

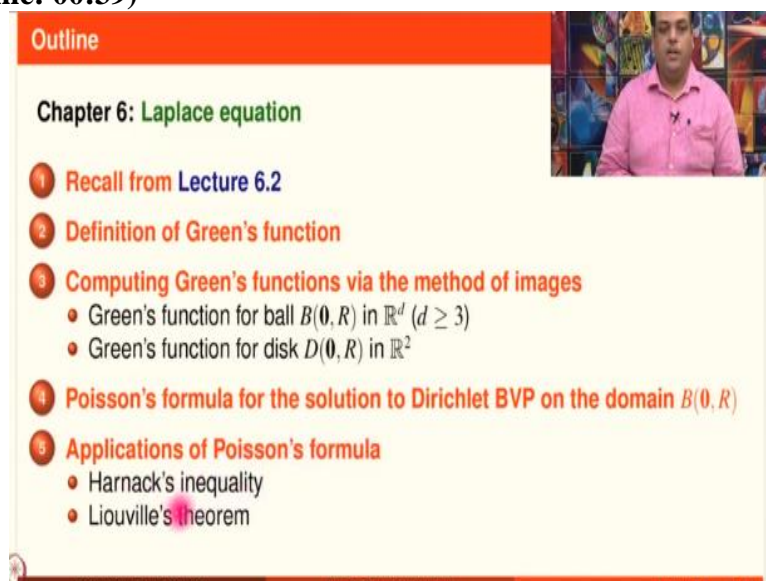
Partial Differential Equations
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Lecture – 6.3

Dirichlet BVP for Laplace Equation – Green's Function and Poisson's Formula

Welcome, in this lecture we are going to consider divisionally boundary value problems for Laplace equation, we will obtain a representative formula for a solution to the division problem which is called Poisson's formula; it can be derived using Green's functions. And towards the end of the lecture, we will see certain applications of Poisson's formula.

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Outline of this lecture is as follows we will recall certain things that we have done in lecture 6.2 namely fundamental solutions and their properties, then we go on to define Green's function. And we compute these functions via what is called a method of images for a ball in \mathbb{R}^d when d is greater than or equal to 3 ball is a disk when it is in \mathbb{R}^2 . Then we write the Poisson's formula for the solution to Dirichlet boundary problem on this domains ball or disk.

As an application of Poisson's formula we will prove Harnack's inequality the further applications have not seen equality is going to be Liouville theorem. Let us recall from lecture 6.2.

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In Lecture 6.2, Fundamental solution for Laplace operator

The fundamental solution for Laplacian is the function

$$K : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \text{Diagonal} \rightarrow \mathbb{R}$$

defined by

$$K(x, \xi) = \begin{cases} \frac{1}{2\pi} \ln \|x - \xi\| & \text{if } d = 2, \\ \frac{1}{\omega_d(2-d)} \|x - \xi\|^{2-d} & \text{if } d \geq 3, \end{cases}$$

where Diagonal stands for the set

$$\{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : x = \xi\}.$$

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In lecture 6.2, fundamental solution for Laplace operator was defined, the fundamental solution for the Laplacian is the function K from $\mathbb{R}^d \times \mathbb{R}^d - \text{diagonal}$ to \mathbb{R} , we will define what is the diagonal once we see the formula for K . K of $x, \xi = 1 / 2\pi$ logarithm norm $x - \xi$ if $d = 2$ and if d is greater than or equal to 3. The formula for K of x, ξ is $1 / \omega_d$ into $2 - d$ into norm $x - \xi$ power $2 - d$. As you see, when $x = \xi$ both these functions have singularities. Therefore, we have to remove that so diagonal stands for the set x, ξ in $\mathbb{R}^d \times \mathbb{R}^d$ such that $x = \xi$.

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The following result was established.

The following equality holds in the sense of distributions on \mathbb{R}^d :

$$\Delta K(x, \xi) = \delta_\xi.$$

i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ the following equality holds.

$$\varphi(\xi) = \int_{\mathbb{R}^d} K(x, \xi) \Delta \varphi(x) dx.$$

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And the following result was established it says that Laplacian of K of $x, \xi = \delta_\xi$, for every ξ fixed in \mathbb{R}^d this Laplacian is in the x coordinates Laplacian in K of $x, \xi = \delta_\xi$. What it means? We have explained that means, actually this you can think this is a notation for the moment, because we do not know what this is, in case you know it is exactly the same as what you know, if you do not know this means this. For every φ which is C infinity

compactly supported function in \mathbb{R}^d ϕ of X_i is given by integral over \mathbb{R}^d of $K \times X_i$ into Laplacian ϕ of x dx .

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Theorem on Logarithmic potential was stated.

- Let $f \in C^2(\mathbb{R}^2)$ having compact support.
- Define the Logarithmic potential on \mathbb{R}^2 by

$$u(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \|\mathbf{x} - \xi\| f(\mathbf{x}) dx.$$

Then the following assertions can be proved.

- Logarithmic potential satisfies $\Delta u = f$.

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Then we have stated this theorem on logarithmic potential, it says that if you have a function which is C^2 and has compact support in \mathbb{R}^2 , then if you define the logarithmic potential by this formula, this u has the property that Laplacian $u = f$.

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Theorem on Logarithmic potential (contd.)

- $u(\xi) \rightarrow \infty$ as $\|\xi\| \rightarrow \infty$. In fact, the Logarithmic potential has the following asymptotic behaviour as $\|\xi\| \rightarrow \infty$:

$$u(\xi) = \frac{M}{2\pi} \ln \|\xi\| + O\left(\frac{1}{\|\xi\|}\right),$$

where $M = \int_{\mathbb{R}^2} f(\mathbf{x}) dx$.

- Logarithmic potential is the only solution to $\Delta u = f$ having the asymptotic behaviour, as described in (2) above. \square

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And u X_i goes to infinity as norm X_i goes to infinity there is a precise asymptotic as it goes to infinity. Logarithmic potential is the only solution to Laplacian $u = f$ having this kind of asymptotic behaviour.

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Theorem on Newtonian potential was s

- Let $f \in C^2(\mathbb{R}^3)$ having compact support.
- Define the Newtonian potential on \mathbb{R}^3 by

$$u(\xi) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{\|\mathbf{x} - \xi\|} f(\mathbf{x}) d\mathbf{x}.$$

Then the following assertions can be proved.

- Newtonian potential satisfies $\Delta u = f$.
- $u(\xi) \rightarrow 0$ as $\|\xi\| \rightarrow \infty$.
- Newtonian potential is the only solution to $\Delta u = f$ that is in $C^2(\mathbb{R}^3)$ and vanishes at infinity.

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Then, we have stated a theorem on Newtonian potential, this is applicable for all d greater than or equal to 3. So, if you have one second C^2 function with compact support and define the Newtonian potential by this formula. Then the following assertions can be proved Laplacian $u = f$, u goes to 0 as norm ξ goes to infinity and this is the only solution to Laplacian $u = f$ which is C^2 and vanishes at infinity.

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Remark

- Fundamental solutions help in obtaining solutions to Poisson's equation

$$\Delta u = f \text{ in } \mathbb{R}^d.$$

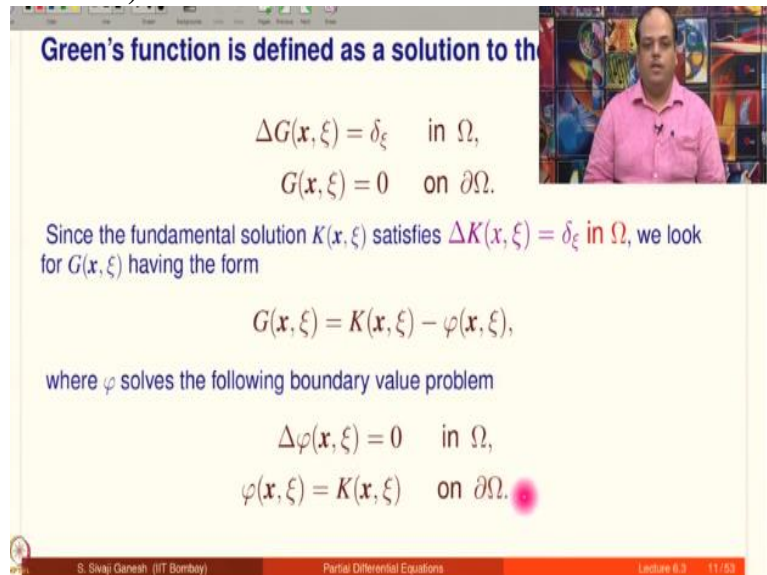
- Once the fundamental solution is known, a single formula gives solution to $\Delta u = f$, for any f suitable!
- Question:** Is there a function (which behaves like Fundamental solution does), by knowing which, ANY Dirichlet problem for Laplace equation can be solved?
- Answer:** Yes, it is called Green's function.

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Remark fundamental solutions help in solving Poisson's equation Laplacian $u = f$ in \mathbb{R}^d this is the essence of the theorems that we saw. So, once the fundamental solution is known, a single formula gives solution to Laplacian $u = f$. It is logarithmic potential if it is $d = 2$ it is Newtonian potential if d is greater than or equal to 3 for any f and suitable. Question, is there a function which behaves like fundamental solution does by knowing which any Dirichlet problem for Laplace equation can be solved?

The answer is yes, it is called Green's function. Understand the difference between Green's function and fundamental solution. These 2 names are often blurred in the literature. So, pay specific attention to the difference between these 2.

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Green's function is defined as a solution to the

$$\begin{aligned} \Delta G(x, \xi) &= \delta_\xi && \text{in } \Omega, \\ G(x, \xi) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Since the fundamental solution $K(x, \xi)$ satisfies $\Delta K(x, \xi) = \delta_\xi$ in Ω , we look for $G(x, \xi)$ having the form

$$G(x, \xi) = K(x, \xi) - \varphi(x, \xi),$$

where φ solves the following boundary value problem

$$\begin{aligned} \Delta \varphi(x, \xi) &= 0 && \text{in } \Omega, \\ \varphi(x, \xi) &= K(x, \xi) && \text{on } \partial\Omega. \end{aligned}$$

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So, definition of Green's function Green's function is defined as a solution to the boundary value problem Laplacian $G = \delta_\xi$ in Ω , $G(x, \xi) = 0$ on boundary of Ω . Since the fundamental solution K already satisfies Laplacian $K = \delta_\xi$ in Ω , we look for G that is a Green's function having this form that $G = K - \varphi$. Now, you know the equation for G you know the equation for K and also the condition for G on the boundary therefore, you know what is the equation φ was satisfied.

So, φ solves Laplacian $\varphi = 0$ in Ω and $\varphi(x, \xi) = K(x, \xi)$ on the boundary of Ω . So, if you want to know the Green's function, since we already know K what remains is to find this φ .

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Does Green's function always exist?

- Laplacian operator is the simplest elliptic operator that we can think of.
- It is a non-trivial matter to show the existence of a **Green's function**, leave alone finding explicit expressions for them **for arbitrary domains Ω** .
 - **P.D. Lax** Functional analysis, **Wiley-Interscience, 2002** for Existence of Green's functions on arbitrary domains in \mathbb{R}^2 .
 - **P.D. Lax** *On the existence of Green's function*, **Proc. AMS, pp. 526-531, 1952** for Existence of Green's functions on arbitrary domains in \mathbb{R}^d .
- For special domains like a Ball, upper half-space, it is easy to construct Green's function.
- We will construct Green's function for a ball in $\mathbb{R}^d, d \geq 2$.

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The question is does Green's function always exist? Laplacian operator is clearly the simplest elliptic operator that we can think of. It is a non-trivial matter to show the existence of a Green's function, leave alone finding explicit expressions for it for arbitrary domains Ω . These are the 2 references I am going to give where you can find the discussion on Green's functions existence one is a book by P.D. Lax on functional analysis. Here you can find the existence of Green's functions and arbitrary domains in \mathbb{R}^2 .

Then there is a paper on the existence of Green's function in proceedings of AMS by P.D. Lax, where he discusses Green's functions on arbitrary domains in \mathbb{R}^d . For special domains like a ball or upper half space, like the upper half plane and so on, it is easy to construct the Green's function, we will construct Green's function for a ball in \mathbb{R}^d, d greater than or equal to 2. So, ball when $d = 2$ is called disk. So, computing Green's functions we use what is called a method of images.

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Method of images

- The method of images (also called, **method of electrostatic images**) used to obtain Green's functions.
- The method prescribes that $\varphi(\mathbf{x}, \xi)$ is the potential due to an imaginary charge q placed at a point ξ^* with $\xi^* \notin \Omega$, and such that
 - For $\mathbf{x} \in \partial\Omega$, the value of $\varphi(\mathbf{x}, \xi)$ equals the potential $K(\mathbf{x}, \xi)$ created by the unit charge at ξ . That is,

$$\varphi(\mathbf{x}, \xi) = K(\mathbf{x}, \xi) \text{ for all } \mathbf{x} \in \partial\Omega.$$

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So, what is method of images? The method of images, it is also called method of electrostatic images will be used to obtain Green's functions. The method prescribes that $\varphi(\mathbf{x}, \xi)$ is the potential due to an imaginary charge q placed at a point ξ^* , which is not in the domain Ω which is outside the domain Ω and such that for every \mathbf{x} on the boundary of Ω , the value of $\varphi(\mathbf{x}, \xi)$ coincides with the value of $K(\mathbf{x}, \xi)$ which is created by the unit charge at ξ that is $\varphi(\mathbf{x}, \xi) = K(\mathbf{x}, \xi)$ for all \mathbf{x} in boundary of Ω .

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Green's function for ball $B(\mathbf{0}, R)$ in \mathbb{R}^d ($d \geq 3$)

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So, Green's function for ball in \mathbb{R}^d , d is greater than or equal to 3, we will deal the case when $d = 2$ separately because of the way the formula for the fundamental division looks is different.

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Method of images for $d \geq 3$

- The potential due to a charge q at a point $\xi^* \notin \Omega$ is given by

$$\frac{1}{\omega_d(2-d)} \frac{q}{\|\mathbf{x} - \xi^*\|^{d-2}}$$
- Thus we take

$$\varphi(\mathbf{x}, \xi) = \frac{1}{\omega_d(2-d)} \frac{q}{\|\mathbf{x} - \xi^*\|^{d-2}}$$
- In order to determine Green's function, we need to find q and ξ^* so that

$$\varphi(\mathbf{x}, \xi) = K(\mathbf{x}, \xi) \text{ for all } \mathbf{x} \in \partial\Omega$$
 is satisfied.

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So, method of images for d is greater than or equal to 3, the potential due to a charge q at point ξ^* which is not in Ω is given by exactly the fundamental solution this is a charge q fundamental solution is for the charge 1. So, now it is q now, location is ξ^* . Thus, we take $\varphi(\mathbf{x}, \xi)$ is equal to this. In order to determine Green's function, we need to find q and ξ^* . So that $\varphi(\mathbf{x}, \xi) = K(\mathbf{x}, \xi)$ for all \mathbf{x} in boundary of Ω . So, using this constraint we have to find both q and ξ^* .

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Green's function for the domain $B(0, R)$

That is, we need to find q and ξ^* such that

$$\frac{1}{\omega_d(2-d)} \frac{q}{\|\mathbf{x} - \xi^*\|^{d-2}} = \frac{1}{\omega_d(2-d)} \frac{1}{\|\mathbf{x} - \xi\|^{d-2}} \text{ for all } \mathbf{x} \text{ s.t. } \|\mathbf{x}\| = R.$$

On simplification, the above equation reduces to

$$q^{1/(d-2)} \|\mathbf{x} - \xi\| = \|\mathbf{x} - \xi^*\| \text{ for all } \mathbf{x} \text{ s.t. } \|\mathbf{x}\| = R.$$

Squaring both sides of the last equation, and re-arranging terms yields

$$R^2 + \|\xi^*\|^2 - q^{2/(d-2)}(R^2 + \|\xi\|^2) = 2\mathbf{x} \cdot (\xi^* - q^{2/(d-2)}\xi)$$

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That is, we need to find q and ξ^* such that this equation holds for every \mathbf{x} such that $\|\mathbf{x}\| = R$, because a boundary of the ball of radius R is a sphere of radius R which is $\|\mathbf{x}\| = R$. On simplification the above equation reduces to this because straightaway you can see that this cancels with this both are same factors and then you get q into $\|\mathbf{x} - \xi\|^{d-2} = \|\mathbf{x} - \xi^*\|^{d-2}$ and take the power $1/(d-2)$ we get this. Squaring both sides of the last

equation and rearranging the terms will give you this. At this point of time pause the video compute for yourself and make sure that you are getting this equation.

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Green's function for the domain $B(0, R)$ (contd.)

Since the LHS of the equation

$$R^2 + \|\xi^*\|^2 - q^{\frac{2}{d-2}}(R^2 + \|\xi\|^2) = 2x \cdot (\xi^* - q^{\frac{2}{d-2}}\xi)$$

is independent of x , and RHS depends on x , both LHS and RHS must be equal to zero (why?) That is,

$$x \cdot (\xi^* - q^{\frac{2}{d-2}}\xi) = 0,$$

$$R^2 + \|\xi^*\|^2 - q^{\frac{2}{d-2}}(R^2 + \|\xi\|^2) = 0.$$

Since the first equation holds for every $x \in S(0, R)$, we get

$$\xi^* - q^{\frac{2}{d-2}}\xi = 0.$$

How?

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Now, assume since the left hand side of this equation is independent of x and right hand side depends on x both must be constant and that constant must be 0. Please justify for yourself why should the constant be 0? That is this is 0 a first equation and LHS are 0 that is a second equation. Now, the first equation holds for every x in S of $0, R$ that is sphere. Therefore $\xi^* - q^{\frac{2}{d-2}}\xi$ that itself is 0, what it says is that division some vector in \mathbb{R}^d whose inner product with the x is 0 for every x on the sphere of radius R , if that happens, then that vector has to be 0. So, I have already given you the hint it is work it out.

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Green's function for the domain $B(0, R)$ (contd.)

Thus q and ξ^* satisfy the conditions

$$\xi^* = q^{\frac{2}{d-2}}\xi,$$

$$R^2 + \|\xi^*\|^2 - q^{\frac{2}{d-2}}(R^2 + \|\xi\|^2) = 0.$$

Substituting the value of ξ^* from the first equation into the second one gives

$$R^2 + q^{\frac{4}{d-2}}\|\xi\|^2 - q^{\frac{2}{d-2}}(R^2 + \|\xi\|^2) = 0.$$

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Thus q and ξ^* satisfy these 2 equations now substitute the value of ξ^* from here into this at this point that will give us an equation like this.

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Green's function for the domain $B(0, R)$ (contd.)

On the last slide, we obtained

$$R^2 + q^{\frac{4}{d-2}} \|\xi\|^2 - q^{\frac{2}{d-2}} (R^2 + \|\xi\|^2) = 0.$$

- If $q = 1$, then $\xi^* = \xi$ and thus $\xi^* \in B(0, R)$. Since ξ^* must be in the complement of $B(0, R)$, $q \neq 1$.
- Thus for $\xi \neq 0$, the above equation yields

$$q^{\frac{1}{d-2}} = \frac{R}{\|\xi\|}$$

on solving a quadratic equation.

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So, on the last slide, we obtained this equation, now we need to solve for q there is no Xi star here only q. So, let us observe that if $q = 1$ then $\xi^* = \xi$ and thus ξ^* belongs to the ball which is not allowed we want ξ^* out at the ball. Therefore q cannot be equal to 1, thus for ξ is nonzero, the above equation yield this how do I get this, I just see that the quantum quantity is q agreed, but then q power $2 / d - 2$ and it is queries here q power $2 / d - 2$ whole square is precisely this. So, it is a quadratic equation in q power $2 / d - 2$ think that is lambda and solve this quadratic equation you will get this.

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Green's function for the domain $B(0, R)$ (contd.)

Summarizing the computations, we obtained

$$q^{\frac{1}{d-2}} = \frac{R}{\|\xi\|}, \quad \xi^* = q^{\frac{2}{d-2}} \xi = \frac{R^2}{\|\xi\|^2} \xi.$$

Recall that

$$\varphi(\mathbf{x}, \xi) = \frac{1}{\omega_d(2-d)} \frac{q}{\|\mathbf{x} - \xi^*\|^{d-2}}.$$

Thus the function $\varphi(\mathbf{x}, \xi)$ for $\xi \neq 0$ is given by

$$\varphi(\mathbf{x}, \xi) = \frac{1}{\omega_d(2-d)} \frac{R^{d-2}}{\|\xi\|^{d-2}} \frac{1}{\|\mathbf{x} - \xi^*\|^{d-2}}.$$

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So, summarizing the computations we have obtained q power $1 / d - 2$ equal to this in other words, we have q and ξ^* is given in terms of q, q is already known from here. Therefore we know q and ξ^* and this is nothing but $R^2 / \|\xi\|^2 \xi$. So, recall that phi of x, ξ we started with this. So, therefore phi of x, ξ is equal to this I have just

substituted what is ξ here and I have not substituted what is ξ^* the value of ξ^* inside this ξ still retain it as ξ^* . $\xi^* = R^2 / \|\xi\|$ into ξ .

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Green's function for the domain $B(0, R)$ (contd.)

Our search for Green's function G proceeded by the decomposition formula

$$G(x, \xi) = K(x, \xi) - \varphi(x, \xi).$$

Thus Green's function $G(x, \xi)$ for $\xi \neq 0$ is given by

$$G(x, \xi) = \frac{1}{\omega_d(2-d)} \left(\frac{1}{\|x - \xi\|^{d-2}} - \frac{R^{d-2}}{\|\xi\|^{d-2}} \frac{1}{\|x - \xi^*\|^{d-2}} \right),$$

where

$$\xi^* = \frac{R^2}{\|\xi\|^2} \xi$$

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So, our search for Green's function proceeded by the decomposition formula $G = K - \varphi$ therefore, K is already known, we have just found φ therefore G is known and that is given by this formula G of x, ξ is equal to this formula where ξ^* is this.

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Green's function for the domain $B(0, R)$ (contd.)

Since

$$\|x - \xi^*\| = \frac{\sqrt{R^4 - 2R^2\xi \cdot x + \|x\|^2\|\xi\|^2}}{\|\xi\|},$$

we get

$$\|\xi\| \|x - \xi^*\| = \sqrt{R^4 - 2R^2\xi \cdot x + \|x\|^2\|\xi\|^2} \xrightarrow{\xi \rightarrow 0} R^2.$$

Thus we may define $G(x, 0)$ by

$$G(x, 0) = \frac{1}{\omega_d(2-d)} \left(\frac{1}{\|x\|^{d-2}} - \frac{1}{R^{d-2}} \right).$$

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Since $\|x - \xi^*\|$, you get this expression compute this then you get that multiply $\|\xi\|$ this side then you have $\|x - \xi^*\|$ equal to this numerator that as ξ goes to 0 goes to R^2 . Therefore, we can define G of $x, 0$ to be this. Now let us turn our attention to finding Green's function for disk in \mathbb{R}^2 .

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Method of images for $d = 2$

- The potential due to a charge q at a point $\xi^* \notin \Omega$ is given by

$$\frac{q}{2\pi} \ln \|\mathbf{x} - \xi^*\|$$
- But we take

$$\varphi(\mathbf{x}, \xi) = \frac{q}{2\pi} \ln \|\mathbf{x} - \xi^*\| + \frac{C}{2\pi}$$
 where $C \in \mathbb{R}$.
- In order to determine Green's function, we need to find q , C , and ξ^* so that

$$\varphi(\mathbf{x}, \xi) = K(\mathbf{x}, \xi) \text{ for all } \mathbf{x} \in \partial\Omega$$
 is satisfied.

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Once again method of images the potential due to a charge q at point ξ^* not in Ω is $q / 2\pi \log \|\mathbf{x} - \xi^*\|$. But we take $\varphi(\mathbf{x}, \xi) = q / 2\pi \log \|\mathbf{x} - \xi^*\| + C / 2\pi$ for convenience. Without this I have tried but I could not get anything. So, we have to add this constant then life is simpler, this idea is taken from all verse partial differential equations book. So, in order to determine Green's function, we are determined what q , ξ^* and C such that $\varphi = K$ on the boundary of the disk.

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Green's function for the domain $D(0, R)$

That is, we need to find q , C , and ξ^* such that

$$\frac{q}{2\pi} \ln \|\mathbf{x} - \xi^*\| + \frac{C}{2\pi} = \frac{1}{2\pi} \ln \|\mathbf{x} - \xi\| \text{ for all } \mathbf{x} \text{ s.t. } \|\mathbf{x}\| = R.$$

On simplification, the above equation reduces to

$$e^C \|\mathbf{x} - \xi^*\|^q = \|\mathbf{x} - \xi\| \text{ for all } \mathbf{x} \text{ s.t. } \|\mathbf{x}\| = R.$$

It is not clear how to solve the above equation for the unknowns q , C , and ξ^* .

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So that is we need to find q , C and ξ^* such that this equation is satisfied for every \mathbf{x} such that $\|\mathbf{x}\| = R$. On simplification exponentiate both sides you get this equation. So, it is not clear how to solve that equation for the unknown q , C and ξ^* . A geometric idea helps here.

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Green's function for the domain $D(0, R)$ (contd.)

Geometric construction

- Choose ξ^* as $\xi^* = c\xi$ in such a way that for all x s.t. $\|x\| = R$,
 - The triangles formed by the vertices $0, x, \xi^*$ and by the vertices $0, \xi, x$ are similar.
 - Similarity requirement yields

$$\frac{\|\xi\|}{\|x\|} = \frac{\|x\|}{\|\xi^*\|} = \frac{\|x - \xi\|}{\|x - \xi^*\|} = \text{constant}$$
- The above equalities suggest that

$$\|\xi^*\| = \frac{R^2}{\|\xi\|}$$

Handwritten notes: ξ^* inversion w.r.t. $S(0, R)$. $\frac{\|\xi^*\| \|\xi\|}{R^2} = 1$

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The geometry construction is as follows choose ξ^* as $\xi^* = c \xi$ that means, it is along the line joining origin and ξ . So, in the direction of ξ in such a way that for all x as the norm $\|x\| = R$. The triangles formed by the vertices $0, x, \xi^*$ and by the vertices $0, x, \xi$ are similar. Similarity requirements yield these equations, let us draw a circle just for explanation, this is our region let us say this point is ξ .

So, now we take a point outside the ball with what property is that you take any point x on the boundary, we have this triangle this now choose ξ^* , which is actually $c \xi$, so, such that this angle equals this angle, note that for both the triangles this and this this angle is common. Now, we have chosen ξ^* such that this angle $\angle 0, \xi, x$ is another angle $\angle 0, x, \xi^*$. Therefore, by similarity what we get is that the sides are proportional. So, whatever side it is opposite to this angle here angle $\angle x, 0, \xi$ or $\angle x, 0, \xi^*$.

One of them is $\|x - \xi\|$, other one is $\|x - \xi^*\|$ that will give us this, what is the side opposite to this angle $\angle 0, \xi, x$ that is $\|x\|$ and what is the side opposite to angle $\angle 0, x, \xi^*$ that is $\|\xi^*\|$, so similarly you get this. So, the above equality suggests that $\|\xi^*\| = R^2 / \|\xi\|$. So, if you observe $\|\xi^*\| \|\xi\| = R^2$. R will be chosen ξ^* with some property correct. So, ξ is here origin is here ξ^* is here.

So, product of the length of ξ^* and length of $\xi = R^2$, hence this ξ^* is called inversion with respect to the sphere in real numbers, let us look at 0, let us say 1. And I take a point here, it is a half what are the inverse of half multiplicative inverses 2. So, why 2 into 1 /

$2 = 1$. So, this is exactly the same thing here that is what is called ξ^* is the inversion of ξ with respect to a S of $0, R$.

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Green's function for the domain $D(0, R)$

- Since
$$\|\xi^*\| = \frac{R^2}{\|\xi\|},$$
 we get $c = \frac{R^2}{\|\xi\|^2}$, and thus
$$\xi^* = \frac{R^2}{\|\xi\|^2} \xi$$
- Thus it follows that
$$\frac{q}{2\pi} \ln \|\mathbf{x} - \xi^*\| + \frac{C}{2\pi} = \frac{1}{2\pi} \ln \|\mathbf{x} - \xi\| \text{ for all } \mathbf{x} \text{ s.t. } \|\mathbf{x}\| = R$$
 holds with $q = 1$, C such that $e^C = \frac{\|\xi\|}{R}$.

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Since $\text{norm } \xi^* = R^2 / \text{norm } \xi$ we get $c = R^2 / \text{norm } \xi^2$ and thus $\xi^* = R^2 / \text{norm } \xi^2 \text{ into } \xi$. Thus it follows that this equality for all x such that $\text{norm } x = R$ holds with $q = 1$ and C is such that $e^C = \text{norms } \xi / R$.

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Green's function for the domain $D(0, R)$ (contd.)

Thus we have

$$\begin{aligned} \varphi(\mathbf{x}, \xi) &= \frac{1}{2\pi} \ln \|\mathbf{x} - \xi^*\| + \frac{1}{2\pi} \ln \frac{\|\xi\|}{R} \\ &= \frac{1}{2\pi} \left(\ln \|\mathbf{x} - \xi^*\| + \ln \frac{\|\xi\|}{R} \right) \\ &= \frac{1}{2\pi} \ln \left(\frac{\|\xi\|}{R} \|\mathbf{x} - \xi^*\| \right) \\ &= \frac{1}{2\pi} \ln \left(\frac{\|\xi\|}{R} \left\| \mathbf{x} - \frac{R^2}{\|\xi\|^2} \xi \right\| \right) \end{aligned}$$

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Therefore, we have $\phi = 1 / 2 \pi \log \text{norm } x - \xi^* + 1 / 2 \pi \log \text{norm } \xi / R$ which if you take $1 / 2 \pi$ common it becomes this now, it is like $\log A + \log B$ that means $\log AB$. So, you have this and substituting the value of ξ^* we get this, so this is ϕ of x, ξ .

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Green's function for the domain $D(0, R)$ (contd.)

Our search for Green's function G proceeded by the decomposition formula

$$G(\mathbf{x}, \xi) = K(\mathbf{x}, \xi) - \varphi(\mathbf{x}, \xi).$$

Thus Green's function $G(\mathbf{x}, \xi)$ for $\xi \neq \mathbf{0}$ is given by

$$G(\mathbf{x}, \xi) = \frac{1}{2\pi} \ln \|\mathbf{x} - \xi\| - \frac{1}{2\pi} \ln \left(\frac{\|\xi\|}{R} \left\| \mathbf{x} - \frac{R^2}{\|\xi\|^2} \xi \right\| \right).$$

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Our search for Green's function proceeded by the decomposition formula $G = K - \varphi$ now, we know φ also K is already known, therefore G is known. So, G has this expression given in this equation. Now, let us discuss Poisson's formula for the solution to Dirichlet boundary value problem on the ball and also on the disk in \mathbb{R}^2 .

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Recall from Lecture 6.2 Remark

The formula

$$u(\xi) = \int_{\Omega} K(\mathbf{x}, \xi) \Delta u \, dx - \int_{\partial\Omega} K(\mathbf{x}, \xi) \partial_{\mathbf{n}} u \, d\sigma + \int_{\partial\Omega} u \partial_{\mathbf{n}} K(\mathbf{x}, \xi) \, d\sigma$$

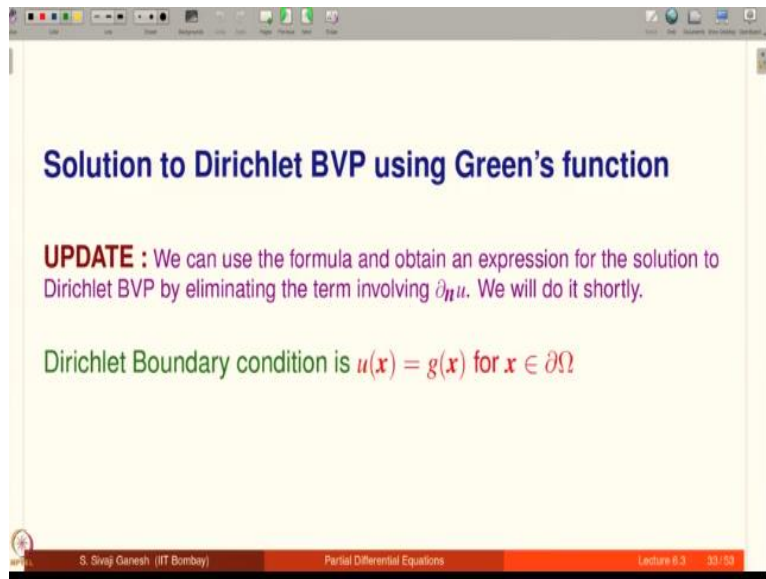
gives a representation of the solution $u(\xi)$ (if exists, in which case it is unique) in terms of values of u and its normal derivative $\partial_{\mathbf{n}} u$ on the boundary $\partial\Omega$.

For Dirichlet problem only u is given on $\partial\Omega$, and $\partial_{\mathbf{n}} u$ is an unknown function on $\partial\Omega$. Thus the formula given above is not useful for computing the solution.

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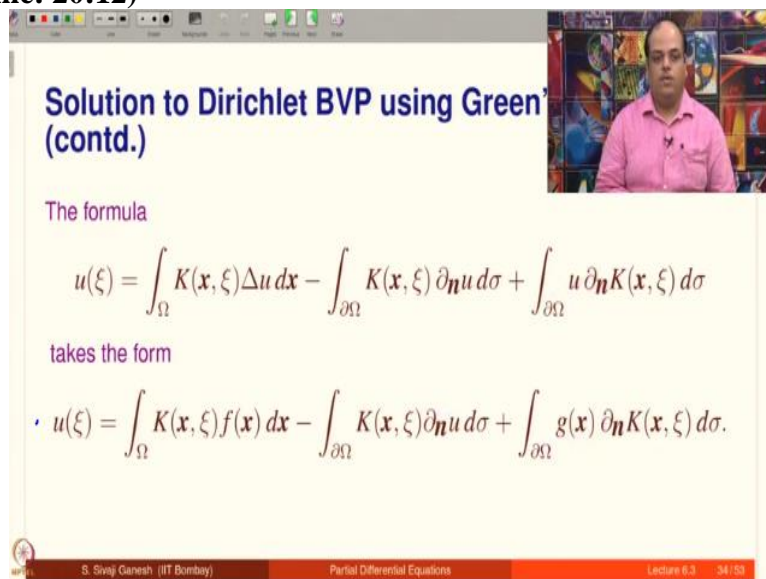
So, recall from lecture 6.2 we made a remark there that this formula which we have proved it gives a representation of the solution u in terms of the Laplacian Δu and Ω that is fine and it is boundary values $\partial_{\mathbf{n}} u$ and u . For Dirichlet problem only u is given and $\partial_{\mathbf{n}} u$ unknown. Therefore, the formula given above is not useful for computing the solution.

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But there is an update now, we can use the formula and obtain an expression for the solution to Dirichlet boundary value problem by eliminating the term involving the normal derivative of u . That is what we are going to do shortly. So, Dirichlet boundary condition is $u = g$ for \mathbf{x} belongs to a boundary of ω .

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So, this formula takes this form if Laplacian $u = f$ you have this and $\partial_{\mathbf{n}}u$ unknown, u is known g .

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Solution to Dirichlet BVP using Green's (contd.)

Applying Green's identity II

$$\int_{\Omega} (v\Delta u - u\Delta v)(\mathbf{x}) \, d\mathbf{x} = \int_{\partial\Omega} (v\partial_{\mathbf{n}}u - u\partial_{\mathbf{n}}v) \, d\sigma$$

with $u = u$ and $v = \varphi(\mathbf{x}, \xi)$ yields

$$0 = - \int_{\Omega} \varphi(\mathbf{x}, \xi) f(\mathbf{x}) \, d\mathbf{x} + \int_{\partial\Omega} \varphi(\mathbf{x}, \xi) \partial_{\mathbf{n}}u \, d\sigma - \int_{\partial\Omega} g(\mathbf{x}) \partial_{\mathbf{n}}\varphi(\mathbf{x}, \xi) \, d\sigma.$$

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Now, let us apply Green's identity to which is given here with $u = u$ and $v = \varphi$ of \mathbf{x}, ξ that will give us an equation which involves $\partial_{\mathbf{n}} u$.

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Solution to Dirichlet BVP using Green's function

Thus we have the two equations

$$u(\xi) = \int_{\Omega} K(\mathbf{x}, \xi) f(\mathbf{x}) \, d\mathbf{x} - \int_{\partial\Omega} K(\mathbf{x}, \xi) \partial_{\mathbf{n}}u \, d\sigma + \int_{\partial\Omega} g(\mathbf{x}) \partial_{\mathbf{n}}K(\mathbf{x}, \xi) \, d\sigma.$$

$$0 = - \int_{\Omega} \varphi(\mathbf{x}, \xi) f(\mathbf{x}) \, d\mathbf{x} + \int_{\partial\Omega} \varphi(\mathbf{x}, \xi) \partial_{\mathbf{n}}u \, d\sigma - \int_{\partial\Omega} g(\mathbf{x}) \partial_{\mathbf{n}}\varphi(\mathbf{x}, \xi) \, d\sigma.$$

Adding the above two equations, and using the definition of $G(\mathbf{x}, \xi)$, we get

$$u(\xi) = \int_{\Omega} G(\mathbf{x}, \xi) f(\mathbf{x}) \, d\mathbf{x} + \int_{\partial\Omega} g(\mathbf{x}) \partial_{\mathbf{n}}G(\mathbf{x}, \xi) \, d\sigma.$$

as $K(\mathbf{x}, \xi) = \varphi(\mathbf{x}, \xi)$ for $\mathbf{x} \in \partial\Omega$.

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So that is we have these 2 equations, if you add these to the term involving $\partial_{\mathbf{n}} u$ gets cancelled, because on the boundary $K - \varphi$ is 0. Thus, we get a formula for u , which involves only g $\partial_{\mathbf{n}} u$ is eliminated.

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Solution to Dirichlet BVP using Green's function

The solution to Dirichlet BVP is given by

$$u(\xi) = \int_{\Omega} G(x, \xi) f(x) dx + \int_{\partial\Omega} g(x) \partial_{\mathbf{n}} G(x, \xi) d\sigma.$$

In the special case, when u is a harmonic function in Ω , i.e., $f = 0$, then the above formula reduces to

$$u(\xi) = \int_{\partial\Omega} g(x) \partial_{\mathbf{n}} G(x, \xi) d\sigma.$$

All that remains is to compute $\partial_{\mathbf{n}} G(x, \xi)$ for $d = 2$ and $d \geq 3$.

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So, the solution to Dirichlet boundary value problem is given by this and the special case when u is harmonic this term drops out and we have just this term. In fact, we are going to solve harmonic functions such that $u = g$ on the boundary. Therefore, this is what we are going to find a new expression first and then $\partial_{\mathbf{n}} G$ is what all needs to be computed. So, compute $\partial_{\mathbf{n}} G$ for $d \geq 3$ and $d = 2$ separately and substitute in this and see what the formula we get. And that formula is called Poisson's formula.

If you are given Laplacian $u = f$, but f is not equal to 0, this will give you the solution, this will be the representation for the solution, we have to be very careful, this will give a solution means we have to prove something, but what we have done so far is if there is a solution, which is smooth enough, then you have this expression, that is what we are doing. So, we are getting an expression for solution if exists, existence is to be proved. Hopefully, the Poisson's formula will help us improve the existence of solutions to Dirichlet boundary value problem.

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Solution to Dirichlet BVP using Green's function ($d \geq 3$)

For $\Omega = B(\mathbf{0}, R)$, the quantity $\partial_n G(x, \xi)$ for $x \in S(\mathbf{0}, R)$ and $\xi \neq \mathbf{0}$ is given by

$$\begin{aligned} \partial_n G(x, \xi) &= \frac{1}{\omega_d(2-d)} \partial_n \left(\frac{1}{\|x - \xi\|^{d-2}} - \frac{R^{d-2}}{\|\xi\|^{d-2}} \frac{1}{\|x - \xi^*\|^{d-2}} \right) \\ &= \frac{1}{\omega_d(2-d)} \nabla x \cdot \left(\frac{1}{\|x - \xi\|^{d-2}} - \frac{R^{d-2}}{\|\xi\|^{d-2}} \frac{1}{\|x - \xi^*\|^{d-2}} \right) \cdot \frac{x}{R} \\ &= \frac{1}{\omega_d} \left(\frac{x - \xi}{\|x - \xi\|^d} - \frac{R^{d-2}}{\|\xi\|^{d-2}} \frac{x - \xi^*}{\|x - \xi^*\|^d} \right) \cdot \frac{x}{R} \end{aligned}$$

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So, for omega equal to ball of radius R center 0, do n g computation, please do this computation by yourself, I just put the value of g it is a constant so this came out. So, I had to find out the normal derivative of this which is nothing but gradient dot, the normal to the ball is along the radius therefore, x, but if you want to unit normal x / R because norm x = R, this will be unit normal outward normal. So, therefore, do n is nothing but grad x of this quantity dot x / R. Now, it is a matter of carrying out the differentiations.

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Solution to Dirichlet BVP using Green's function ($d \geq 3$)

Recall

$$\|x - \xi^*\| = q^{\frac{1}{d-2}} \|x - \xi\| = \frac{R}{\|\xi\|} \|x - \xi\|.$$

Thus

$$\begin{aligned} \partial_n G(x, \xi) &= \frac{1}{\omega_d} \left(\frac{x - \xi}{\|x - \xi\|^d} - \frac{R^{d-2}}{\|\xi\|^{d-2}} \frac{x - \xi^*}{\|x - \xi^*\|^d} \right) \cdot \frac{x}{R} \\ &= \frac{1}{\omega_d \|x - \xi\|^d R} \left(x - \xi - \frac{\|\xi\|^2}{R^2} (x - \xi^*) \right) \cdot x \\ &= \frac{R^2 - \|\xi\|^2}{\omega_d R} \frac{1}{\|x - \xi\|^d} \end{aligned}$$

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Please do the computations on your own I will skip the details, but I will keep the slide for some time so that you can note down. So, this is the final expression we get for do n G here, it is simplified very nicely. In fact, even d we are going to see that same formula will come for do n G.

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Solution to Dirichlet BVP using Green's function

Substituting the value of $\partial_n G(\mathbf{x}, \xi)$ from

$$\partial_n G(\mathbf{x}, \xi) = \frac{R^2 - \|\xi\|^2}{\omega_d R} \frac{1}{\|\mathbf{x} - \xi\|^d}$$

into equation

$$u(\xi) = \int_{\partial\Omega} g(\mathbf{x}) \partial_n G(\mathbf{x}, \xi) d\sigma$$

gives

$$u(\xi) = \frac{R^2 - \|\xi\|^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{g(\mathbf{x})}{\|\mathbf{x} - \xi\|^d} d\sigma.$$

The last expression is known as **Poisson's formula for $d \geq 3$** .

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So, substituting the value of $\partial_n G$ into this equation will give us this formula which is called Poisson's formula of course we have handled $d \geq 3$.

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Poisson's formula for $d = 2$

Computing $\partial_n G(\mathbf{x}, \xi)$ for $d = 2$ is left as an exercise.

Substituting the value of $\partial_n G(\mathbf{x}, \xi)$ into equation

$$u(\xi) = \int_{\partial\Omega} g(\mathbf{x}) \partial_n G(\mathbf{x}, \xi) d\sigma$$

gives

$$u(\xi) = \frac{R^2 - \|\xi\|^2}{2\pi R} \int_{S(\mathbf{0}, R)} \frac{g(\mathbf{x})}{\|\mathbf{x} - \xi\|^2} d\sigma.$$

The last expression is known as **Poisson's formula for $d = 2$** .

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Now computing $\partial_n G$ is left an exercise when $d = 2$ and substituting in this formula for $\partial_n G$ we get this expression look exactly the same formula $d = 2$ that is it.

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Poisson's formula

Poisson's formula has the same form for all d :

$$u(\xi) = \frac{R^2 - \|\xi\|^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{g(\mathbf{x})}{\|\mathbf{x} - \xi\|^d} d\sigma.$$

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So, Poisson's formula has the same form for all $d = 2$ which is $u(\xi) = \frac{R^2 - \|\xi\|^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{g(\mathbf{x})}{\|\mathbf{x} - \xi\|^d} d\sigma$. You may also write instead of g u itself after all it is a notation for values of u on the boundary. Now let us look at Harnack's inequality.

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Theorem. Harnack's inequality

Let $B(\mathbf{0}, R) \subset \mathbb{R}^d$ be the open ball with center at $\mathbf{0}$ and having radius R .

Let $u : B(\mathbf{0}, R) \rightarrow [0, \infty)$ be a harmonic function. Let $u \in C(\overline{B(\mathbf{0}, R)})$.

Then for any $x \in B(\mathbf{0}, R)$, the following inequalities hold:

$$\frac{R^{d-2}(R - \|x\|)}{(R + \|x\|)^{d-1}} u(\mathbf{0}) \leq u(x) \leq \frac{R^{d-2}(R + \|x\|)}{(R - \|x\|)^{d-1}} u(\mathbf{0}).$$

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So, let $B(\mathbf{0}, R)$ be open ball with center at $\mathbf{0}$ and having radius R in \mathbb{R}^d d greater than or equal to 2. And u be a harmonic function which is non negative. It takes values which are greater than or equal to 0 that is very important. And let u be continuous on the closed ball of radius R that means u is continuous up to the boundary. Then for any x in $B(\mathbf{0}, R)$ following inequalities hold so the value of $u(x)$ and the value of u at the center of this ball are tied by these inequalities or they satisfy these inequalities.

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Proof of Harnack's inequality

Poisson's formula has the same form for all d :

$$u(\mathbf{x}) = \frac{R^2 - \|\mathbf{x}\|^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{u(\mathbf{y})}{\|\mathbf{y} - \mathbf{x}\|^d} d\sigma.$$

Applying triangle inequality, for $\mathbf{y} \in S(\mathbf{0}, R)$ the following inequalities hold:

$$R - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| \leq R + \|\mathbf{x}\|.$$

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Proof is very simple, it follows just from the Poisson's formula, look at this Poisson's formula and then triangle inequality whenever \mathbf{y} is on the sphere of radius R that means norm \mathbf{y} is R . So, norm $\mathbf{x} - \mathbf{y}$ by triangle inequality is less than a norm $\mathbf{x} + \text{norm } \mathbf{y}$, but norm \mathbf{y} is R therefore we have this, here norm $\mathbf{x} - \mathbf{y}$ is greater than or equal to norm $\mathbf{y} - \text{norm } \mathbf{x}$, which is a consequence of triangle inequality. Therefore, you get this now use these inequalities in this denominator, you get Harnack's inequality.

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Proof of Harnack's inequality (contd.)

Using the inequalities

$$R - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| \leq R + \|\mathbf{x}\|.$$

in the Poisson's formula

$$u(\mathbf{x}) = \frac{R^2 - \|\mathbf{x}\|^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{u(\mathbf{y})}{\|\mathbf{y} - \mathbf{x}\|^d} d\sigma.$$

yields

$$u(\mathbf{x}) \leq \frac{R + \|\mathbf{x}\|}{(R - \|\mathbf{x}\|)^{d-1}} \frac{1}{\omega_d R} \int_{S(\mathbf{0}, R)} u(\mathbf{y}) d\sigma,$$

$$u(\mathbf{x}) \geq \frac{R - \|\mathbf{x}\|}{(R + \|\mathbf{x}\|)^{d-1}} \frac{1}{\omega_d R} \int_{S(\mathbf{0}, R)} u(\mathbf{y}) d\sigma.$$

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So, this is what we get one, these other side, we get this harnack's inequality did not have this expression. What was there? Here is R power $d - 2$ into u of $\mathbf{0}$. Similarly, here, it was R power $d - 2$ into u of $\mathbf{0}$.

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Proof of Harnack's inequality (contd.)

Proof of Harnack's inequality is complete if we knew that

$$\frac{1}{\omega_d R} \int_{S(\mathbf{0}, R)} u(y) dy = R^{d-2} u(\mathbf{0}).$$

The last equation may be re-written as

$$\frac{1}{\omega_d R^{d-1}} \int_{S(\mathbf{0}, R)} u(y) dy = u(\mathbf{0}).$$

RHS is the mean value of u on the sphere (circle if $d = 2$) $S(\mathbf{0}, R)$. This follows from Poisson's formula.

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So, proof of harmonics inequalities complete if we knew that this quantity which appeared on the last slide is R power $d - 2$ into u of 0 . The last equation may be rewritten as this, I divide both sides with R power $d - 2$ so we get this. Now, if you look at what is the left hand side, it is your integrating u on the sphere. And this is the surface area of the sphere you are dividing with that so this is the spherical average spherical mean of u or this sphere. Now, we are asking whether that is equal to u of 0 this follows from Poisson's formula.

(Refer Slide Time: 26:47)

Proof of Harnack's inequality (contd.)

Recall Poisson's formula:

$$u(x) = \frac{R^2 - \|x\|^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{u(y)}{\|y - x\|^d} d\sigma.$$

Hence

$$u(\mathbf{0}) = \frac{R^2}{\omega_d R} \int_{S(\mathbf{0}, R)} \frac{u(y)}{\|y\|^d} d\sigma = \frac{R^2}{\omega_d R^{1+d}} \int_{S(\mathbf{0}, R)} u(y) d\sigma.$$

Thus

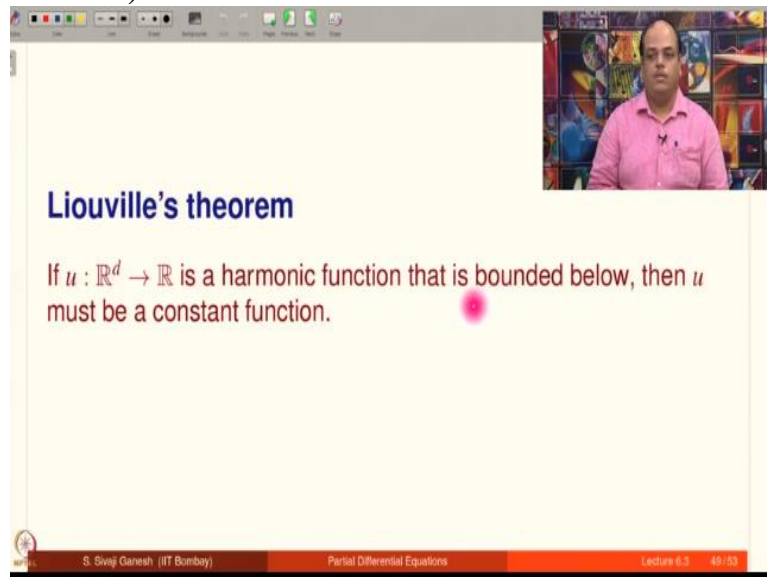
$$\frac{1}{\omega_d R} \int_{S(\mathbf{0}, R)} u(y) dy = R^{d-2} u(\mathbf{0}). \quad \square$$

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So, recall Poisson's formula u of $x = R$ square - norm x square / ω_d times R integral over the sphere of radius R u of y / norm $y - x$ power d d sigma. Therefore, u of $0 = R$ square / ω_d d R , because norm x is 0 into the integral on the sphere of radius R u of y / norm y power d d sigma because x is 0 , but when y belongs to the circle norm y is R therefore, we get R square / ω_d d R power $1 + d$ into integral of u on the circle. Thus, we have this

equation is exactly what we wanted to show. So, this concludes the proof of Harnack's inequality.

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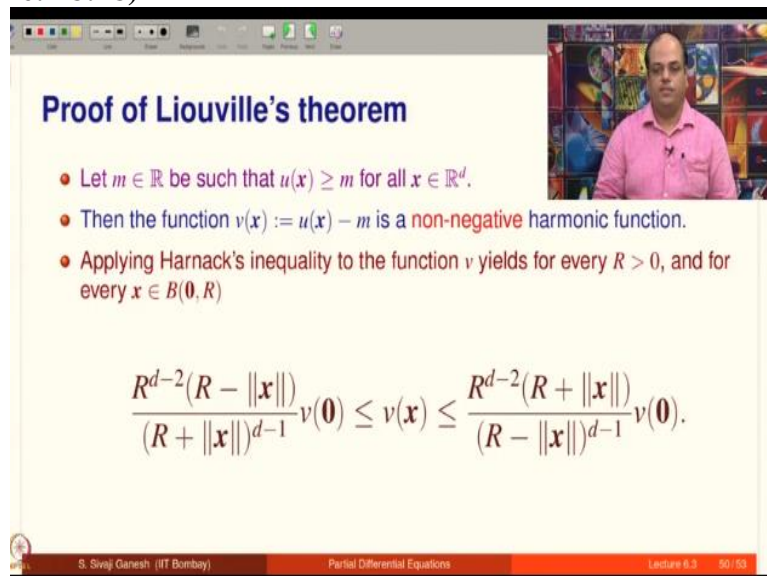
Liouville's theorem

If $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a harmonic function that is bounded below, then u must be a constant function.

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Now, as a consequence of Harnack's inequality, we are going to prove Liouville's theorem, what is Liouville's theorem? If you have a harmonic function on \mathbb{R}^d which is bounded below then it must be constant it means no non constant harmonic functions can be bounded below. Now, as a corollary we can also get that if your harmonic function is bounded above then also it must be constant.

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Proof of Liouville's theorem

- Let $m \in \mathbb{R}$ be such that $u(x) \geq m$ for all $x \in \mathbb{R}^d$.
- Then the function $v(x) := u(x) - m$ is a non-negative harmonic function.
- Applying Harnack's inequality to the function v yields for every $R > 0$, and for every $x \in B(0, R)$

$$\frac{R^{d-2}(R - \|x\|)}{(R + \|x\|)^{d-1}}v(\mathbf{0}) \leq v(x) \leq \frac{R^{d-2}(R + \|x\|)}{(R - \|x\|)^{d-1}}v(\mathbf{0}).$$

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So, let m belongs to \mathbb{R} be such that u is greater than or equal to m for all x in \mathbb{R}^d because we are assuming u is a function which is bounded below. Then look at this function $u(x) - m$ this will be harmonic and it will be non-negative therefore, Harnack's inequalities applicable for this v . Therefore, for every R and for every x in $B(0, R)$ we have these inequalities.

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Proof of Liouville's theorem (contd.)

$$\frac{R^{d-2}(R - \|x\|)}{(R + \|x\|)^{d-1}}v(\mathbf{0}) \leq v(x) \leq \frac{R^{d-2}(R + \|x\|)}{(R - \|x\|)^{d-1}}v(\mathbf{0}).$$

- Fix an $x \in \mathbb{R}^d$, and $R > 0$ such that $R > \|x\|$.
- Passing to the limit as $R \rightarrow \infty$ in the inequalities above, we get

$$v(\mathbf{0}) \leq v(x) \leq v(\mathbf{0}).$$

- Thus we get $v(x) = v(\mathbf{0})$.
- Since x is arbitrary, we conclude that v is a constant function. As a consequence, u is a constant function.

These are the inequalities coming from Harnack's inequality. So, fix an x in \mathbb{R}^d take R which is positive so that R bigger than norm x and pass to the limit as R goes to infinity because when if you want to pass the limit as R goes to infinity, whatever x you take there will be a time after which your R becomes bigger than norm x anyway so we have this. So, in the inequalities table let us pass to the limit as R goes to infinity, what we get is v of 0 less than or equal $v(x)$ less than or equal to v of 0 .

Rough understanding of this is this is R power $d - 2$ into 1 here is also the leading term R power $d - 1$. Therefore, this limit will be 1 . Similarly, this limit will be 1 as R goes to infinity. So, we have this now what does this mean $v(x) = v(0)$ since x is arbitrary, what we have shown is that v is a constant function. As a consequence, u is a constant function.

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Summary

- Defined Green's function for Dirichlet BVPs.
 - Obtained **Poisson's formula**, a representative formula for a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ when Ω is a ball (or disk) in \mathbb{R}^d .
 - It is not clear if Dirichlet BVP admits a solution! It can be shown that Poisson's formula defines such a solution.
- As a consequence of Poisson's formula, we have established
 - **Harnack's inequality**
 - **Liouville theorem.**

Let us summarize we have defined Green's function for Dirichlet boundary value problems. We have obtained Poisson's formula which is a representative formula for a solution which is C^2 in ω and continuous up to the boundary of ω . When ω is a ball or a disk in \mathbb{R}^d . It is not clear if Dirichlet boundary value problem admits a solution. It can be shown that Poisson's formula defines such a solution. As a consequence of Poisson's formula, we have established Harnack's inequality and Liouville theorem. Thank you.