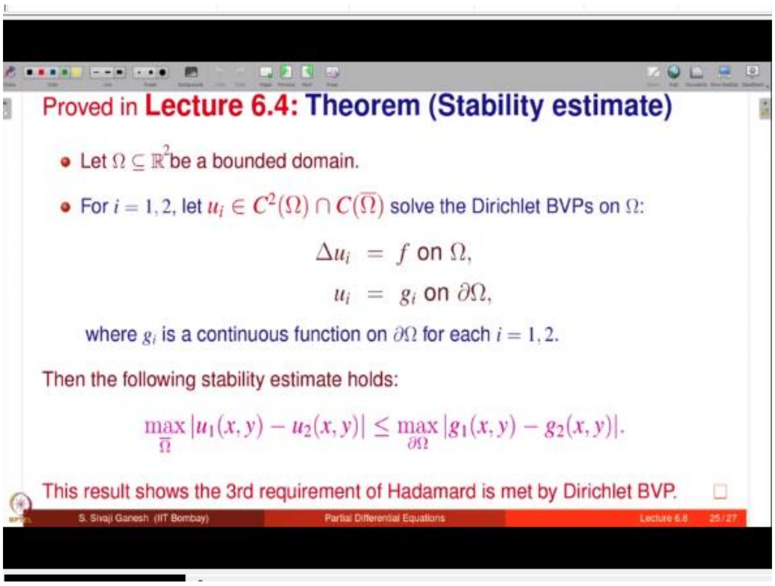

We have proved in lecture 6.4 it was called stability estimate this theorem. So, let omega inside R 2 be a bounded domain. For i = 1 to 2 like u i belongs to C to omega intersection C of omega bar solve the Dirichlet boundary value problem given here Laplacian u i = f on omega, u i = g i on boundary of omega. Then we have proved this stability estimate. This result shows the third requirement of Hadamard is met by Dirichlet BVP because if this data is close solutions are close. These exactly the continuous dependence on the boundary data for the Dirichlet boundary value problem.

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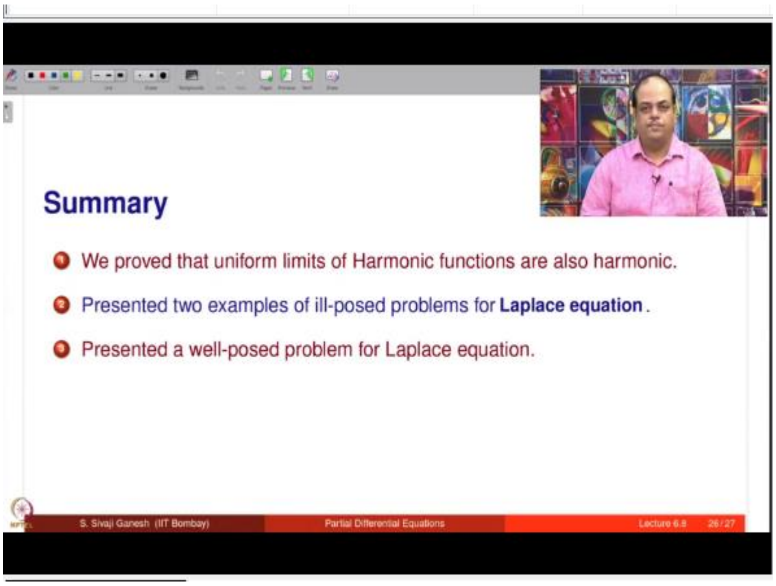

Let us summarize what is done in this lecture. We proved that uniform limits of harmonic functions are also harmonic. Presented 2 examples of ill-posed problems for Laplace equation, both of them are like initial value problems or Cauchy problems presented a well-posed problem for the Laplace equation. Thank you.