Partial Differential Equations
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Lecture - 6.4
Laplace Equation

Welcome to this lecture on weak maximum principle and its applications. The outline for today is lecture is first we introduce weak maximum principle and prove it and then we look at some consequences of weak maximum principle we will discuss weak maximum principle only for domains in R 2 for R n it is similar.

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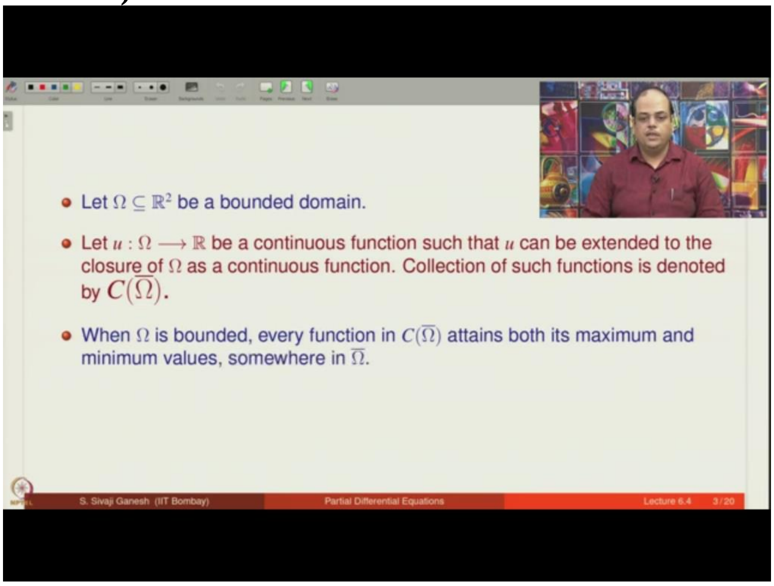

So, let omega inside R 2 be a bounded domain. Let u be a continuous function defined on omega such that u can be extended to the closure of omega as a continuous function. That means, u makes sense for points on the boundary of omega and that too in a continuous manner. So, a collection of such functions is denoted by C of omega bar or omega closure when omega is bounded every function in C of omega bar attains both its maximum.

And minimum values because omega bar is a compact set continuous function and a compact set attains both minimum and maximum values, but somewhere in omega bar the location is not known, but somewhere in omega bar definitely both maximum and minimum values are attained.

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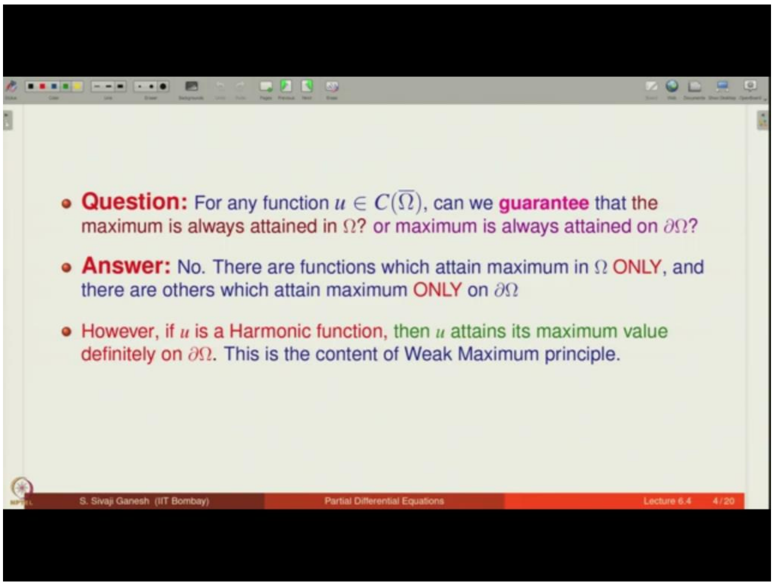

Now, we have a question for any function u that is for any given function u in C of omega bar, can we guarantee that the maximum is always attained in the domain omega or maximum is always attained on boundary of omega. So, can we answer this question for any arbitrary continuous function on omega closer? Answer no, we cannot say this, because there are functions which attain maximum in omega only and not on the boundary.

And there are others which attain maximum only on the boundary of omega not in omega. However, if u is a harmonic function that is u is a solution of Laplacian u = 0 then u attains its maximum value definitely on boundary of omega. This is a content of weak maximum principle. So, let is describe, what is weak maximum principle?

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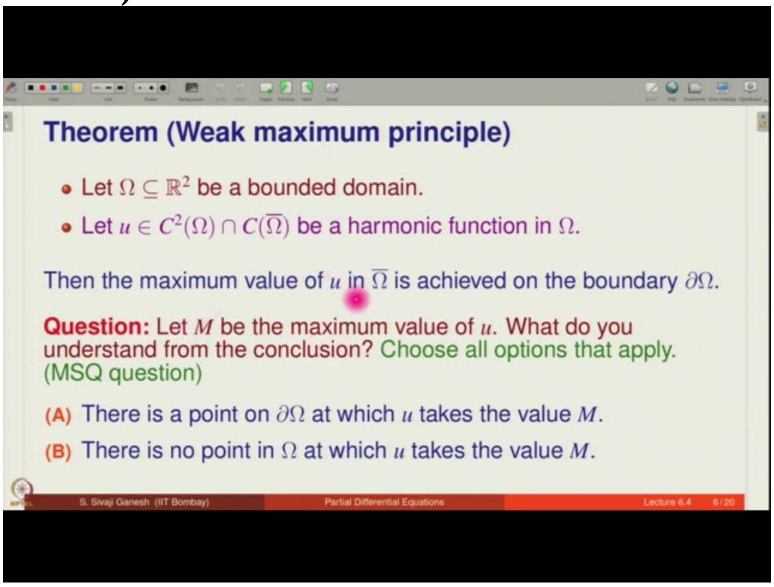

Let us take it as a theorem proof is very simple, let omega inside R 2 be a bounded domain, let u be in C 2 of omega intersection C of omega bar and u is a harmonic function in omega, then the maximum value of u in omega bar is achieved on the boundary omega. We have a question here, let M be the maximum value of u on omega bar. What do you understand from the conclusion of this weak maximum principle which is here?

Choose all options that apply, it is a MSQ question, which you are already very familiar with that means more than 1 answer can be correct. First option, A is there is a point on boundary of omega at which u takes the value M. second option, there is no point in omega at which u takes the value M. Definitely the conclusion is saying that maximum value of u in omega bar is achieved on the boundary that means, there is a point on the boundary at which u takes a value m.

So, A is correct, what about B? B is correct or not? B is true or false? When we write statements in English, we have to be very careful that the maximum value of u is achieved on the boundary. So, if you put extra emphasis on this word is, it is achieved on the boundary that gives an impression that it is not achieved in omega, whereas the statement does not say so, you are likely to tick B also, but that is not true.

There are harmonic functions, which are time maximum inside omega as well as on the boundary of omega as guaranteed by this theorem. So, you have to be very careful with the conclusions which are written in a language like this, you should not give extra emphasis on these kinds of words in the language. Be careful we have come across such statements in the first order PDEs also.

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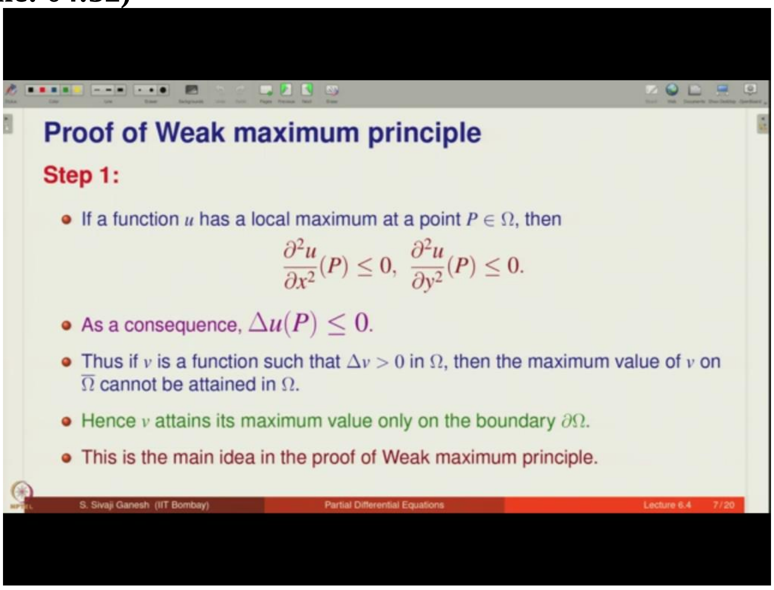

Proof of weak maximum principle step 1, if a function u has a local maximum at a point in omega, omega is an open set. Then, the second order partial derivatives are always less than or equal to 0. So dou 2 u / dou x square at the point P as well as dou 2 u / dou y square at the point P is less than or equal to 0. Both of them are less than or equal to 0 as a consequence Laplacian u at the point P is less than or equal to 0.

Because Laplacian u after all by definition is u xx at P + u yy at P both of them are less than or equal to 0. Therefore, Laplacian u at P is less than or equal to 0. Thus if v is a function such that Laplacian v is positive in omega that means, this kind of condition cannot be satisfied by v Laplacian v at P is always greater than 0. Therefore, it is never less than or equal to 0. Therefore, we can conclude that maximum value of v on omega bar is never achieved in omega it is achieved in boundary of omega.

Because it is definitely achieved somewhere in omega closure it is not achieved in omega therefore, it has to attain on the boundary of omega this is the main idea in the proof of weak maximum principle what we are given is a harmonic function. So, Laplacian u = 0 therefore, this idea cannot be implemented straight away. So, we construct a v such that Laplacian v is positive and hence v has a maximum on the boundary of omega and using that information we show that u also has maximum on the boundary of omega.

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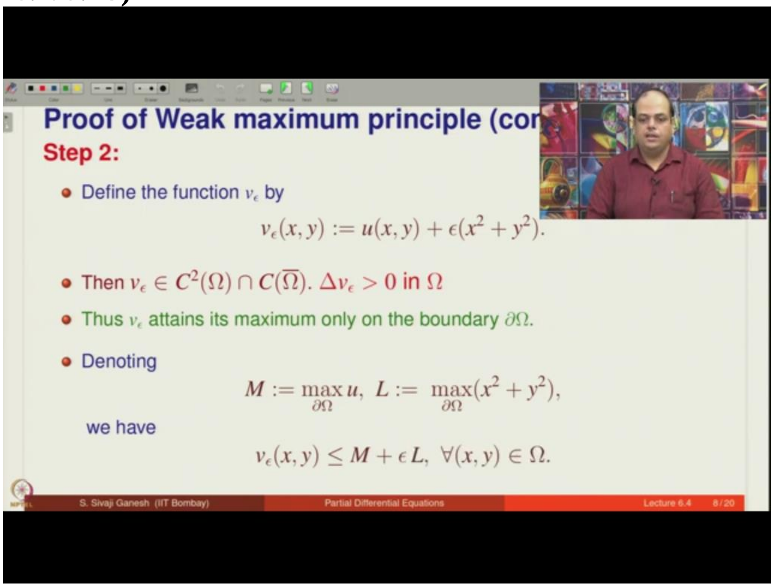

So, define the function v epsilon / u of x y + epsilon times x square + y square. So, when epsilon = 0 you are at u of x y. So, this is you can think these are perturbation of this function, but what kind of perturbation? I am going to consider the epsilon positive x square + y square is always greater than or equal to 0, therefore, this term is always greater than or equal to 0, in particular, v epsilon of x y is always greater than or equal to u of x y.

We will use this observation later and v epsilon is a C 2 function in the domain omega because u is a C 2 function. What we are adding is definitely a C 2 function for the same reason v epsilon is also C of omega bar the reason being that u is in C of omega bar this function is continuous everywhere in R 2 in particular and omega bar. So, Laplacian v epsilon is positive, because what is the Laplacian v epsilon is the Laplacian u plus Laplacian of this, what is the Laplacian of this? It is epsilon times Laplacian of this.

So, v epsilon having this property that Laplacian v epsilon is positive in omega it attains its maximum only on the boundary of omega let us denote by M maximum of u on the boundary and the L to be maximum of x square + y square on the boundary. So, we have v epsilon of x y less than or equal to M + epsilon L for every x y in omega. Why is that? Because M + epsilon L is precisely the maximum of v epsilon on maximum v epsilon is maximum of u plus maximum of epsilon times maximum of x square + y square.

That is the reason, because we have observed that v epsilon maximum is attained only on the boundary. So, therefore, the values of v epsilon for x y in omega is always less than or equal to maximum of v epsilon on the boundary of omega which is less than or equal to M + epsilon L.

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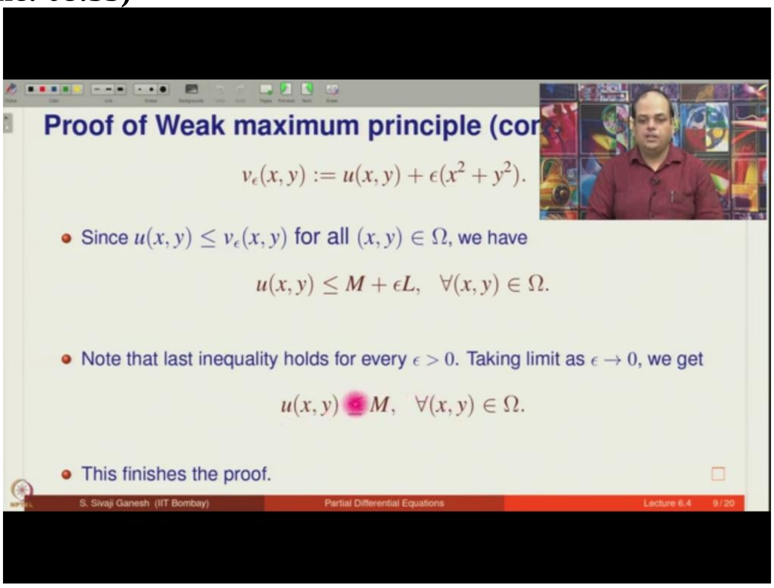

Since u of x y is less than or equal to v epsilon of x y, because this is what we are adding is nonnegative quantity. So, epsilon is positive, therefore, we have this inequality. And v epsilon of x y, we know is less than or equal to M + epsilon L, therefore, u is less than or equal to M + epsilon L for every x y in omega. Note that the last inequality holds for every epsilon positive and decide there is no epsilon.

So, as epsilon goes to 0, what we get is u of x y is less than or equal to M and that is what we want to show. What is M? It is the maximum of on the boundary. So, what we have shown by this inequality is the value of u at any point in omega is less than or equal to the maximum value of u on the boundary. In other words, maximum is definitely achieved on the boundary that completing the proof of weak maximum principle.

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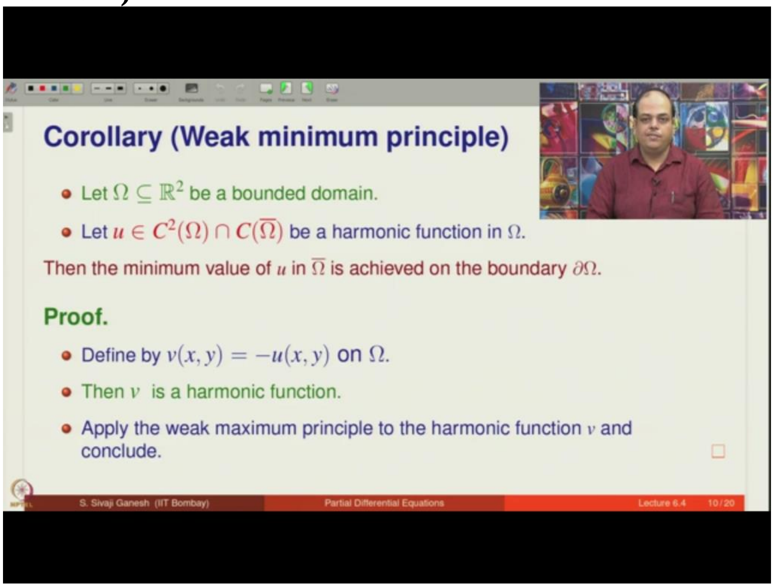

A quick corollary of weak maximum principle is what is called weak minimum principle. Let omega be a bounded domain in R 2. Let u be C 2 of omega intersection C of omega bar and harmonic function, then the minimum value of u in omega bar is achieved on the boundary. Proof is very simple if u is a harmonic function minus u is also a harmonic function. So let us consider v = - u, then v is a harmonic function, apply the weak maximum principle to the harmonic function v and conclude. So make drawing the conclusion is left an exercise to you it is a very simple thing.

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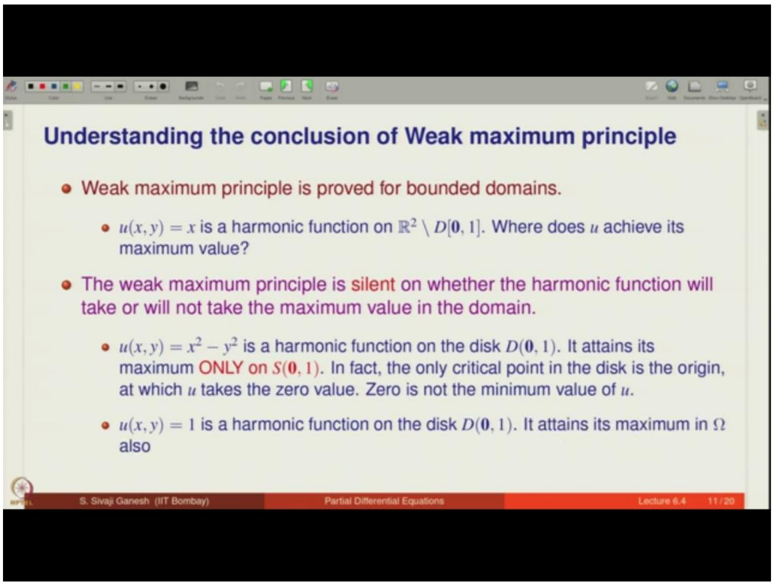

So let is understand the conclusion of weak maximum principle. What does it say and what it does not say. So, weak maximum principle is proved for bounded domains; that is to be remembered. Look at this function u of x y = x. Of course, it is a harmonic functional Laplacian u is 0. Throughout R 2, but I am now considering this only on this domain R 2 from which I have removed the closed disk with center at origin and radius 1.

So, this is not a bounded domain, in fact, is what is called an exterior domain as we discussed earlier, where does u achieve its maximum value? Of course, u does not have a maximum at all. The weak maximum principle is silent on whether the harmonic function will take or will not take the maximum value in the domain is always saying maximum value is taken on the boundary, but never says a sentence about what happens in omega.

Let us look at this function u of x y = x square - y square these are harmonic function is a polynomial. So this is also called harmonic polynomial sometimes. It is harmonic function everywhere, because Laplacian, u will be 2 - 2 that will be 0. So Laplacian u is 0 everywhere, but I consider as D 0, 1 the disk of radius 1 with center at the origin, it attains its maximum only on the circle as 0, 1.

Why is that? If at all it attains maximum inside the open disk, we know that derivative of this function must be 0 gradient must be 0. What is the gradient of this function? It is 2 x, 2 y where is it 0 at the origin. So at the origin value is actually u of x y = 0 at the origin, but 0 is not maximum. Because clearly, you can see, if you are looking at the point u of 1, 0, that is actually 1 thus, 0 is not a maximum value similarly it is not a minimum value also.

So the only critical point that is where the gradient is 0, by definition in the disk is the origin at which u takes 0 value, 0 is not the minimum value of u also. So these are function where maximum minimum is not attained inside the disk, it is attained only on the boundary. On the other hand, if you look at u of x y = 1, it is a harmonic function, it is a constant function. So maximum and minimum both are 1, it is assumed everywhere in the domain as well as on the boundary. Let us look at some consequences of weak maximum principle.

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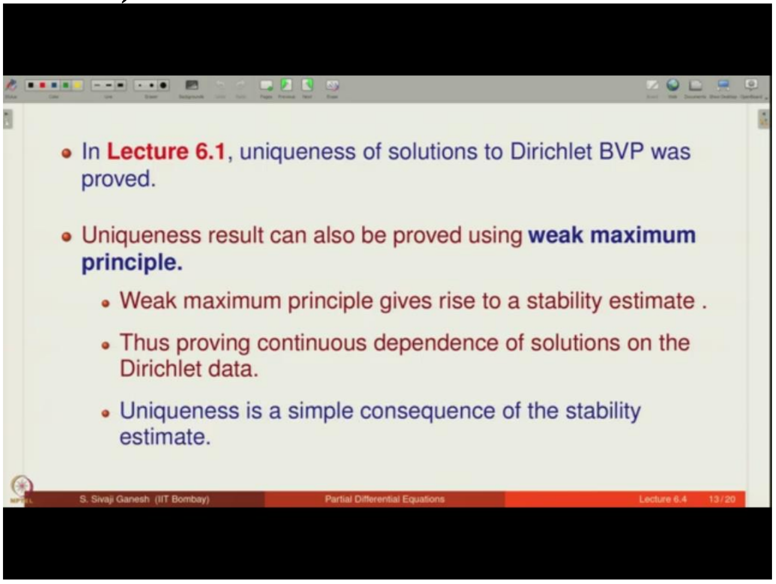

Let us look at some consequences of weak maximum principle. In lecture 6.1, uniqueness of solutions to Dirichlet boundary value problem was proved. In fact, we proved uniqueness for the Robin boundary value problem and Dirichlet boundary value problem turned out to be a special case. Uniqueness result can also be proved using weak maximum principle. In fact, weak maximum principle, because it says maximum of u is attained on the boundary.

It compares 2 quantities, maximum of u in omega is less than or equal to maximum of u on the boundary. Because it is in the form of inequality, we get some estimates. So weak maximum principle gives rise to a stability estimate. We are going to see that so that is proving continuous dependence of solutions on the Dirichlet boundary data. Uniqueness is a simple consequence of the stability estimate.

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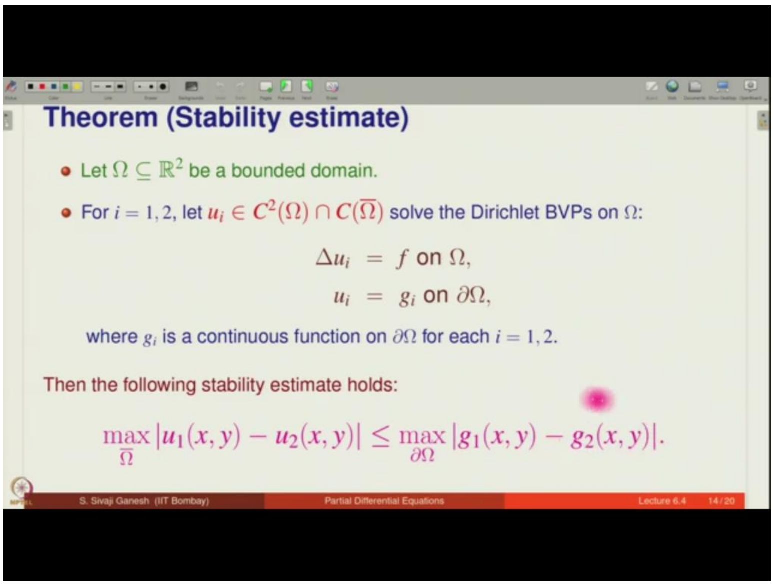

Let us state the stability estimate let omega inside R 2 be a bounded domain and for i = 1 to 2, let u i be C 2 omega intersection C of omega bar solve the Dirichlet problem with right hand side same right hand side f and g i has the boundary data. Then the following stability estimate holds that is maximum of mod u 1 of x y - u 2 of x y as x y vary in omega closure is less than or equal to maximum over boundary of omega of mod g 1 of x y - g 2 of x y. If you want this is the distance between the Dirichlet data and this is the distance between the solutions in omega bar.

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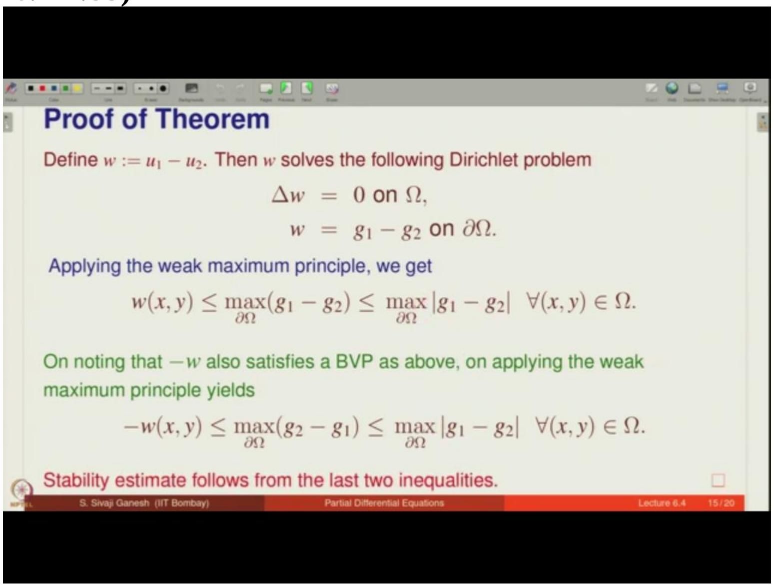

So, define w = u 1 - u 2 look at what problem w solves Laplacian w is 0. Because Laplacian of u 1 as well as Laplacian of u 2 is f and w = g 1 - g 2 on boundary of omega. Applying the weak maximum principle, we get w of x y is less than or equal to maximum of the Dirichlet data the boundary data on the boundary. Because w is a harmonic function maximum is attained on the boundary.

Therefore, we have this inequality of course, this is less than or equal to maximum boundary of omega mod g 1 - g 2 because g 1 - g 2 is always less than or equal to mod g1 - g 2. On noting that minus w also satisfies a BVP as above what does minus w satisfy Laplacian minus w = 0 and minus w = g 2 - g1 unbound. So, we can apply weak maximum principle for minus w and what is a boundary data for minus w g 2 - g 1.

So, therefore, weak maximum principle gives this now, we know g 2 - g 1 is less than or equal to mod g 2 - g 1, which is same as mod g 1 - g 2. Therefore, we have this now, what do we have w of x y is less than or equal to a certain quantity minus w of x y is also less than or equal to the same quantity for every x y in omega. If a number and a negative of that number both of them are less than or equal to the same quantity it means, the modulus of this number namely mod w of x y is less than or equal to that number. So stability estimate follows from the last 2 inequalities.

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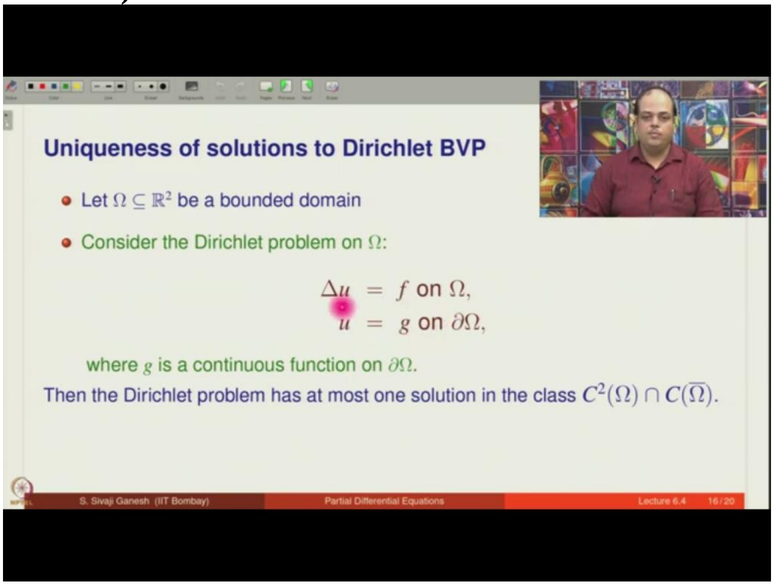

Now, let us look at uniqueness of solutions to the Dirichlet bounded value problem. Let omega be a bounded domain in R 2 and consider this Dirichlet problem. We want to show uniqueness how do we show that? We have to take u 1 u 2 satisfying this boundary value problem, subtract look at the boundary value problem that is solved by the difference if u 1 and u 2 are solutions to this problem, u 1 - u 2 let us call it w it satisfies Laplacian w = 0 and w = 0 on the boundary. So, from the stability estimate, we get the uniqueness of solutions to Dirichlet boundary value problem.

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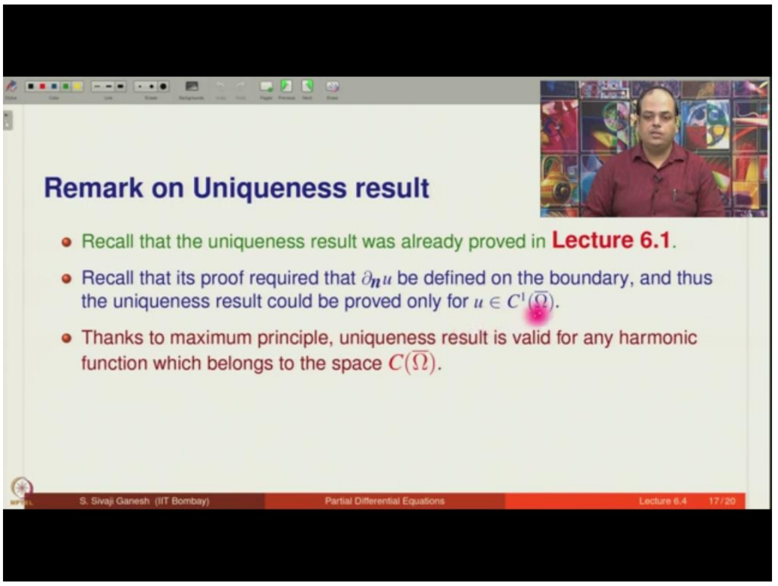

Remark on uniqueness result, recall that the uniqueness result was already proved in lecture 6.1. Recall that its proof required that the normal derivative of u is defined on the boundary and thus uniqueness result could only be proved for u in C 1 of omega bar. That is u in C 2 of omega intersection C 1 of omega bar. Thanks to maximum principle uniqueness result is valid for any harmonic function which belongs to the space C of omega bar. In other words, C 2 of omega intersection C of omega bar. Here I am mentioning only the smoothness what is required on the boundary in the domain of course, u is in C 2 of omega.

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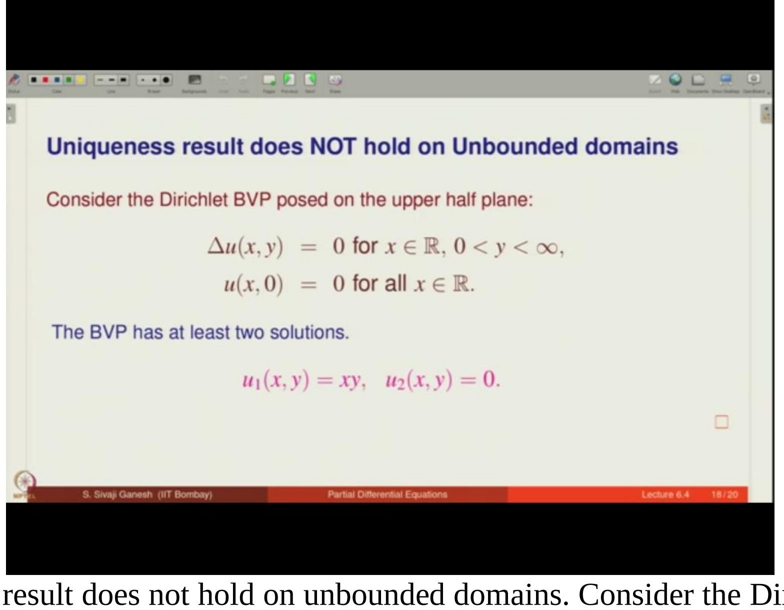

And uniqueness result does not hold on unbounded domains. Consider the Dirichlet boundary value problem posed on the upper half plane Laplacian u is 0. And u is 0 for all x in R so upper half plane boundary is the x axis. So, this BVP has at least 2 solutions u 1 of x y = x y, x y Laplacian will be 0. And when I put y = 0, this is 0 so it is a solution to this boundary value problem. Of course, 0 is also a solution so we do not have uniqueness for Dirichlet boundary value problem on unbounded domains in general.

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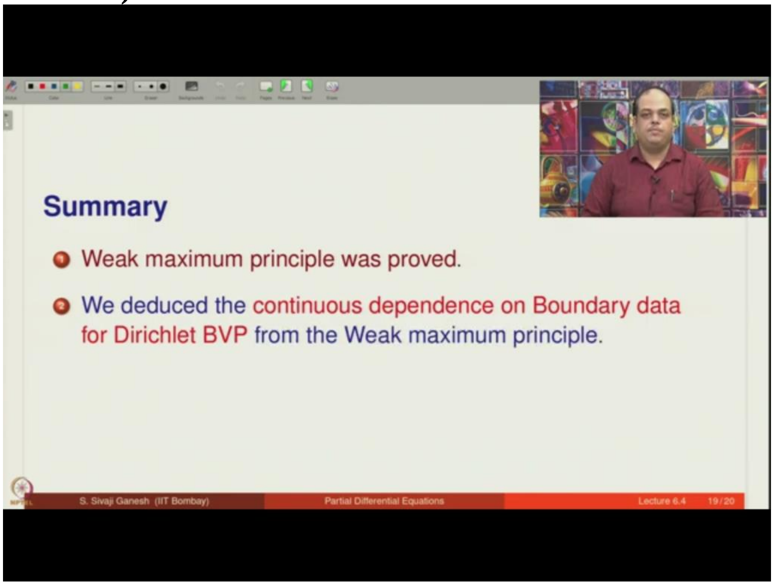

Summary weak maximum principle was proved we deduced continuous dependence on boundary data for Dirichlet boundary value problem from the weak maximum principle. Thank you.