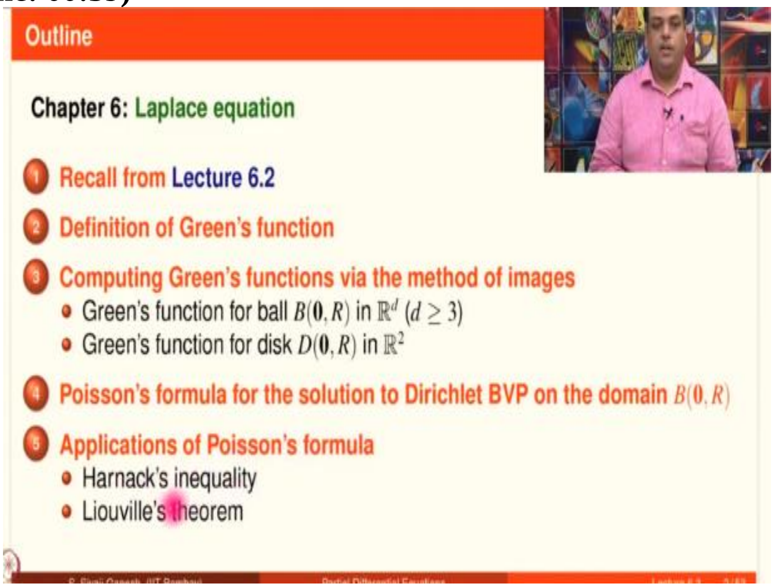
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 6.3
Dirichlet BVP for Laplace Equation – Green's Function and Poisson's Formula

Welcome, in this lecture we are going to consider divisionally boundary value problems for Laplace equation, we will obtain a representative formula for a solution to the division problem which is called Poisson's formula; it can be derived using Green's functions. And towards the end of the lecture, we will see certain applications of Poisson's formula.

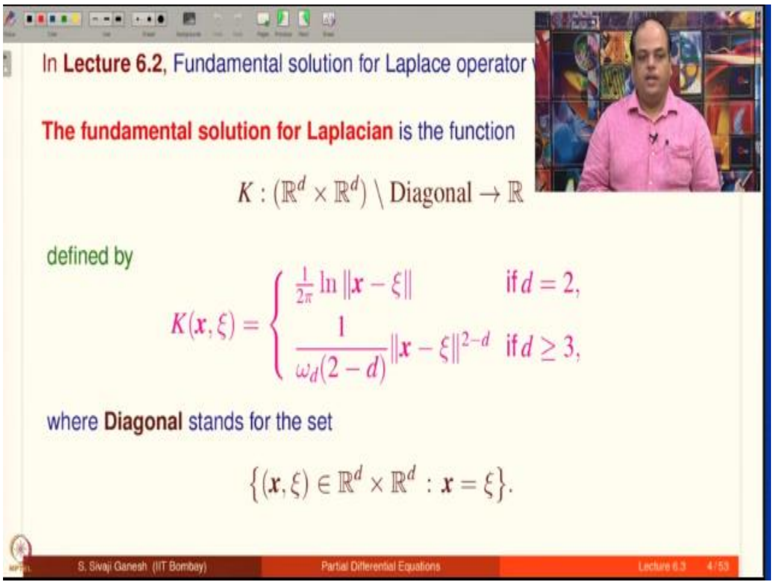
(Refer Slide Time: 00:39)



Outline of this lecture is as follows we will recall certain things that we have done in lecture 6.2 namely fundamental solutions and their properties, then we go on to define Green's function. And we compute these functions via what is called a method of images for a ball in R d when D is greater than or equal to 3 ball is a disk when it is in R 2. Then we write the Poisson's formula for the solution to Dirichlet boundary problem on this domains ball or disk.

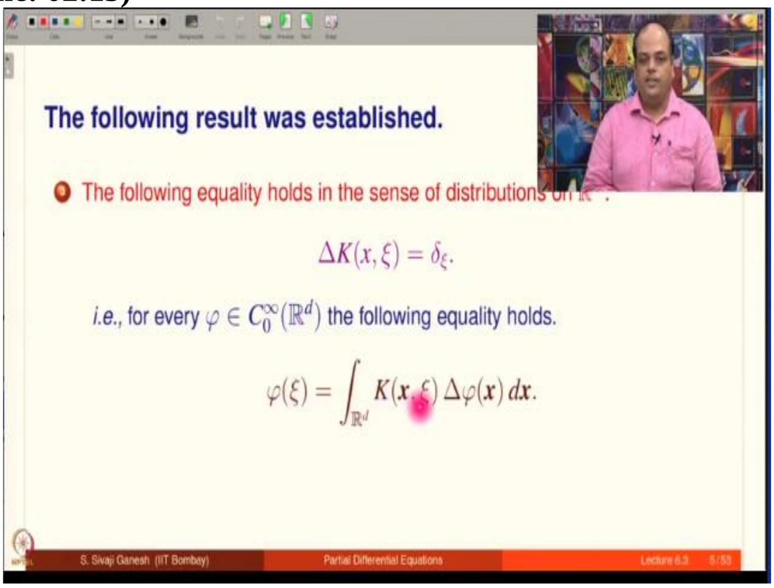
As an application of Poisson's formula we will prove Harnack's inequality the further applications have not seen equality is going to be Louiville theorem. Let us recall from lecture 6.2.

(Refer Slide Time: 01:22)



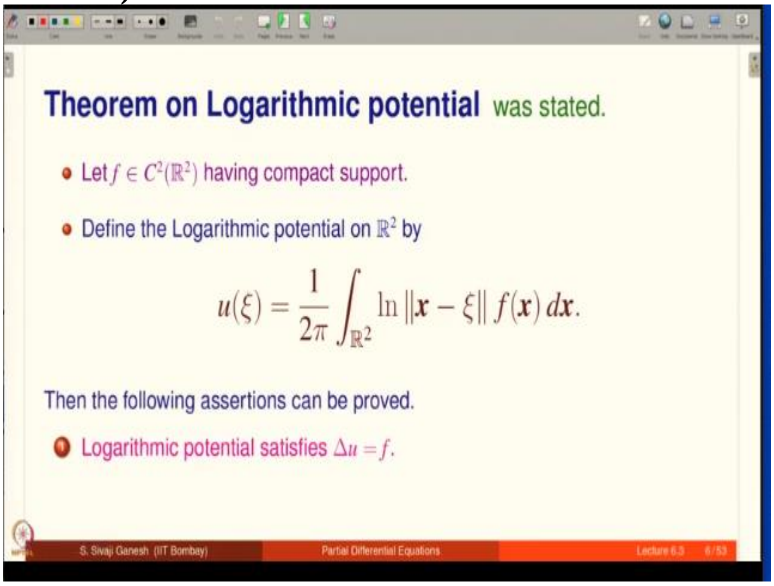
In lecture 6.2, fundamental solution for Laplace operator was defined, the fundamental solution for the Laplacian is the function K from R d cross R d - diagonal to R, we will define what is the diagonal once we see the formula for K. K of x Xi = 1 / 2 pi logarithm norm x - Xi if d = 2 and if d is greater than or equal to 3. The formula for K x Xi is 1 / omega d into 2 - d into norm x - Xi power 2 – d. As you see, when x = Xi both these functions have singularities. Therefore, we have to remove that so diagonal stands for the set x, Xi in R d cross R d such that x = Xi.

(Refer Slide Time: 02:15)



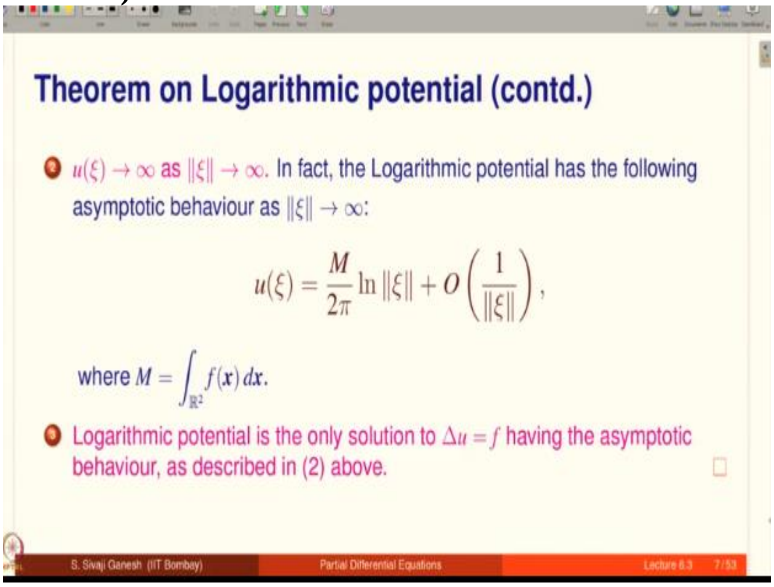
And the following result was establish it says that Laplacian of K of x, Xi = delta Xi, for every Xi fixed in R d this Laplacian is in the x coordinates Laplacian in K of x, Xi = delta z. What it means? We have explained that means, actually this you can think this is a notation for the moment, because we do not know what this is, in case you know it is exactly the same as what you know, if you do not know this means this. For every phi which is C infinity compactly supported function in R d phi of Xi is given by integral over R d of K x Xi into Laplacian phi of x dx.

(Refer Slide Time: 02:56)



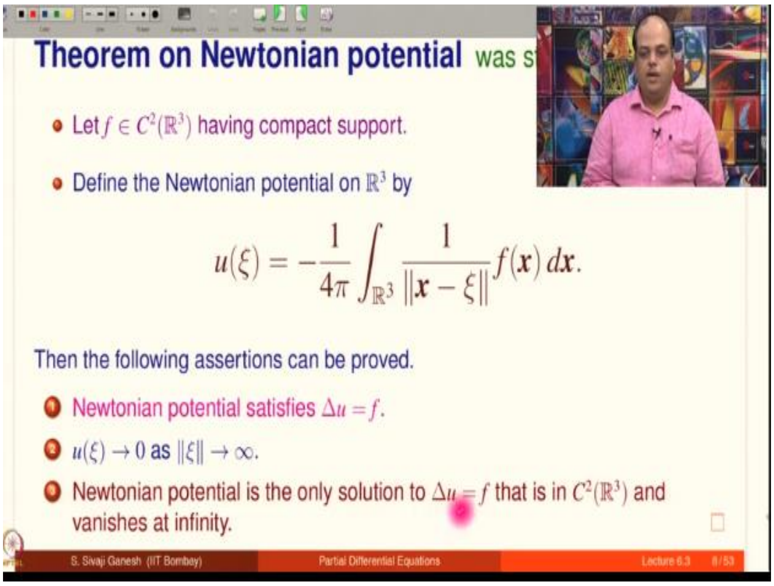
Then we have stated this theorem on logarithmic potential, it says that if you have a function which is C 2 and has compact support in R 2, then if you define the logarithmic potential by this formula, this u has the property that Laplacian u = f.

(Refer Slide Time: 03:13)



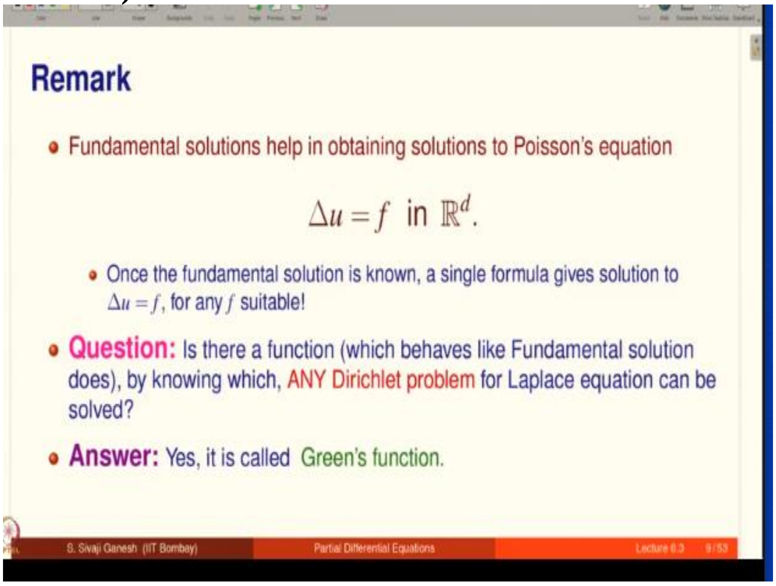
And u Xi goes to infinity as norm Xi goes to infinity there is a precise asymptotic as it goes to infinity. Logarithmic potential is the only solution to Laplacian u = f having this kind of asymptotic behaviour.

(Refer Slide Time: 03:31)



Then, we have stated a theorem on Newtonian potential, this is applicable for all d greater than or equal to 3. So, if you have one second C 2 function with compact support and define the Newtonian potential by this formula. Then the following assertions can be proved Laplacian u = f, u goes to 0 as norm Xi goes to infinity and this is the only solution to Laplacian u = f which is C 2 and vanishes at infinity.

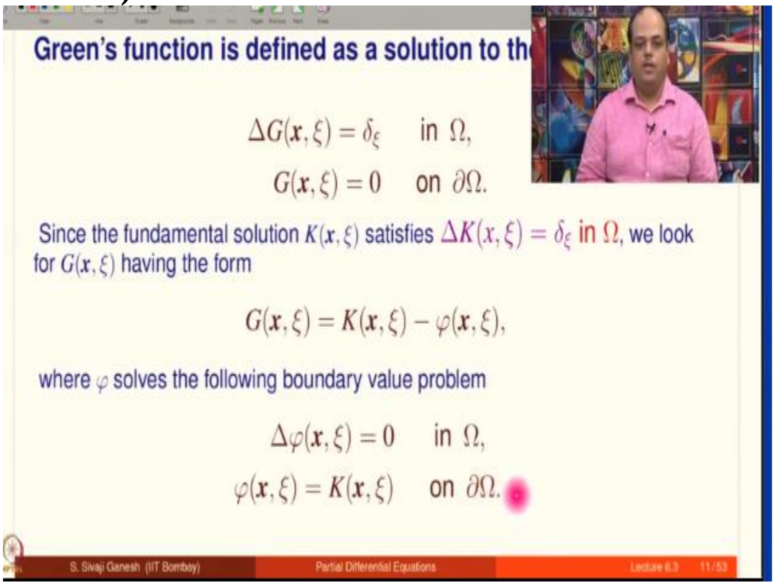
(Refer Slide Time: 04:00)



Remark fundamental solutions help in solving Poisson's equation Laplacian u = f in R d this is the essence of the theorems that we saw. So, once the fundamental solution is known, a single formula gives solution to Laplacian u = f. Tt is logarithmic potential if it is t = 2 it is Newtonian potential if d is greater than or equal to 3 for any f and suitable. Question, is there a function which behaves like fundamental solution does by knowing which any Dirichlet problem for Laplace equation can be solved?

The answer is yes, it is called Green's function. Understand the difference between Green's function and fundamental solution. These 2 names are often blurred in the literature. So, pay specific attention to the difference between these 2.

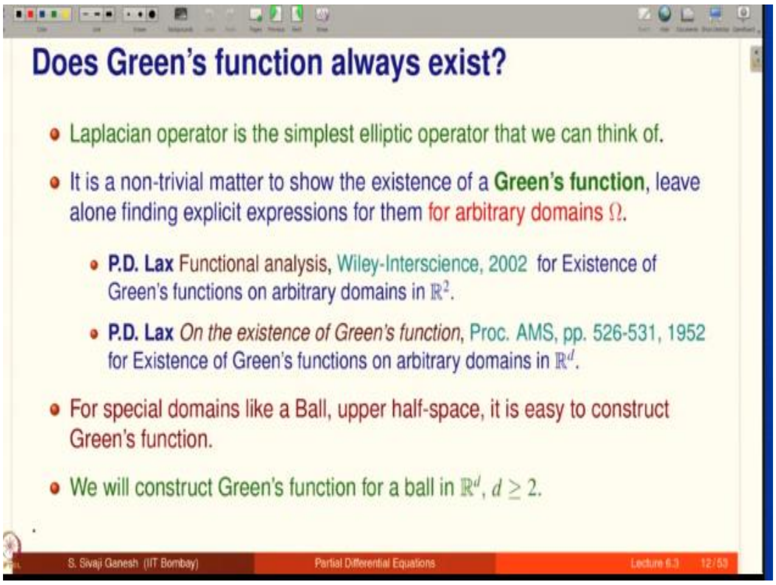
(Refer Slide Time: 04:58)



So, definition of Green's function Green's function is defined as a solution to the boundary value problem Laplacian G =delta Xi in omega, G of x, Xi = 0 on boundary of omega. Since the fundamental solution K already satisfies Laplacian K = delta Xi in omega, we look for G that is a Green's function having this form that G = K - phi. Now, you know the equation for G you know the equation for K and also the condition for G on the boundary therefore, you know what is the equation phi was satisfied.

So, phi solves Laplacian phi = 0 in omega and phi of x, Xi = K of x, Xi on the boundary of omega. So, if you want to know the Green's function, since we already know K what remains is to find this phi.

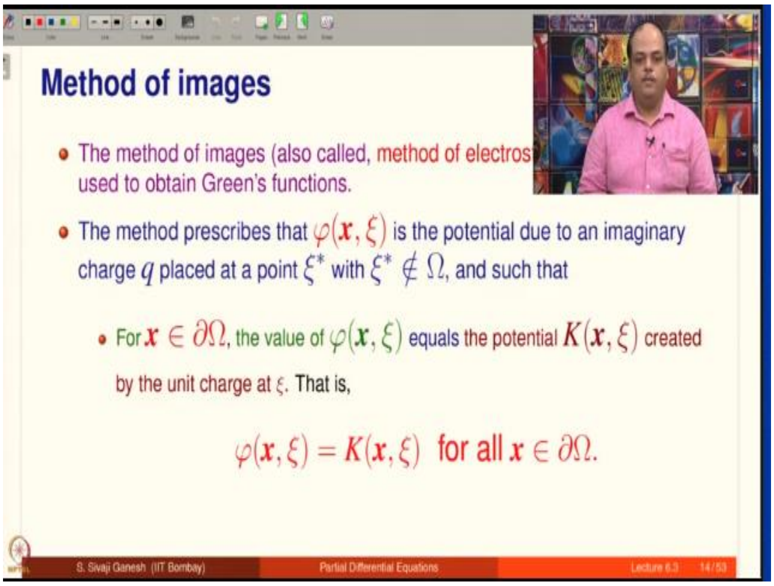
(Refer Slide Time: 05:52)



The question is does Green's function always exist? Laplacian operator is clearly the simplest elliptic operator that we can think of. It is a non-trivial matter to show the existence of a Green's function, leave alone finding explicit expressions for it for arbitrary domains omega. These are the 2 references I am going to give where you can find the discussion on Green's functions existence one is a book by P.D. Lax on functional analysis. Here you can find the existence of Green's functions and arbitrary domains in R 2.

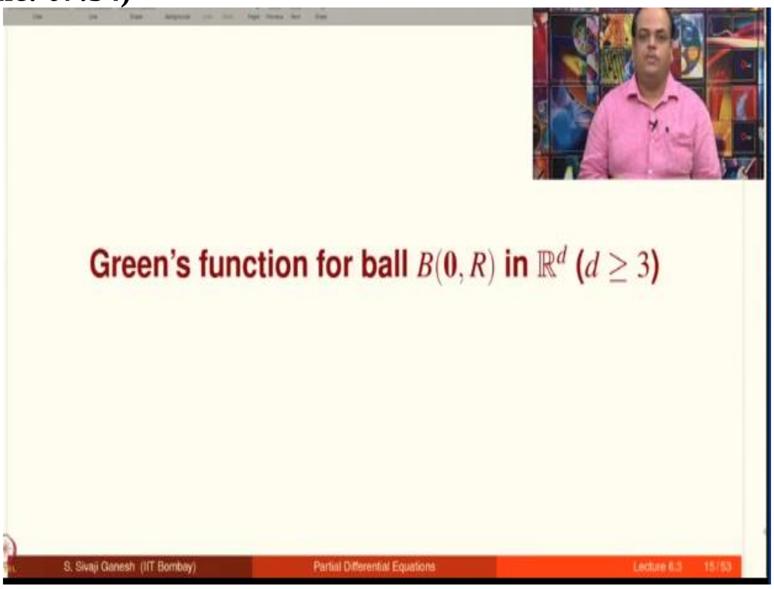
Then there is a paper on the existence of Green's function in proceedings of AMS by P.D. Lax, where he discusses Green's functions on arbitrary domains in R d. For special domains like a ball or upper half space, like the upper half plane and so on, it is easy to construct the Green's function, we will construct Green's function for a ball in R d, d greater than or equal to 2. So, ball when d = 2 is called disk. So, computing Green's functions we use what is called a method of images.

(Refer Slide Time: 07:07)



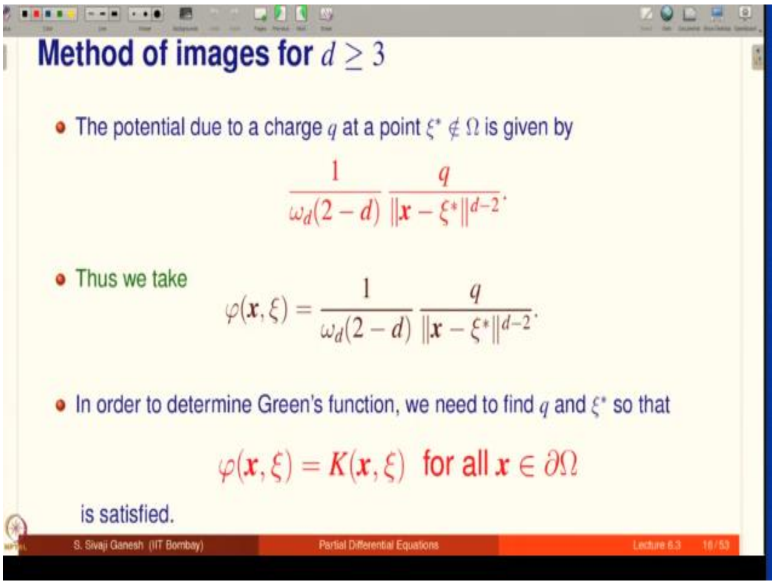
So, what is method of images? The method of images, it is also called method of electrostatic images will be used to obtain Green's functions. The method prescribes that phi of x, Xi is the potential due to an imaginary charge q placed at a point Xi star, which is not in the domain omega which is outside the domain omega and such that for every x on the boundary of omega, the value of phi of x, Xi coincides with the value of K x, Xi which is created by the unit charge at Xi that is phi x, Xi = K x, Xi for all x in boundary of omega.

(Refer Slide Time: 07:54)



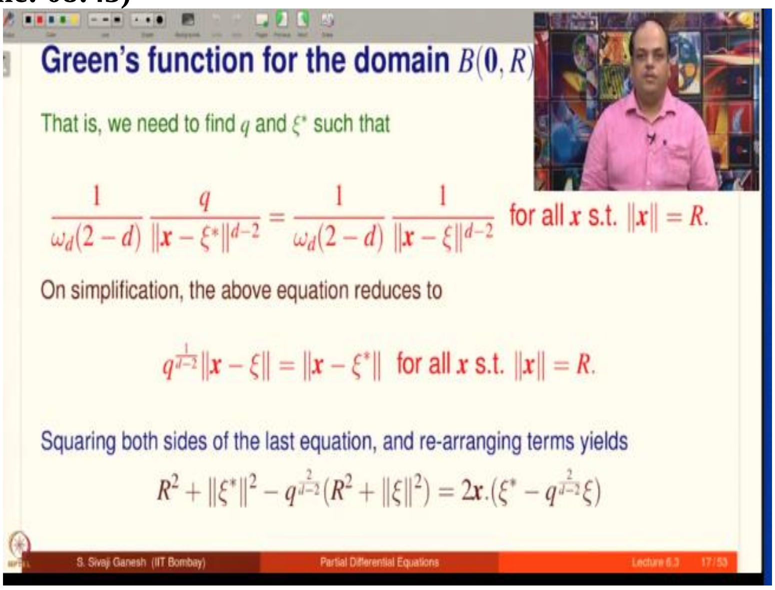
So, Green's function for ball in R d, d is greater than or equal to 3, we will deal the case when d = 2 separately because of the way the formula for the fundamental division looks is different.

(Refer Slide Time: 08:05)



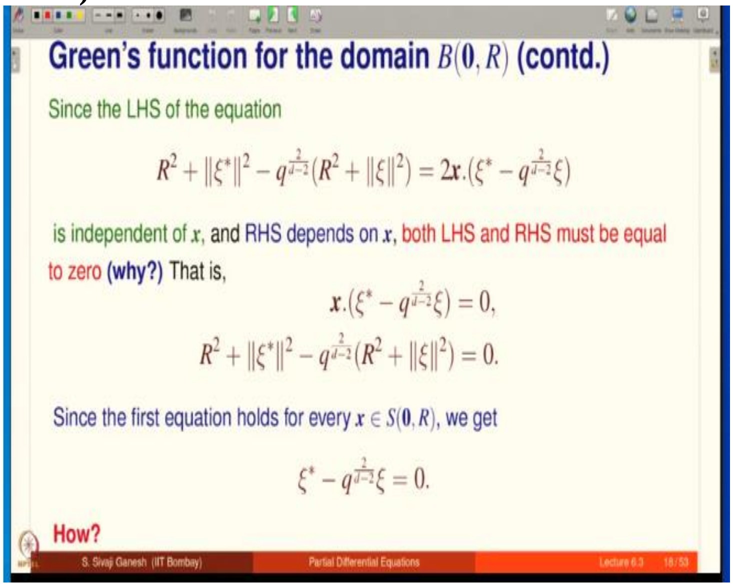
So, method of images for d is greater than or equal to 3, the potential due to a charge q at point Xi star which is not in omega is given by exactly the fundamental solution this is a charge q fundamental solution is for the charge 1. So, now it is q now, location is Xi star. Thus, we take phi x, Xi is equal to this. In order to determine Green's function, we need to find q and Xi star. So that phi x, Xi = K x, Xi for all x in boundary of omega. So, using this constraint we have to find both q and Xi star.

(Refer Slide Time: 08:43)



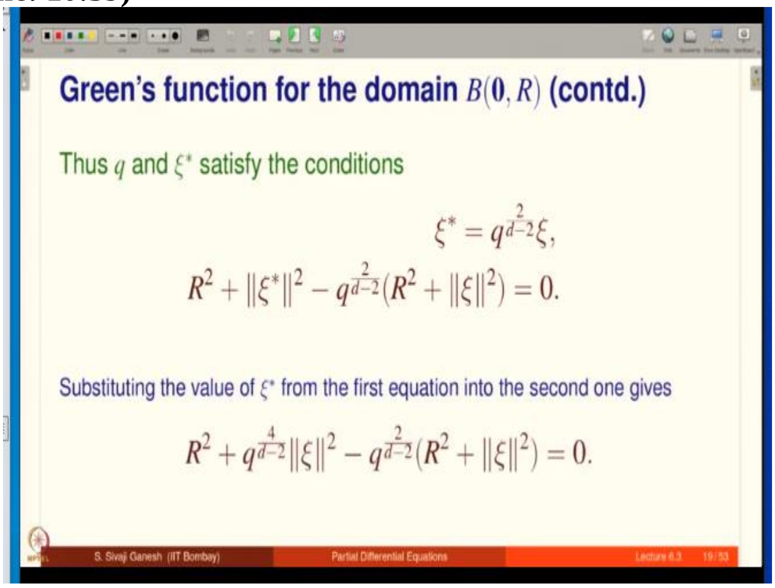
That is, we need to find q and Xi star such that this equation holds for every x such that norm x = R, because a boundary of the ball of radius R is a phi of radius R which is norm x = R. On simplification the above equation reduces to this because straightaway you can see that this cancels with this both are same factors and then you get q into norm x - Xi d - 2 = norm x - Xi star power d - 2 and take the power 1 / d - 2 we get this. Squaring both sides of the last equation and rearranging the terms will give you this. At this point of time pause the video compute for yourself and make sure that you are getting this equation.

(Refer Slide Time: 09:40)



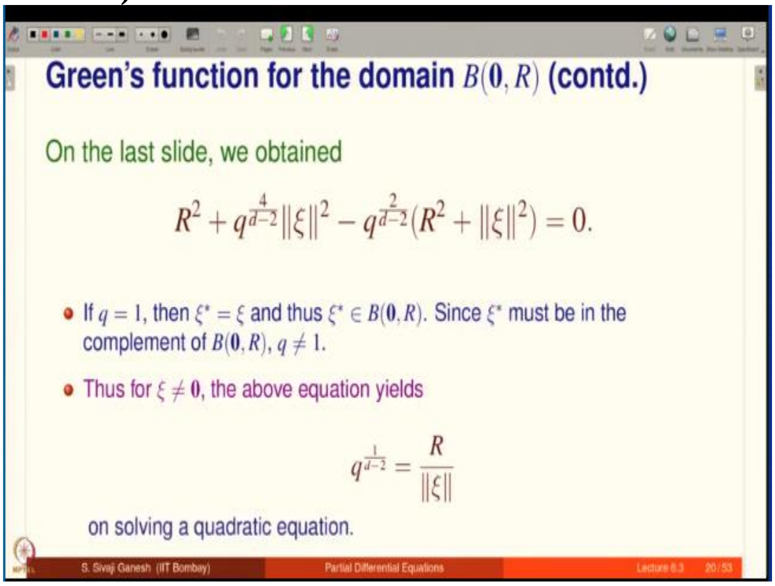
Now, assume since the left hand side of this equation is independent of x and right hand side depends on x both must be constant and that constant must be 0. Please justify for yourself why should the constant be 0? That is this is 0 a first equation and LHs are 0 that is a second equation. Now, the first equation holds for every x in S of 0, R that is sphere. Therefore Xi star - q power 2 / d - 2 Xi that itself is 0, what it says is that division some vector in R d whose inner product with the x is 0 for every x on the sphere of radius R, if that happens, then that vector has to be 0. So, I have already given you the hint it is work it out.

(Refer Slide Time: 10:39)



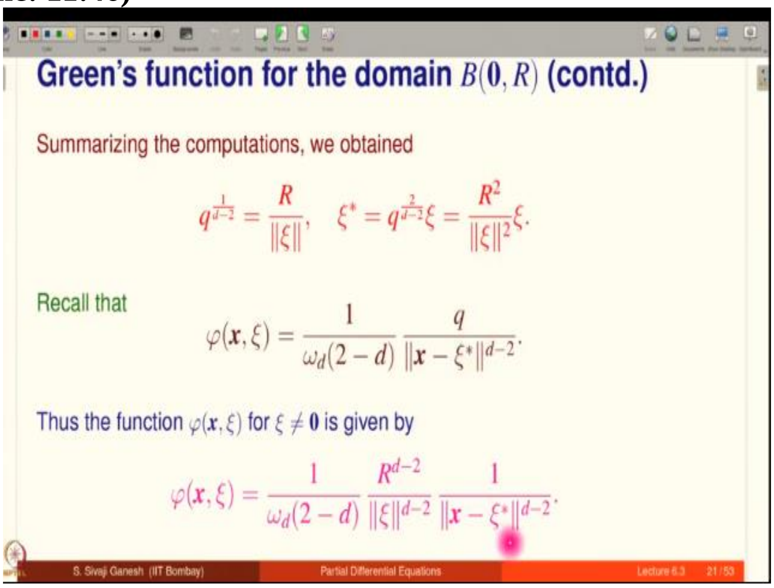
Thus q and Xi star satisfy these 2 equations now substitute the value of Xi star from here into this at this point that will give us an equation like this.

(Refer Slide Time: 10:54)



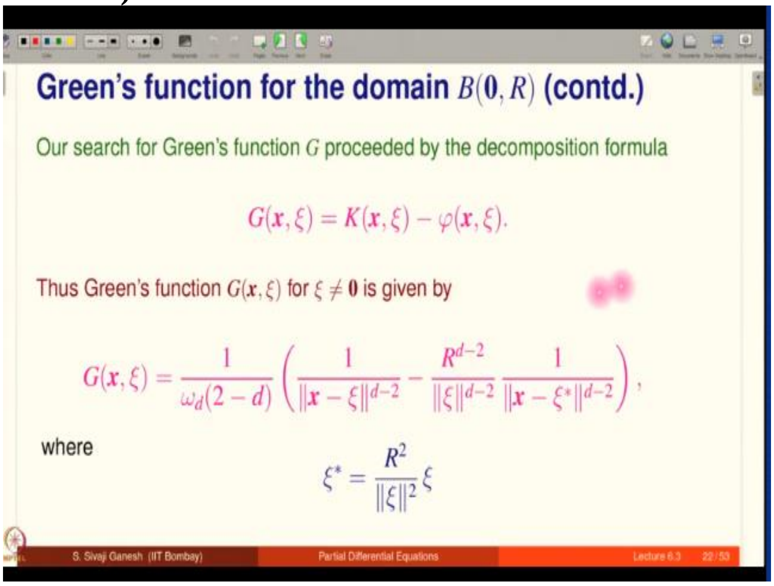
So, on the last slide, we obtained this equation, now we need to solve for q there is no Xi star here only q. So, let us observe that if q = 1 then Xi star = Xi and thus Xi star belongs to the ball which is not allowed we want to Xi star out at the ball. Therefore q cannot be equal to 1, thus for Xi is nonzero, the above equation yield this how do I get this, I just see that the quantum quantity is q agreed, but then q power 2 / d - 2 and it is queries here q power 2 / d - 2 whole square is precisely this. So, it is a quadratic equation in q power 2 / d - 2 think that is lambda and solve this quadratic equation you will get this.

(Refer Slide Time: 11:46)



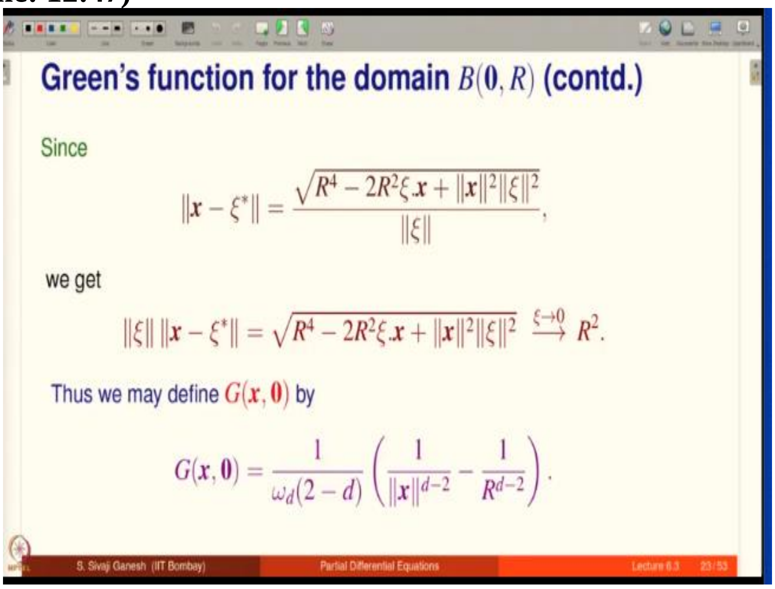
So, summarizing the computations we have obtained q power 1 / d - 2 equal to this in other words, we have q and Xi star is given in terms of q, q is already known from here. Therefore we know q and Xi star and this is nothing but R square / norm Xi square into Xi. So, recall that phi of x, Xi we started with this. So, therefore phi of x, Xi is equal to this I have just substituted what is q here and I have not substituted what is Xi star the value of Xi star inside this Xi still retain it as Xi star. Xi star = R square / norm Xi square into Xi.

(Refer Slide Time: 12:25)



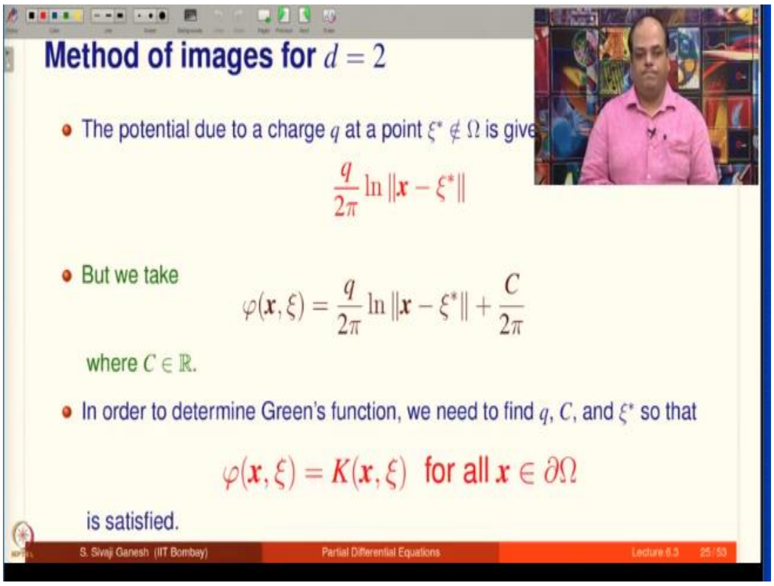
So, our search for Green's function proceeded by the decomposition formula G = K – phi therefore, K is already known, we have just found phi therefore G is known and that is given by this formula G of x, Xi is equal to this formula where Xi star is this.

(Refer Slide Time: 12:47)



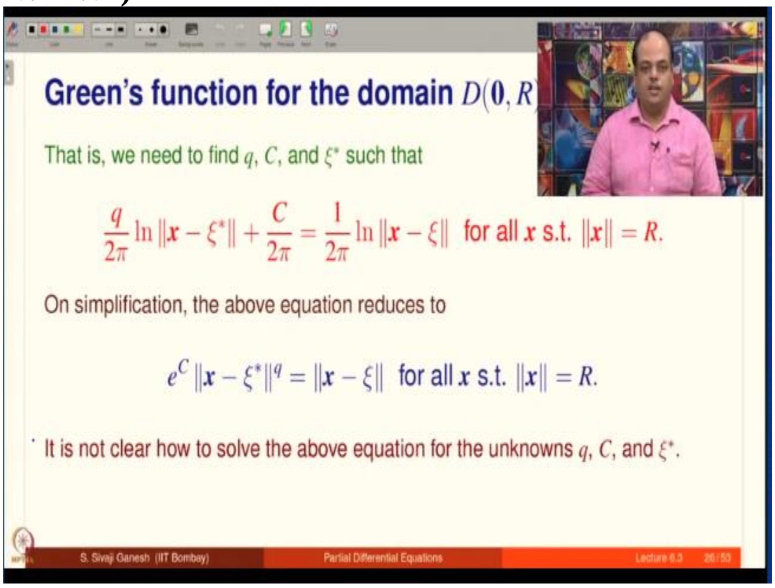
Since norm x - Xi star, you get this expression compute this then you get that multiply norm Xi this side then you have norm x in norm x - Xi star equal to this numerator that as I goes to 0 goes to R square. Therefore, we can define G of x, 0 to be this. Now let us turn our attention to finding Green's function for disk in R 2.

(Refer Slide Time: 13:14)



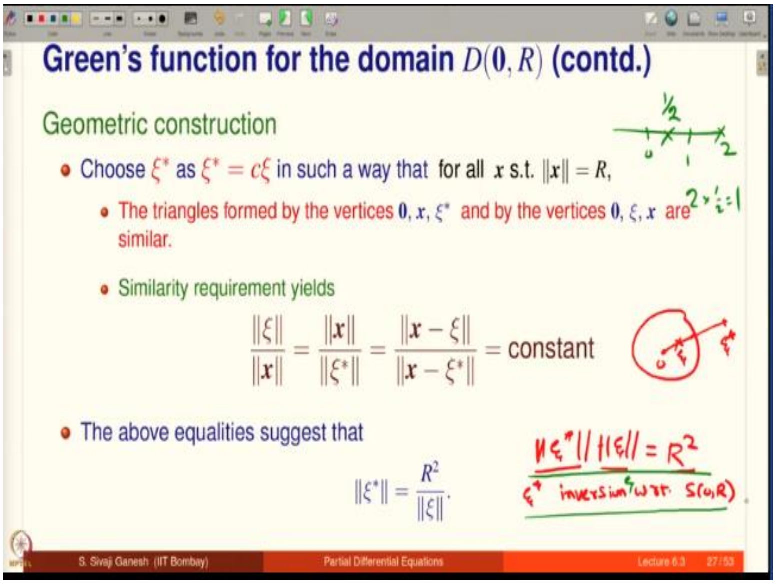
Once again method of images the potential due to a charge q at point Xi star not in omega q / 2 pi log norm x - Xi star. But we take phi of x, Xi = q / 2 pi log norm x - Xi star as above plus a constant which I have written as C / 2 pi for convenience. Without this I have tried but I could not get anything. So, we have to add this constant then life is simpler, this idea is taken from all verse partial differential equations book. So, in order to determine Green's function, we are determined what q Xi star and C such that phi = K on the boundary of the disk.

(Refer Slide Time: 14:02)



So that is we need to find q, C and Xi star such that this equation is satisfied for every x such that norm x = R. On simplification exponentiate both sides you get this equation. So, it is not clear how to solve that equation for the unknown q, C and Xi star. A geometric idea helps here.

(Refer Slide Time: 14:32)



The geometry construction is as follows choose Xi star as Xi star = c Xi that means, it is along the line joining origin and Xi. So, in the direction of Xi in such a way that for all x as the norm x = R. The triangles formed by the vertices 0, x, Xi star and by the vertices 0, x, Xi are similar. Similarity requirements yield these equations, let us draw a circle just for explanation, this is our region let us say this point is Xi.

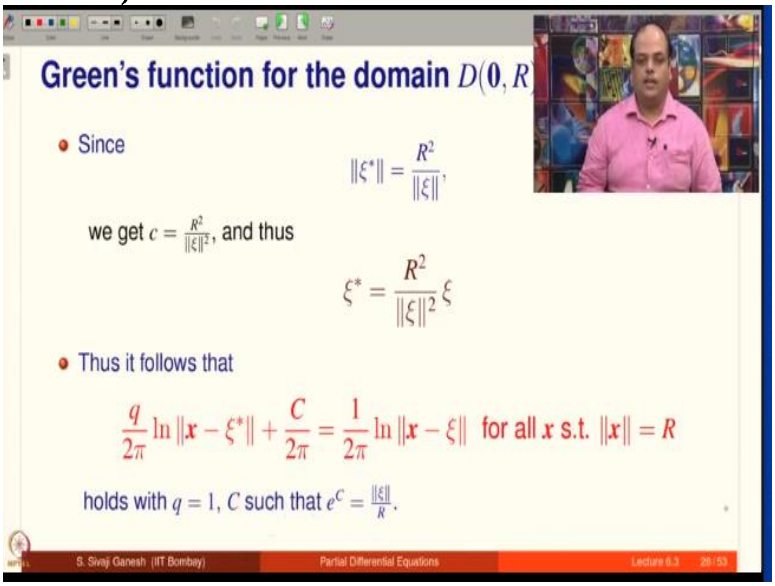
So, now we take a point outside the ball with what property is that you take any point x on the boundary, we have this triangle this now choose Xi star, which is actually C Xi, so, such that this angle equals this angle, note that for both the triangles this and this this angle is common. Now, we have chosen Xi star such that this angle o, Xi, x is another angle o, x, Xi star. Therefore, by similarity what we get is that the sides are proportional. So, whatever side it is opposite to this angle here angle x, o, Xi or x, o, Xi star.

One of them is norm x - Xi, other one is norm x - Xi star that will give us this, what is the side opposite to this angle o, Xi x that is x norm x and what is the side opposite to angle o, x, Xi star that is Xi star, so similarly you get this. So, the above equality suggests that norms Xi star = R square / norm Xi. So, if you observe norm Xi star into norm Xi = R square. R will be chosen Xi star with some property correct. So, Xi is here origin is here Xi star is here.

So, product of the length of Xi star and length of Xi = R square, hence this Xi star is called inversion with respect to the sphere in real numbers, let us look at 0, let us say 1. And I take a point here, it is a half what are the inverse of half multiplicative inverses 2. So, why 2 into 1 /

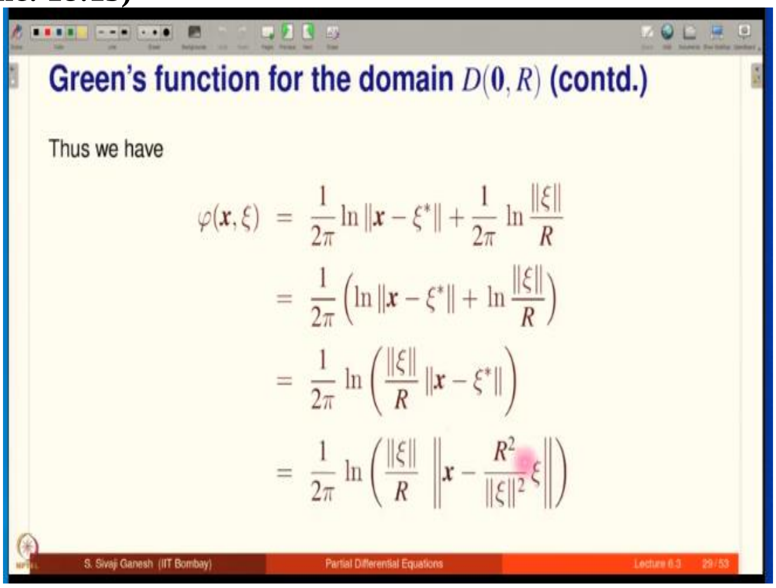
2 = 1. So, this is exactly the same thing here that is what is called Xi star is the inversion of Xi with respect to a S of 0, R.

(Refer Slide Time: 17:44)



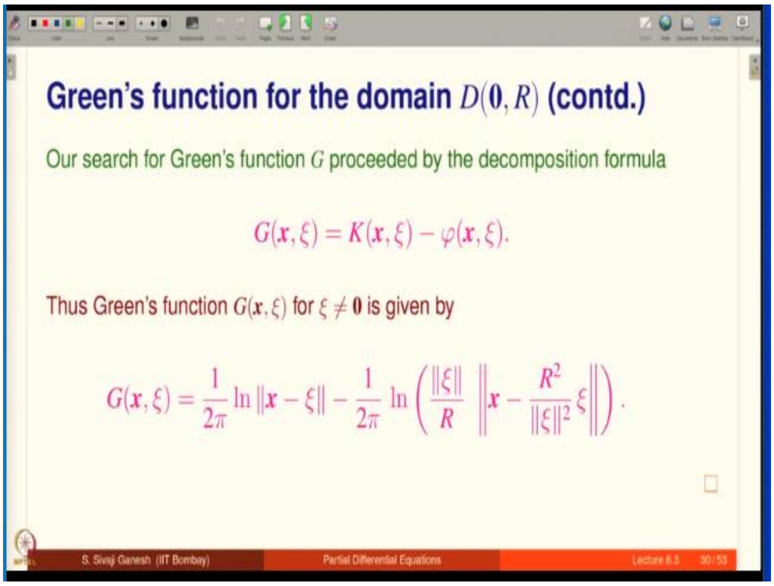
Since norm Xi star = R square / norm Xi we get c = R square / norm Xi square and thus Xi star = R square / norm Xi square into Xi. Thus it follows that this equality for all x such that norm x = R holds with q = 1 and C is such that e power C = norms Xi / R.

(Refer Slide Time: 18:15)



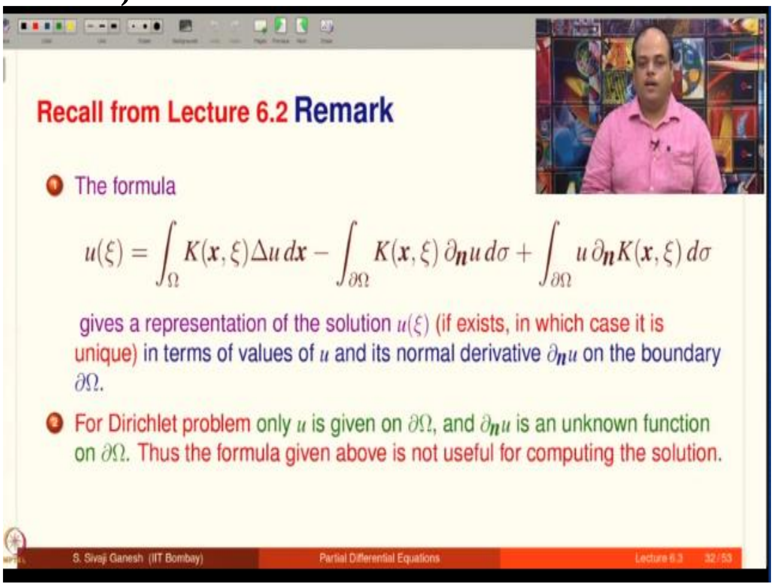
Therefore, we have phi = 1 / 2 pi log norm x - Xi star + 1 / 2 pi log norm Xi / R which if you take 1 / 2 pi common it becomes this now, it is like log A + log B that means log AB. So, you have this and substituting the value of Xi star we get this, so this is phi of x, Xi.

(Refer Slide Time: 18:41)



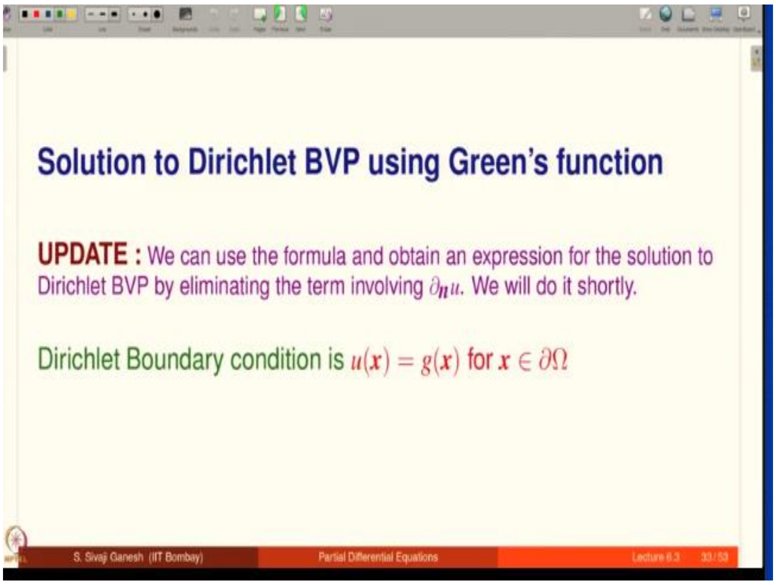
Our search for Green's function proceeded by the decomposition formula G = K – phi now, we know phi also K is already known, therefore G is known. So, G has this expression given in this equation. Now, let us discuss Poisson's formula for the solution to Dirichlet boundary value problem on the ball and also on the disk in R 2.

(Refer Slide Time: 19:06)



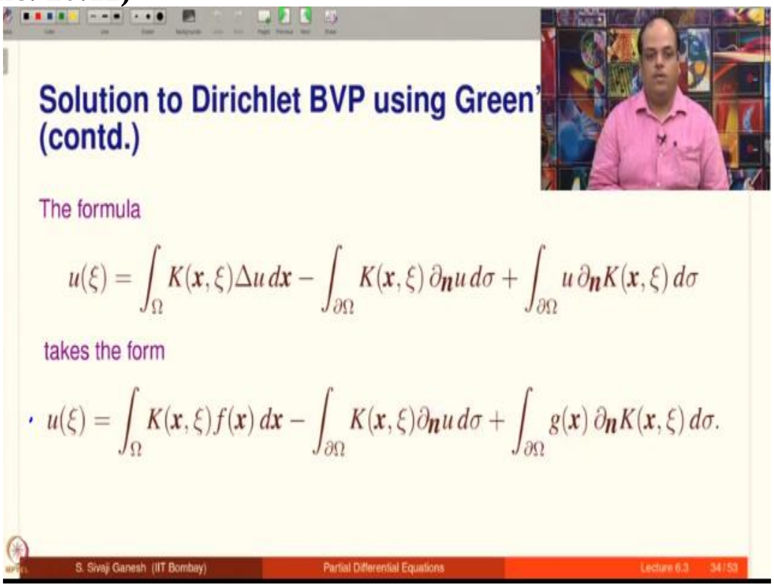
So, recall from lecture 6.2 we made a remark there that this formula which we have proved it gives a representation of the solution u in terms of the Laplacian u and omega that is fine and it is boundary values dou n u and u. For Dirichlet problem only u is given and dou n u unknown. Therefore, the formula given above is not useful for computing the solution.

(Refer Slide Time: 19:49)



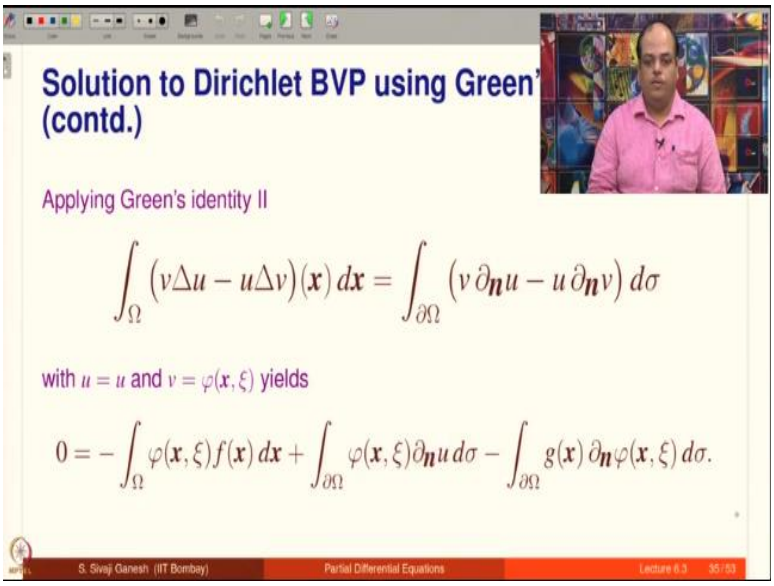
But there is an update now, we can use the formula and obtain an expression for the solution to Dirichlet boundary value problem by eliminating the term involving the normal derivative of u. That is what we are going to do shortly. So, Dirichlet boundary condition is u = g for x belongs to a boundary of omega.

(Refer Slide Time: 20:12)



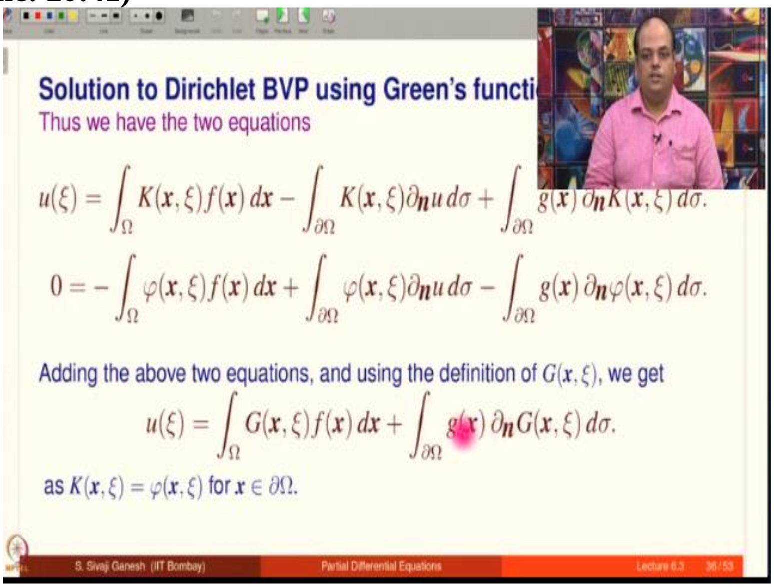
So, this formula takes this form if Laplacian u = f you have this and dou n u unknown, u is known g.

(Refer Slide Time: 20:27)



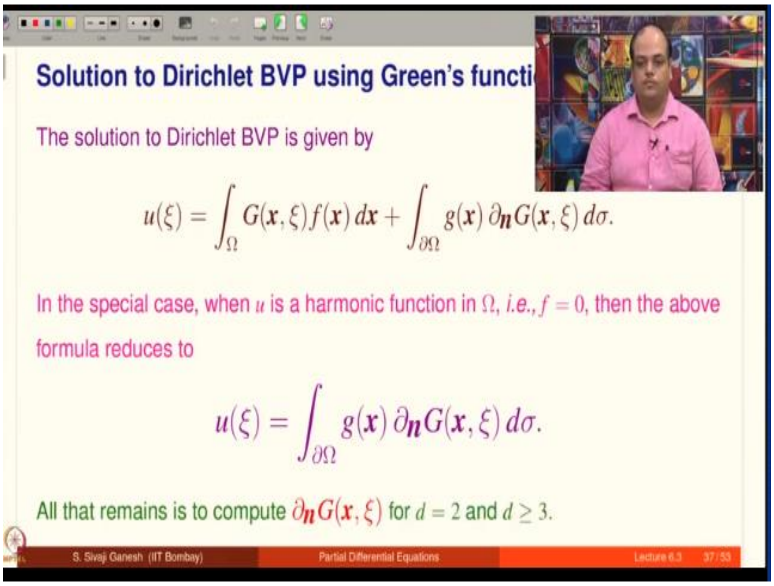
Now, let us apply Green's identity to which is given here with u = u and v = phi of x, Xi that will give us an equation which involves dou n u.

(Refer Slide Time: 20:41)



So that is we have these 2 equations, if you add these to the term involving dou n u gets cancelled, because on the boundary K - phi is 0. Thus, we get a formula for u, which involves only g dou n u is eliminated.

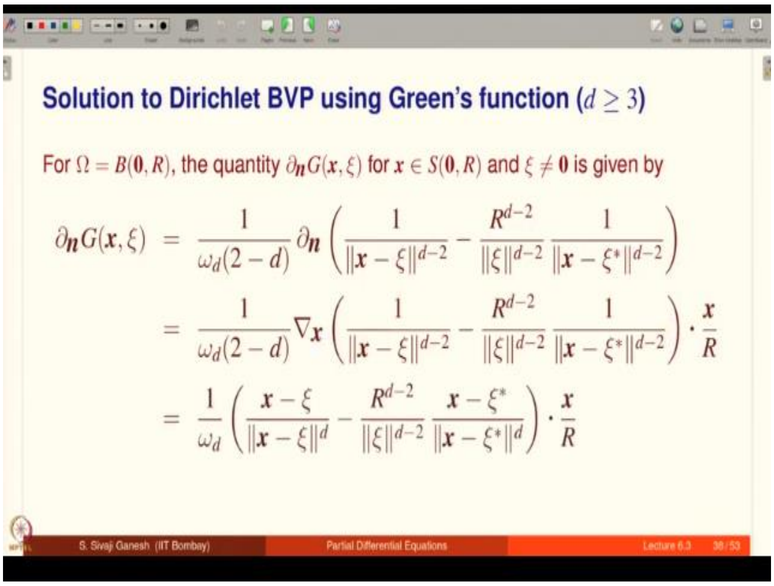
(Refer Slide Time: 21:08)



So, the solution to Dirichlet boundary value problem is given by this and the special case when u is harmonic this term drops out and we have just this term. In fact, we are going to solve harmonic functions such that u = g on the boundary. Therefore, this is what we are going to find a new expression first and dou n G is what all needs to be computed. So, compute dou n G for d greater than equal to 3 n = 2 separately and substitute in this and see what the formula we get. And that formula is called Poisson's formula.

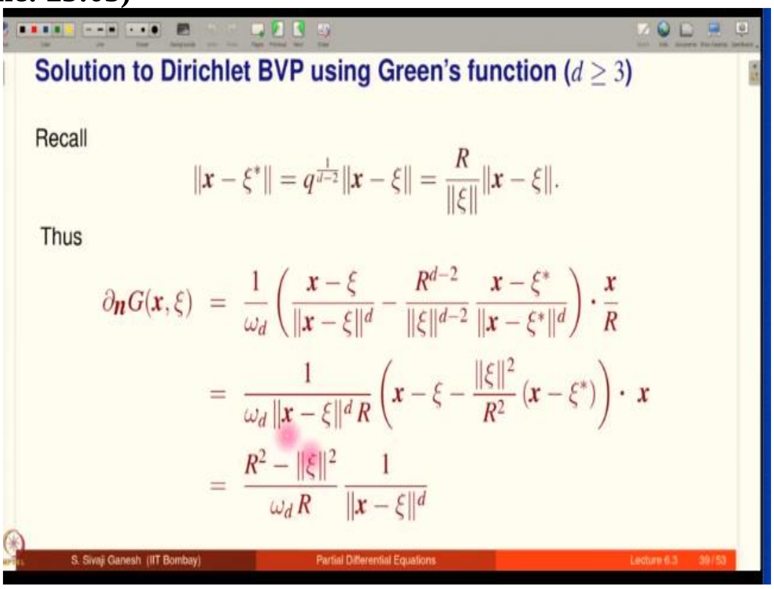
If you are given Laplacian u = f, but f is not equal to 0, this will give you the solution, this will be the representation for the solution, we have to be very careful, this will give a solution means we have to prove something, but what we have done so far is if there is a solution, which is smooth enough, then you have this expression, that is what we are doing. So, we are getting an expression for solution if exists, existence is to be proved. Hopefully, the Poisson's formula will help us improve the existence of solutions to Dirichlet boundary value problem.

(Refer Slide Time: 22:24)



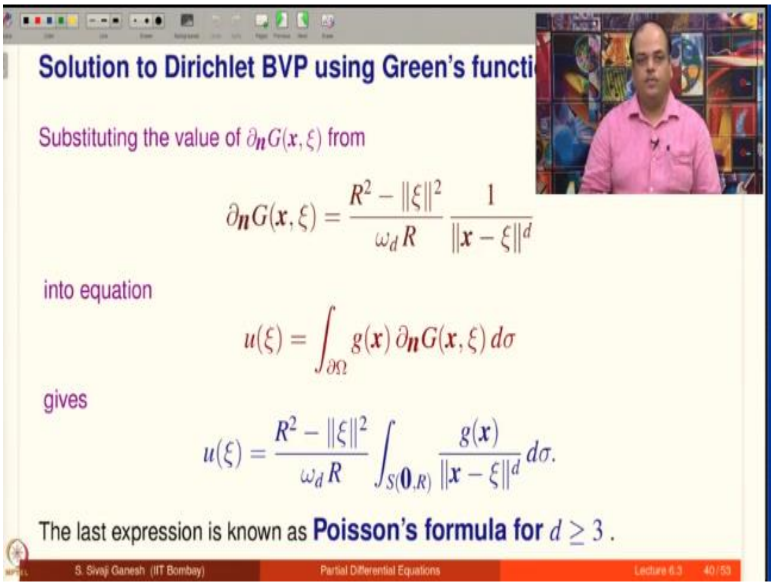
So, for omega equal to ball of radius R center 0, dou n g computation, please do this computation by yourself, I just put the value of g it is a constant so this came out. So, I had to find out the normal derivative of this which is nothing but gradient dot, the normal to the ball is along the radius therefore, x, but if you want to unit normal x / R because norm x = R, this will be unit normal outward normal. So, therefore, dou n is nothing but grad x of this quantity dot x / R. Now, it is a matter of carrying out the differentiations.

(Refer Slide Time: 23:03)



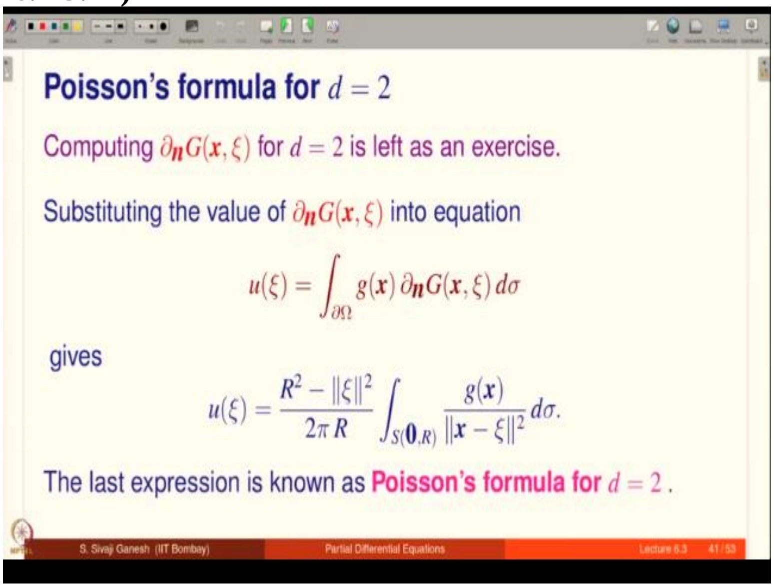
Please do the computations on your own I will skip the details, but I will keep the slide for some time so that you can note down. So, this is the final expression we get for dou n G here, it is simplified very nicely. In fact, even d we are going to see that same formula will come for dou n G.

(Refer Slide Time: 23:29)



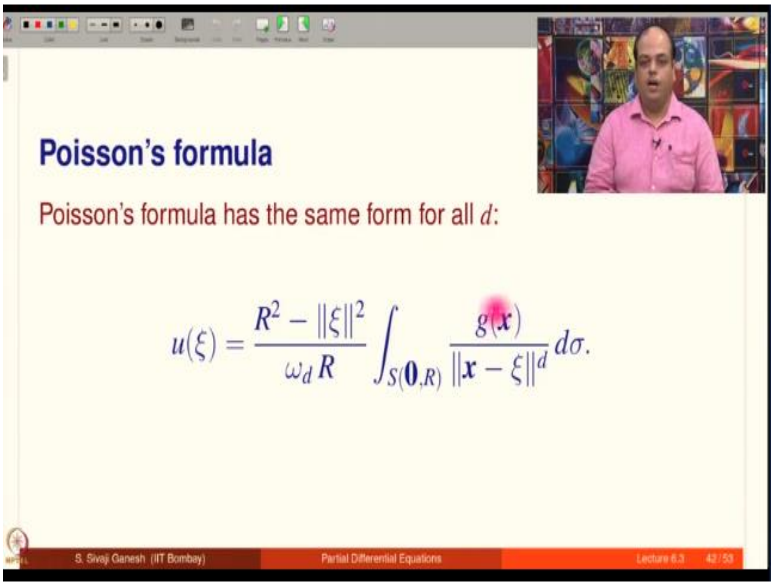
So, substituting the value of dou n G into this equation will give us this formula which is called Poisson's formula of course we have handled d greater than equal to 3.

(Refer Slide Time: 23:41)



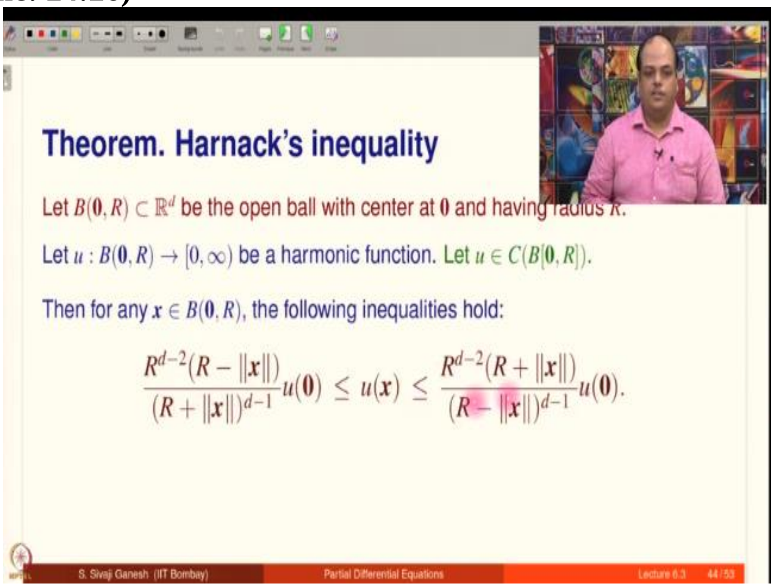
Now computing dou n G is left an exercise when d = 2 and substituting in this formula for dou n G we get this expression look exactly the same formula d = 2 that is it.

(Refer Slide Time: 23:56)



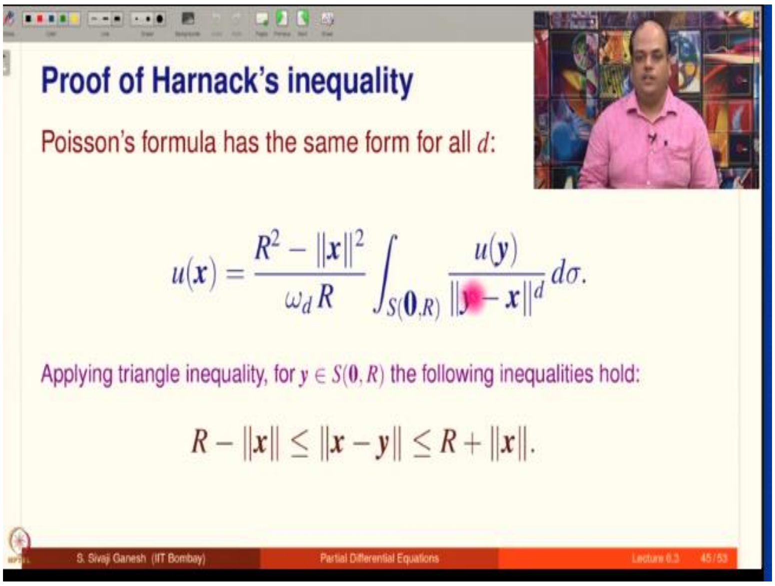
So, Poisson's formula has the same form for all d = 2 which is u Xi = R square – norm Xi square / omega d R integral over the sphere or circle if it is 2 dimensions of g / norm x - Xi power d d sigma, you may also write instead of g u itself after all it is a notation for values of u on the boundary. Now let us look at Harnack's inequality.

(Refer Slide Time: 24:26)



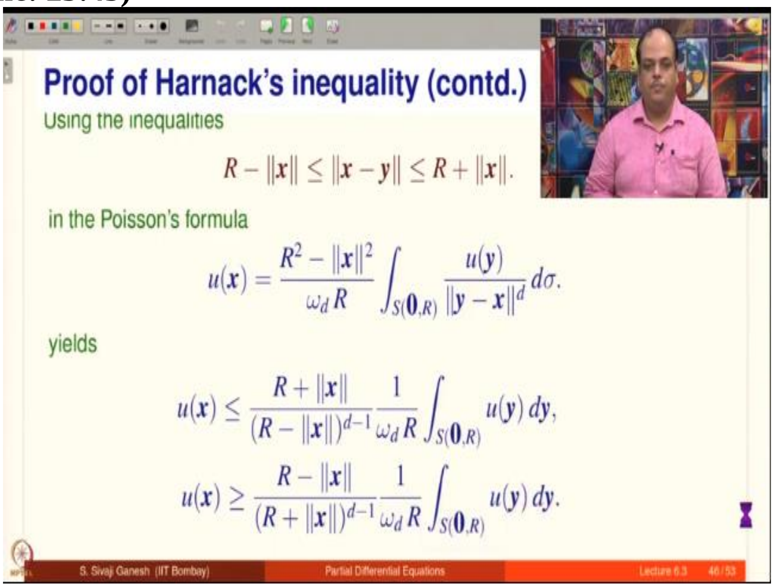
So, let B of 0, R be open ball with center at 0 and having radius R in R d d greater than or equal to 2. And u be a harmonic function which is non negative. It takes values which are greater than or equal to 0 that is very important. And let u be continuous on the closed ball of radius R that means u is continuous up to the boundary. Then for any x in B of 0, R following inequalities hold so the value of u x and the value of u at the center of this ball are tied by these inequalities or they satisfy these inequalities.

(Refer Slide Time: 25:11)



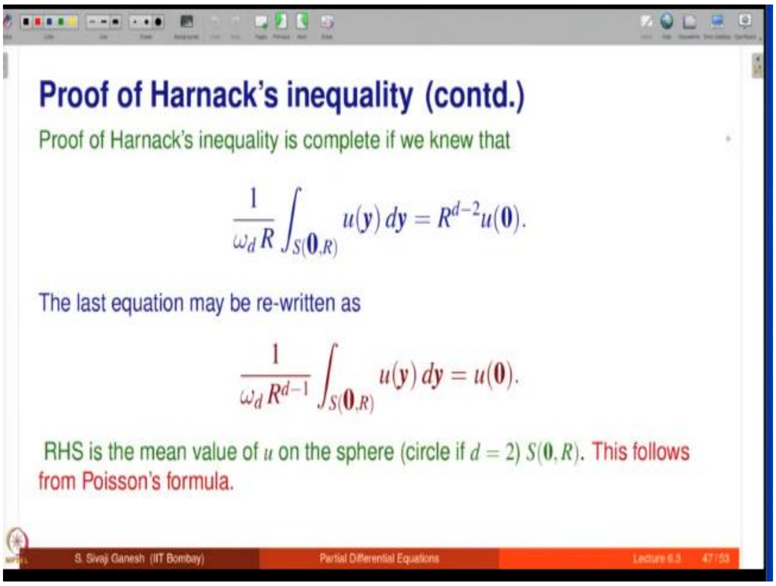
Proof is very simple, it follows just from the Poisson's formula, look at this Poisson's formula and then triangle inequality whenever y is on the sphere of radius R that means norm y is R. So, norm x - y by triangle inequality is less than a norm x + norm y, but norm y is R therefore we have this, here norm x - y is greater than or equal to norm y - norm x, which is a consequence of triangle inequality. Therefore, you get this now use these inequalities in this denominator, you get Harnack’s inequality.

(Refer Slide Time: 25:45)



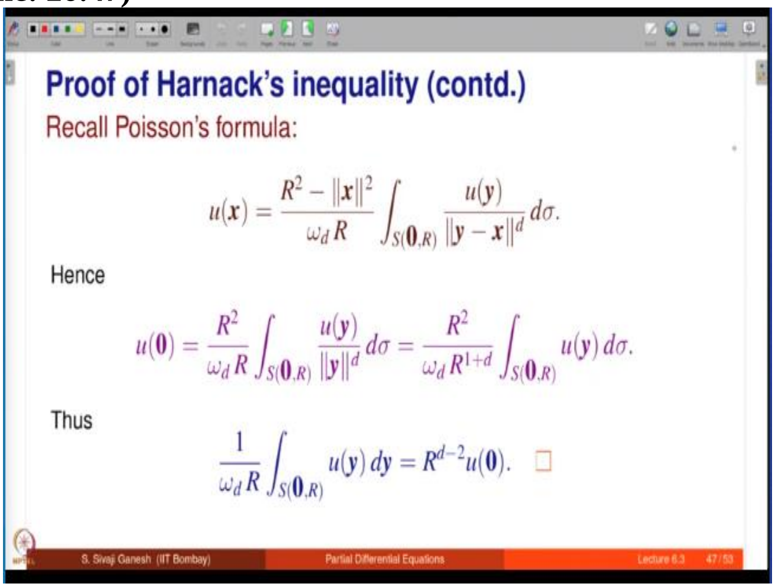
So, this is what we get one, these other side, we get this harnack’s inequality did not have this expression. What was there? Here is R power d - 2 into u of 0. Similarly, here, it was R power d - 2 into u of 0.

(Refer Slide Time: 26:05)



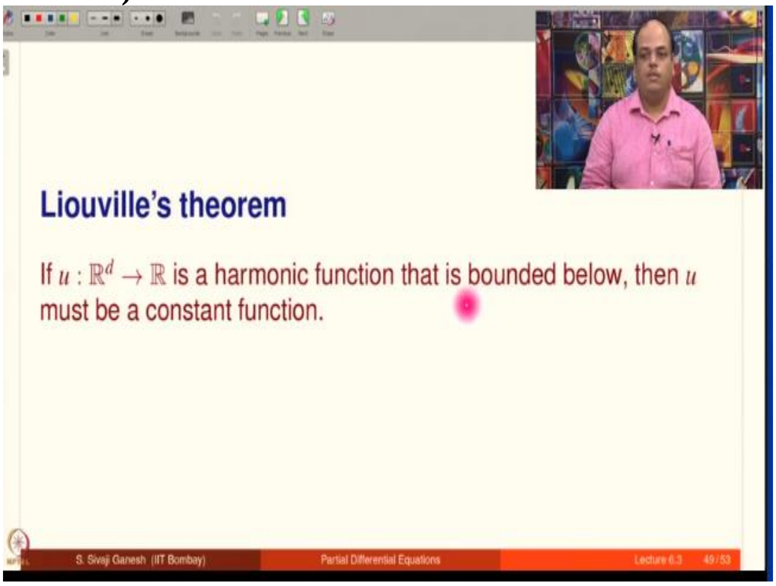
So, proof of harmonics inequalities complete if we knew that this quantity which appeared on the last slide is R power d - 2 into u of 0. The last equation may be rewritten as this, I divide both sides with R power d – 2 so we get this. Now, if you look at what is the left hand side, it is your integrating u on the sphere. And this is the surface area of the sphere you are dividing with that so this is the spherical average spherical mean of u or this sphere. Now, we are asking whether that is equal to u of 0 this follows from Poisson's formula.

(Refer Slide Time: 26:47)



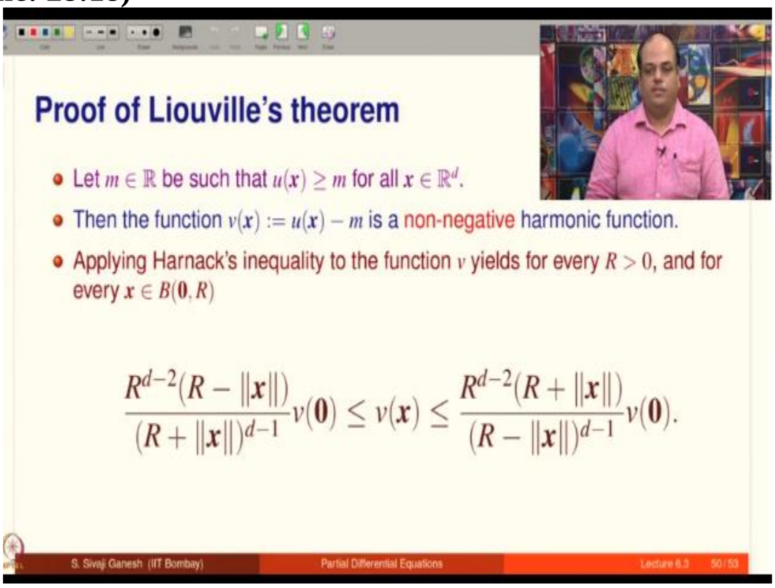
So, recall Poisson's formula u of x = R square - norm x square / omega d times R integral over the sphere of radius R u of y / norm y - x power d d sigma. Therefore, u of 0 = R square / omega d R, because norm x is 0 into the integral on the sphere of radius R uy / norm y power d d sigma because x is 0, but when y belongs to the circle norm y is R therefore, we get R square / omega d R power 1 + d into integral of u on the circle. Thus, we have this equation is exactly what we wanted to show. So, this concludes the proof of Harnack's inequality.

(Refer Slide Time: 27:54)



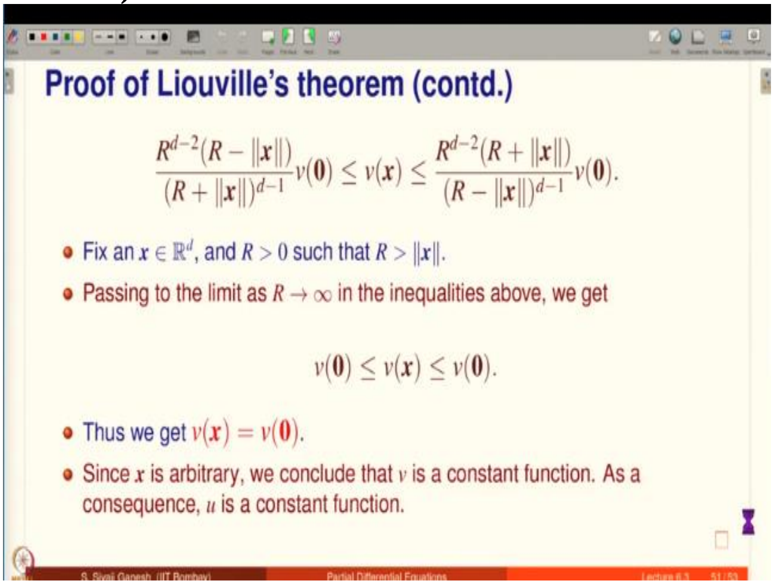
Now, as a consequence of Harnack's inequality, we are going to prove Liouville's theorem, what is Liouville's theorem? If you have a harmonic function on R d which is bounded below then it must be constant it means no non constant harmonic functions can be bounded below. Now, as a corollary we can also get that if your harmonic function is bounded above then also it must be constant.

(Refer Slide Time: 28:18)



So, let m belongs to R be such that u is greater than or equal to m for all x in R d because we are assuming u is a function which is bounded below. Then look at this function ux - m this will be harmonic and it will be non-negative therefore, Harnack's inequalities applicable for this v. Therefore, for every R and for every x in B of 0, R we have these inequalities.

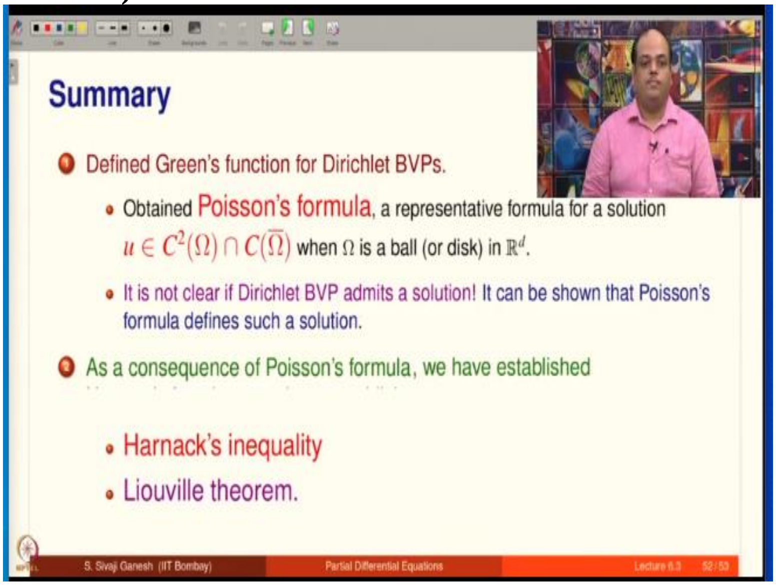
(Refer Slide Time: 28:48)



These are the inequalities coming from harnack's inequality. So, fix an x in R d take R which is positive so that R bigger than norm x and pass to the limit as R goes to infinity because when if you want to pass the limit as R goes to infinity, whatever x you take there will be a time after which your R becomes bigger than norm x anyway so we have this. So, in the inequalities table let us pass to the limit as R goes to infinity, what we get is v of 0 less than or equal vx less than or equal to v of 0.

Rough understanding of this is this is R power d - 2 into 1 here is also the leading term R power d - 1. Therefore, this limit will be 1. Similarly, this limit will be 1 as R goes to infinity. So, we have this now what does this mean vx = v0 since x is arbitrary, what we have shown is that v is a constant function. As a consequence, u is a constant function.

(Refer Slide Time: 29:47)



Let us summarize we have defined Green's function for Dirichlet boundary value problems. We have obtained Poisson's formula which is a representative formula for a solution which is C 2 in omega and continuous up to the boundary of omega. When omega is a ball or a disk in R d. It is not clear if Dirichlet boundary value problem admits a solution. It can be shown that Poisson's formula defines such a solution. As a consequence of Poisson's formula, we have established Harnack’s inequality and Liouville theorem. Thank you.