Partial differential Equations

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Lecture – 5.4
Qualitative Analysis of wave Equation
Uniqueness by Energy Method

In our studies of wave equation, we have solved the Cauchy problem for the wave equation in dimensions 1, 2, 3 and also initial boundary value problem when d is equal to 1. We have proved uniqueness of solutions by at least 2 methods so far. In this lecture, we are going to see a third method which is known as energy method. So, that is the topic of this lecture is uniqueness by energy method.

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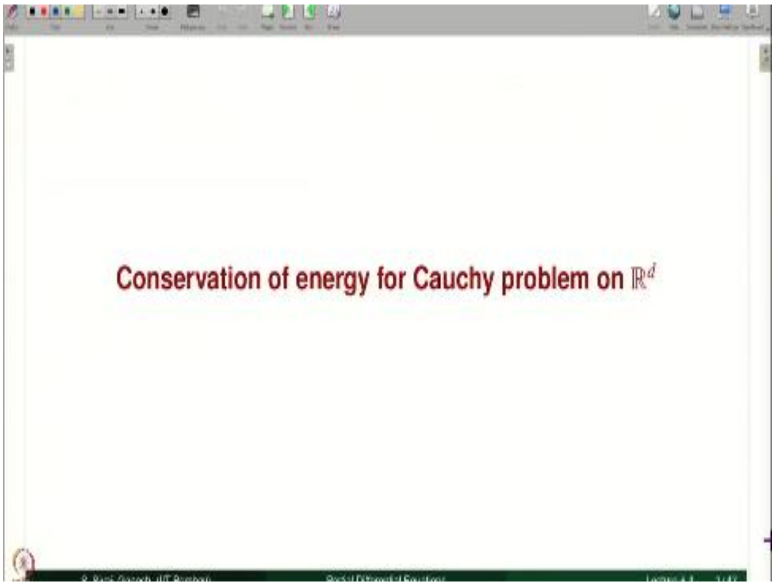

So, first we look at Cauchy problem on R d and we show that energy is conserved. We will define what is energy here.

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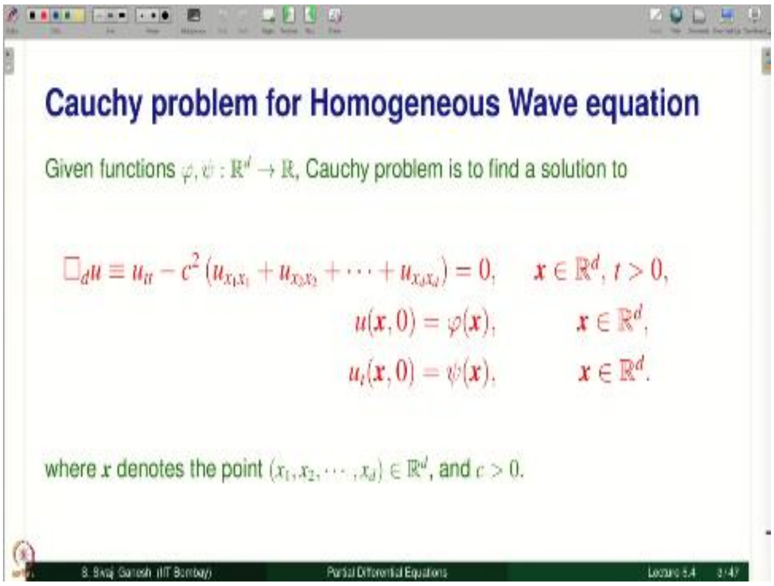

So, Cauchy problem for homogeneous wave equation is; we are in d space dimensions. So, square d'Alembert in d dimensions equal to 0 and this is a Cauchy data phi and psi.

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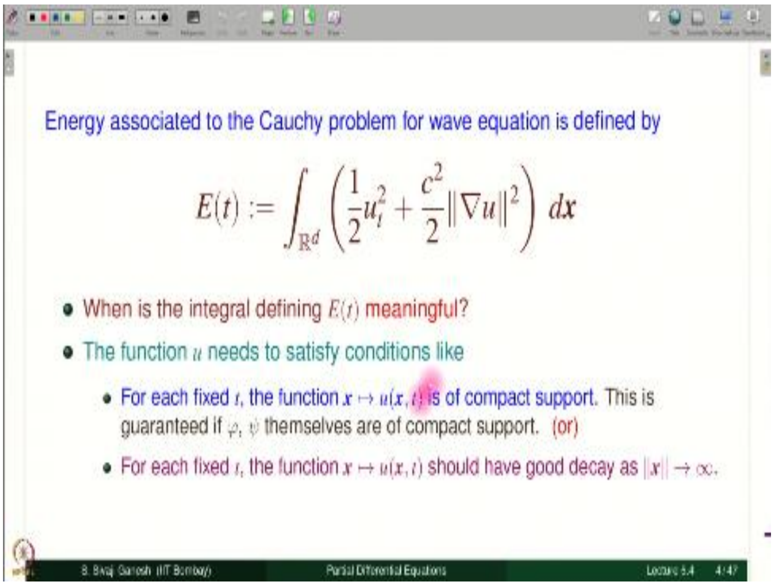

Energy associated to the Cauchy problem for the wave equation is defined by E of t equal to this, where u is the solution to the Cauchy problem. When is this integral meaningful? Because the integral is in R d, we have to ask is this integral meaningful. Is it a finite real number for each fix t? Of course, that would require that u t to be square integrable on R d and non-grad u square equivalently all partial derivatives square of the all partial derivatives must be integrable on R d.

A simple condition which guarantees this is this. For each fix t, you should have compact support the function x going to u of x t is of compact support, then this integral is really on a bounded set on a compact set and these are continuous functions they in fact, C 2 functions; u is a C 2 function, therefore, u t is C 1 and u will also be C 1. Therefore, this integral, there is no problem.

They are definitely continuous functions and we are integrating on a compact set instead of R d because of this hypothesis being satisfied. So, then E of t makes sense. Of course, this is in turn guarantee when the Cauchy data itself is a compact support. We have seen this already. Cauchy data compact support solution to the homogeneous wave equation is also of compact support for each fixed t or we need to assume things like this.

For each fix t, this function should have some good decay properties, so, that these integrals will be finite. We will not elaborate more on this point. We are going to deal with these kinds of assumptions in this lecture.

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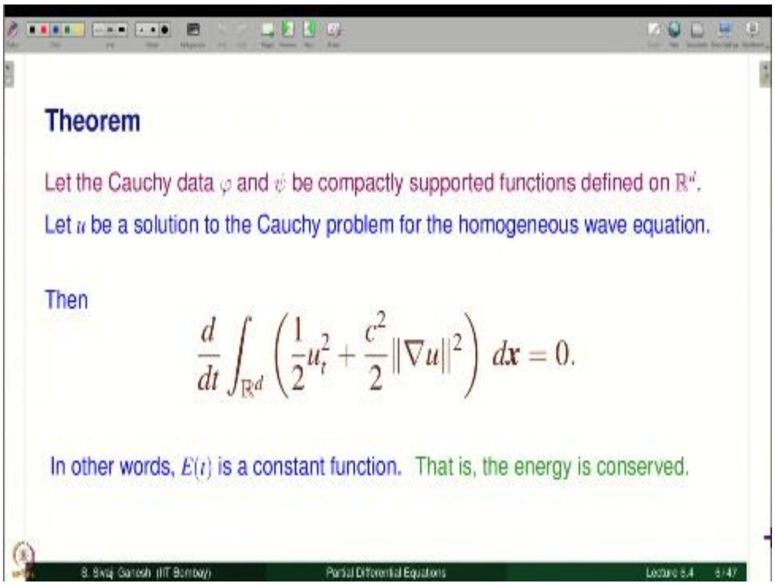

So, let the Cauchy data be compactly supported functions on R d and let u be solution to the Cauchy problem for the homogeneous wave equation. Then the assertion is d by dt of this quantity which we call it energy is 0 that means energy is constant function of t energy depends on t. It is independent of t and hence, it is actually equal to energy at 0. What is the energy at time 0? What is u t at x, 0? It is psi.

And what is grad u? That is simply grad phi when time is 0; u is equal to phi. In other words, E t is a constant function that is the energy is conserved.

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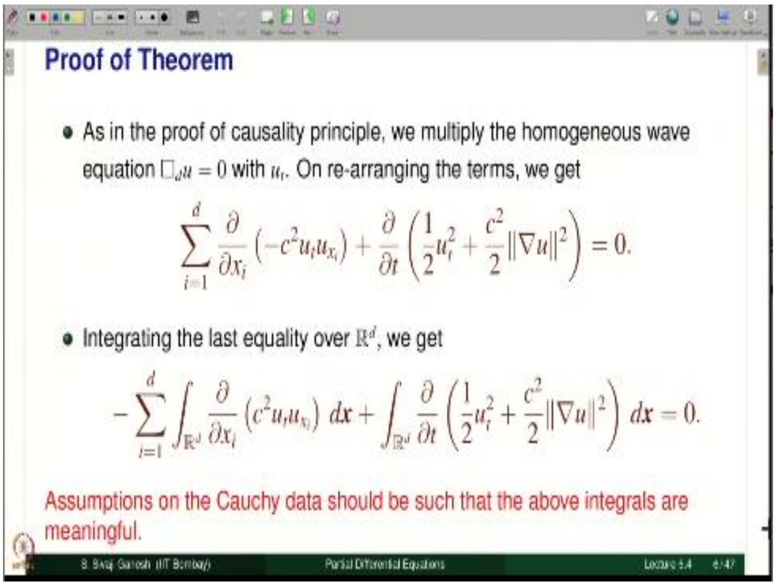

Let us prove this theorem. The proofs are by energy method invariably go through the same first step which is to multiply the given equation with the u t. This was also done exactly in that causality principle proof multiply the equation with u t, then we rearrange, you get this way. Now, we are planning to integrate this on R d. So, exactly the same equation on R d. Now, usually this term is actually energy d by dt of the energy.

This term, so everything depends on this term now. How the energy behaves? Here, this is dou by dou x i is there. So, we plan to do integration by parts in this term and then conclude things about the energy. As mentioned before, assumptions on the Cauchy data should be such that these integrals are meaningful.

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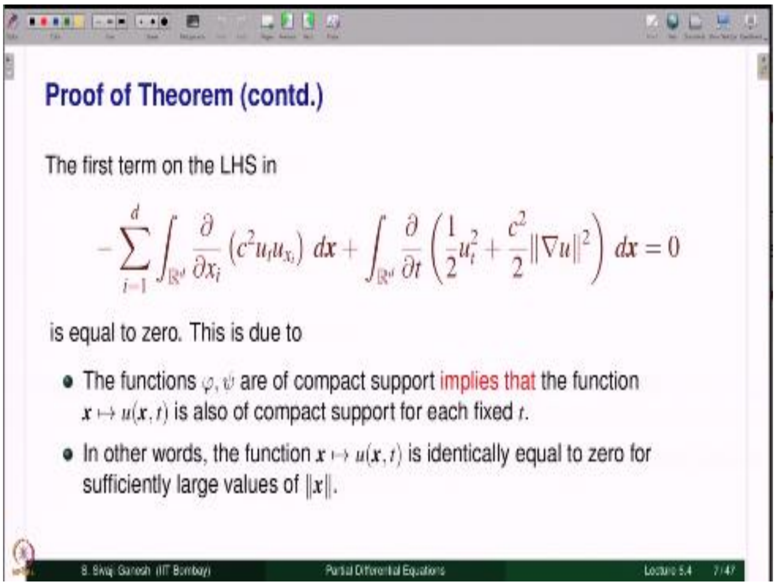

The first time on the LHS in this equation, namely this term is 0. This is because the functions phi and psi have compact support that implies that the function x going to u of x t is also of compact support for each fixed t. In other words, the function x going to u of x t is identically equal to 0 for sufficiently large values of norm x.

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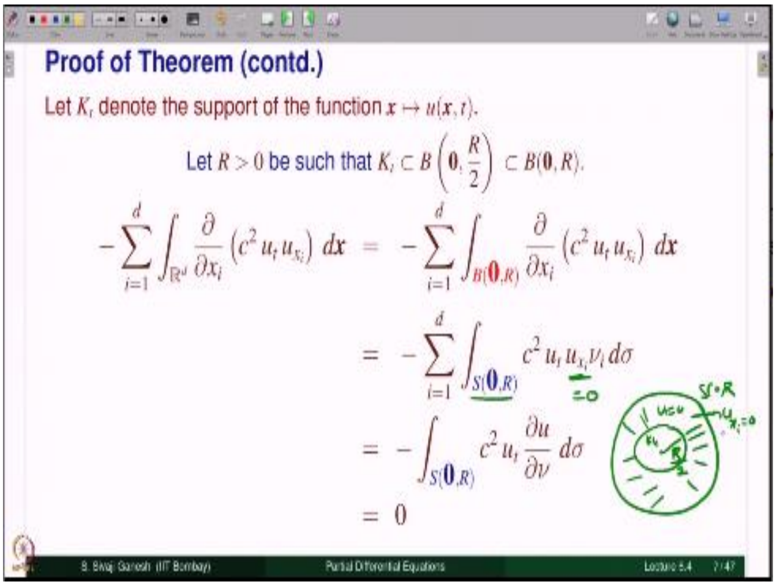

Let K t denote the support or the function x going into u of x t. So, K t suggests it will depend on T. Of course, this function also depends on t. For each fixed t, K t denotes the support of this function. Let R positive be such that k t is contained in this ball of radius R by 2 with centre at the origin because it is a compact set. We can always find such an R. Of course, ball of radius R by 2 is contained ball of radius R having the same centre.

Now, the first term that we want to show is equal to 0, look at this integral; it is an R d because the support of x going to u x t is compact, this integral is really on K t or let us say integral on this ball or maybe on this ball. So, for convenience, we write it as the integral under bigger ball. Now, we are going to do integration by parts. Integration by parts in this integral will give you 1 domain integral and 1 boundary integral.

The domain integral here will be 0 because it is dou by dou x of c squared u t u x i into 1. So, when the dou by dou x shifts to 1, it is 0. So, what we are left with will only be the surface integral or the boundary integral. So, you have this integrand because dou by dou x I, you get the ith component of an outward normal neu i d sigma. Now, if you see a summation over i = 1 to d of u x i new i is nothing but the normal derivative dou by dou neu and this is equal to 0.

Why is that? In fact, we need not write this step. I am writing this because for a future user, we are going to use it later on in this lecture. Why is this 0? Because u x i is 0. Why is u x i 0? Because see, u is supported in K t and K t is contained in the ball of radius R by 2 and this integral that we have is on the sphere s of 0, R in particular u is 0 in this annular region. Therefore, all the derivatives will be 0 here.

Here, u x i will be 0 and hence, this term will be 0 because u x i on the sphere is 0 and hence, we have this equal.

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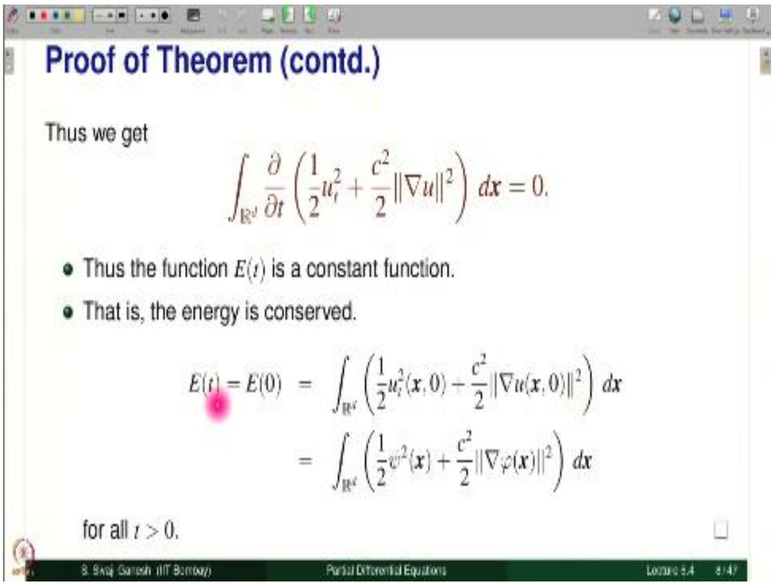

So, thus we get the second term which remains in the equation equals 0, but second term is nothing but d by dt of E t. So, d by dt of E t = 0 that means E is a constant function that means, energy is conserved. So, E of t = E of 0; but, what is E of 0? It is this expression. I have to put t equal to 0 in the definition of E of t. But, I know what these are that is u t is psi and u of 0 is phi x therefore, grad u will be grant phi. So, therefore, energy for all times positive is actually equal to the energy when t equals 0 which is given by Cauchy data.

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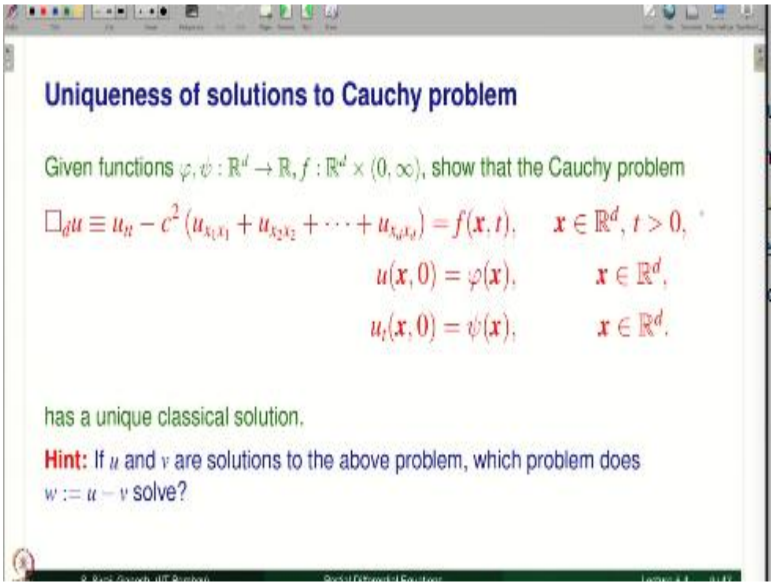

So, now, uniqueness of solutions to Cauchy problem. So, let us consider a non-homogeneous equation with the Cauchy data. We want to show it as a unique classical solution. So, what is the general strategy for showing any problem has a unique solution is let u and v are solutions, consider the difference and show that the difference is 0. So, we would like to see what is the equation that u – v solves.

u satisfies this equation; v satisfies exactly the same equation; instead of u, I have a v here equal to f, v x 0 is phi x, v t x 0 is psi x. When I subtract, the operator is linear here, the delamination operator is linear, these conditions are also linear in u and u t. So, w will satisfy homogeneous wave equation with 0 Cauchy data.

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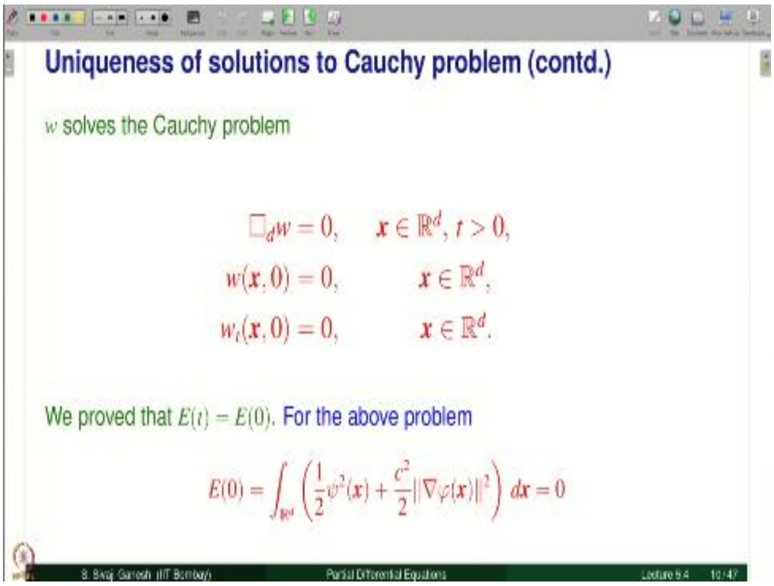

But, we prove the u of t equal to u of 0 for homogeneous wave equation. Therefore, we have to see what is E of 0 here. E of 0 is given in terms of psi and phi which is 0. Therefore, for this problem, E of 0 is 0.

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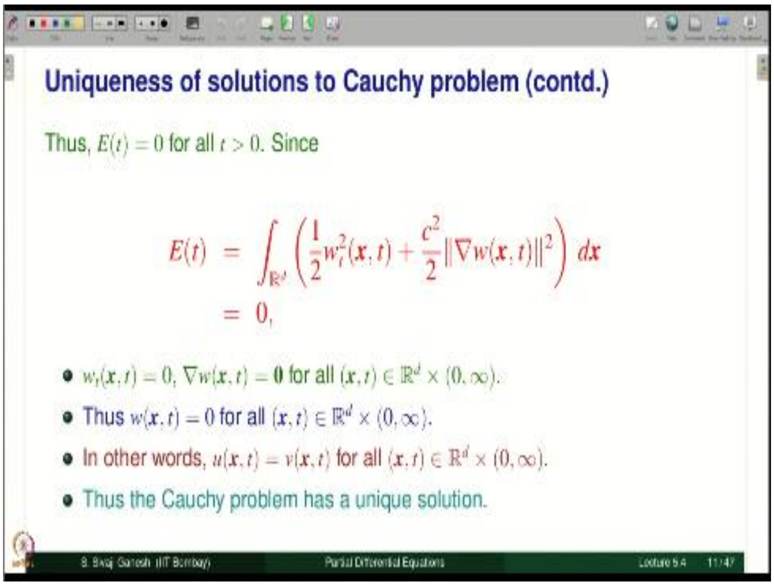

And therefore, E of t is 0 for all t by what is E of T? It is this expression; this expression equal to 0, integrand is always non-negative, it is sum of 2 non-negative quantities. So, that is 0 if and only if each of the terms is 0 that means w t is 0 and grad w is 0; this implies w is a constant function. And w is a constant function and it should be 0 function because it is 0 at time t = 0. So, we are assuming the solutions to be classical solutions.

Therefore, w of x t equal to 0 for all x t in R d cross 0 infinity. In other words, u of x t equal to v of x t that means, we are shown uniqueness to the Cauchy problem.

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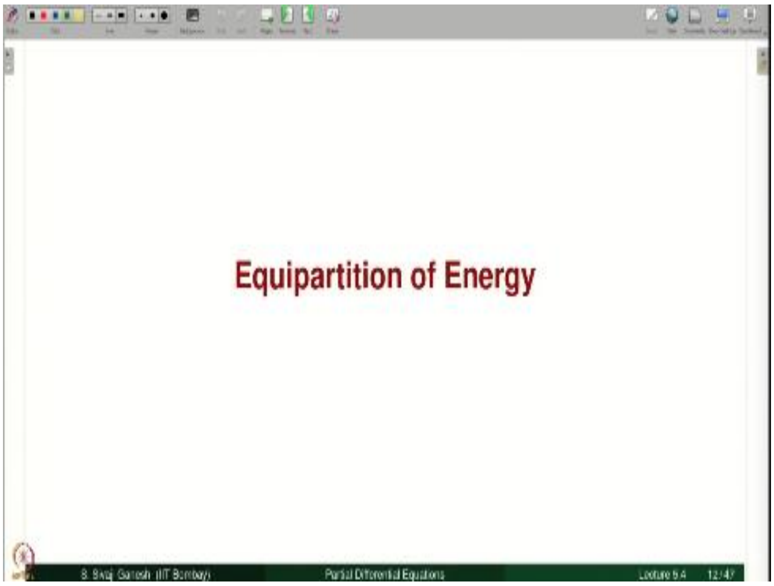

Let us look at the second problem which is called equipartition of energy. It means energy is partitioned into 2 equal parts. Let us see what are the 2 parts.

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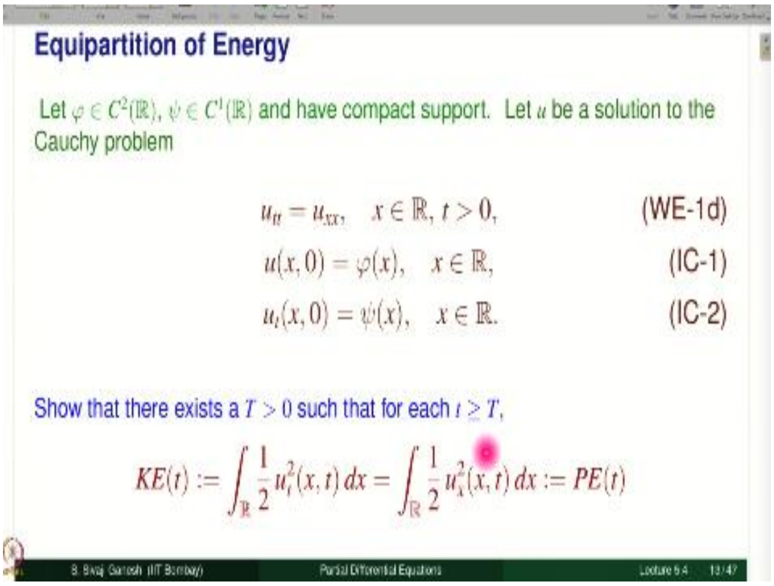

Like phi and psi have a compact support with the usual regularity assumptions and let u be a solution to the homogeneous wave equation with phi and psi as the Cauchy data. show that there exists a T, there is a time t such that for all times after that t greater than or equal to T, we have the kinetic energy which is given by half u t square equal to potential energy which is half u x square.

Energy is sum of kinetic energy and potential energy and what we are showing here is that there is a time T after which, kinetic energy equals potential energy that means, energy is equally partitioned equipartition of energy. How do we show this? We will use. We know the formula for the solution. We compute u t; we compute u x, substitute here and see what we get.

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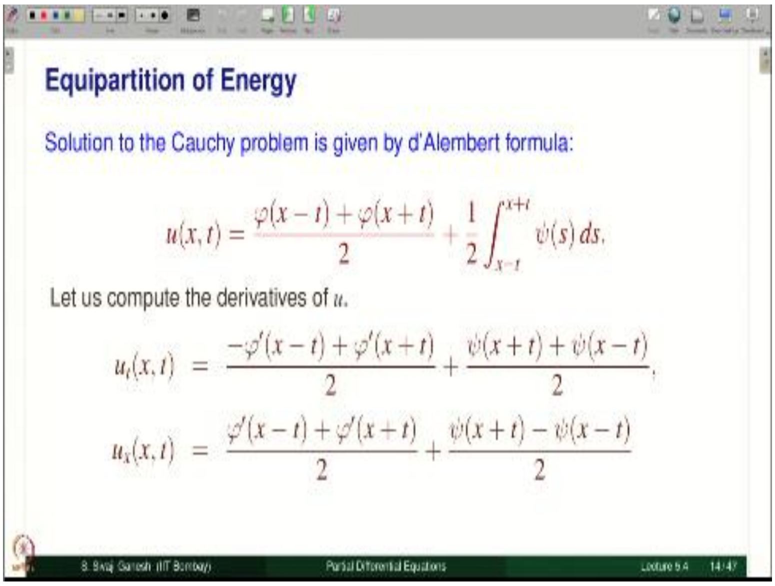

So, this is the formula called a d'Alembert formula which gives solution to the Cauchy problem for the homogeneous wave equation. Let us come to the derivatives of u. Let us compute u t. First, I had to differentiate phi and then differentiate x – t with respect to T which will give me a minus sign and again differentiate phi, differentiate t with respect to T therefore, you have 1 + 1 by 2.

Now, here we need to differentiate, here x and t both are in the limits of integration. So, we have to use what is called a Leibniz rule for differentiation of integrals. So, this is as we discussed earlier, it is a combination of fundamental theorem of calculus and chain rule. So, psi of x + t into derivative of this with respect to T which is 1 minus psi at this point x – t into derivative of this quantity x – t with respect to T which is – 1.

Therefore, we get a plus here and this quantity is u t. Similarly, you can compute u x. Here, you see that phi prime plus psi x + t is there, here phi prime plus psi x + t. So, let us separate those term, rearrange.

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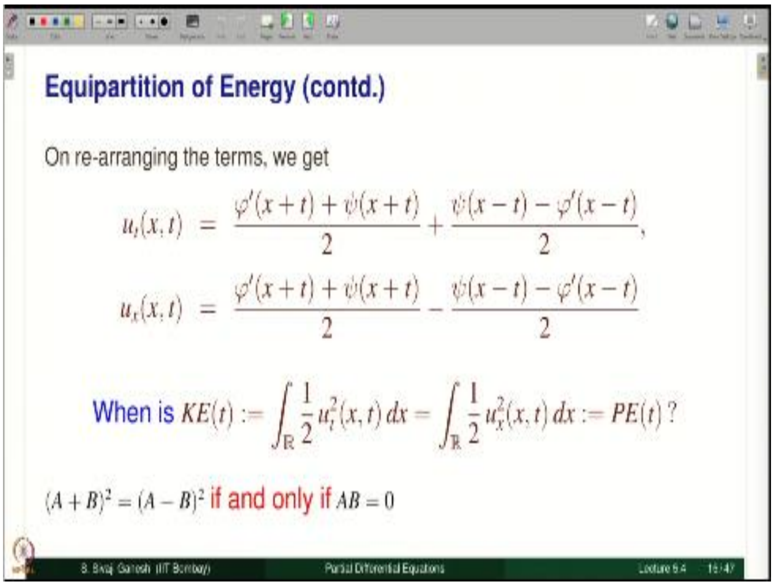

When we rearrange u t has this expression and u x has this separate this expression. If you notice, the first 2 terms in both expressions are the same; second 2 terms, this second term and this second term are also same, but for sign. So, it looks like a plus b. This looks like A – B. It is interesting to keep this analogy in mind, because now we are going to ask when is kinetic energy equal to potential energy.

Half is always equal half, you cancel half; u t square and u x square, u t square as I mentioned a + b whole square A – B whole square. So, A + B whole square equals A – B whole square if and only if AB 0. Therefore, when you substitute u t inside this and u x inside this, the A square term, B square term will get cancelled because it is the same on both sides. Here A square is this on the left side.

On the right side in the expansion for u x square, you get A square here. So, they get cancelled because they are the same. Similarly B is same. So, B square gets cancelled. So, what remains is integral product of this quantity into this quantity equal to 0.

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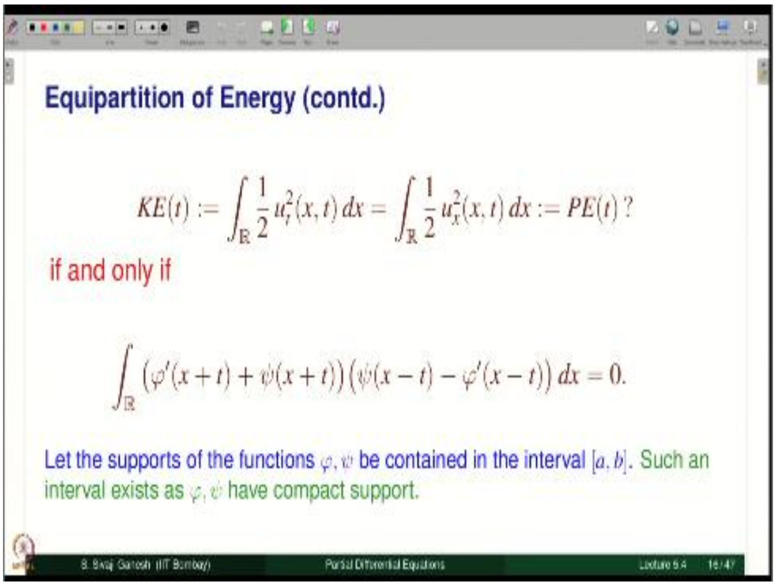

Please do this computation by yourself. Pause the slide here and then do the computation. Now, we have assumed phi and psi compact supports. So, let them be contained in an interval a b. In a compact set, you can always put it inside some interval a b closed and bounded interval. So, let a b be such that support of phi and psi are contained in a b. It means outside a b, phi and psi are 0.

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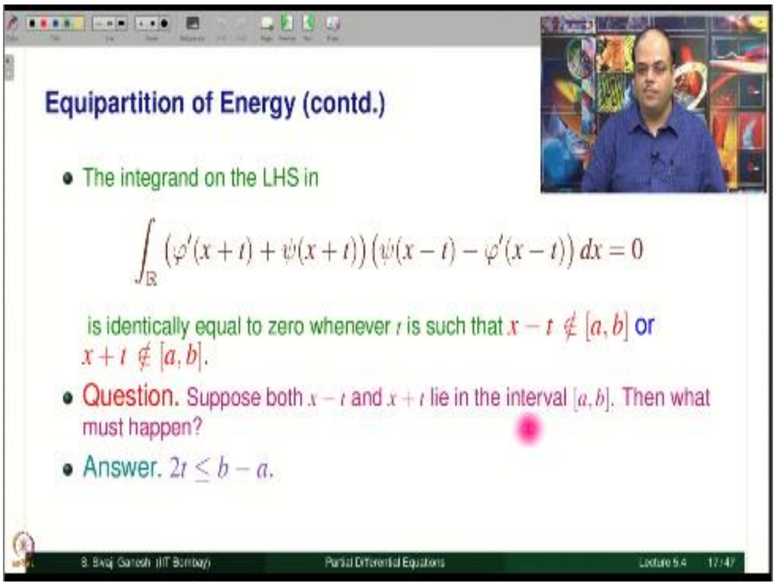

Now, the integrand on the LHS is identical equals 0, because, see, one way of assuring that integral is 0 is to assure the integrand is 0. So, that is why we are interested in making the integrand equal to 0. Whenever t such that either x – t is not in a b or x + t is not in a b. Imagine x – t is not in a b, what will happen? x – t is not in the a b, therefore, psi will be 0; phi dash will also be 0.

Therefore, this term is 0. Therefore, this integral is 0. Similarly, if x + t is not in a b, this first quantity, quantity in the first brackets is 0. It does not matter what this is. Product will be 0 and hence, integral will be 0. So, therefore, the integrand of this integral and hence, the integral itself is 0 whenever t such that x – t or x + t, one of them at least is outside the interval a b. So, therefore, we ask the opposite question.

Suppose, both of them are there in this interval a b, then what can we say? x – t, x + t interval is lying in interval a b. So, we should be able to say something. a here, b here, x – t, x + t, length of this interval is 2t. Length of the interval a b is b – a, therefore, what should happen is; 2t should be less than or equal to b – a. Suppose, t such that 2t is bigger than b – a, what does it mean?

Both x – t and x + t cannot lie in the interval a b that means, at least one of them is outside interval a b and hence, we have this integral equals 0 as a consequence KE = P. Now, we asked the question.

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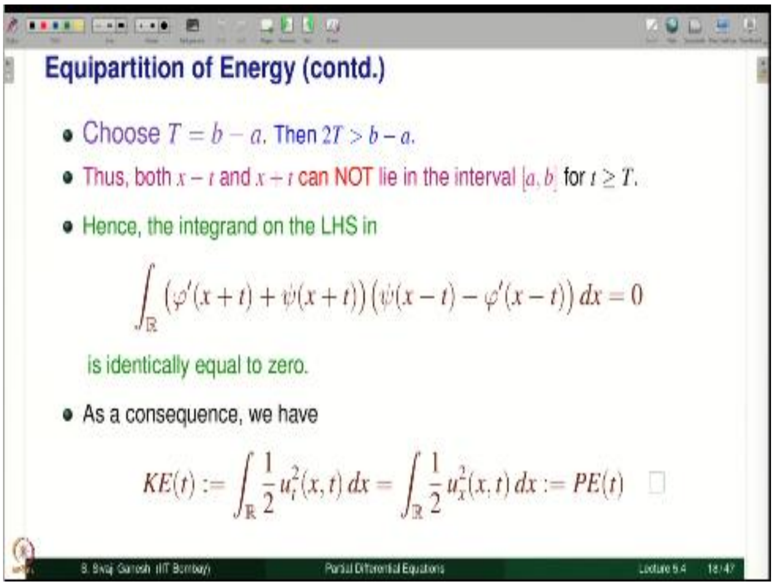

So, therefore, we choose t = b – a, then 2t will be bigger than b – a, therefore, x – t and x + t both of them cannot lie simultaneously in the interval a b for any t bigger than or equal to T. If one lies, other cannot lie, because both of them lie it means 2t is less than or equal to b – a, but I have chosen here 2 T is bigger than b – a therefore, 2 t, 2 times small t is greater than or equal to 2 times T and that is bigger than b – a.

So, both of them cannot lie in interval a b whenever t is bigger than equal to T and hence, what we wanted the integrand is identical equal to 0 and hence integral is 0. Therefore, we have KE = PE for all t bigger than or equal to T.

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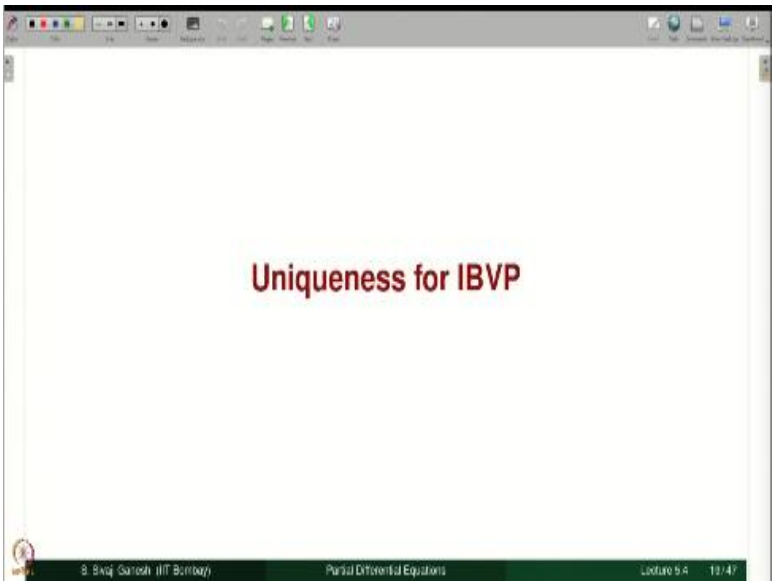

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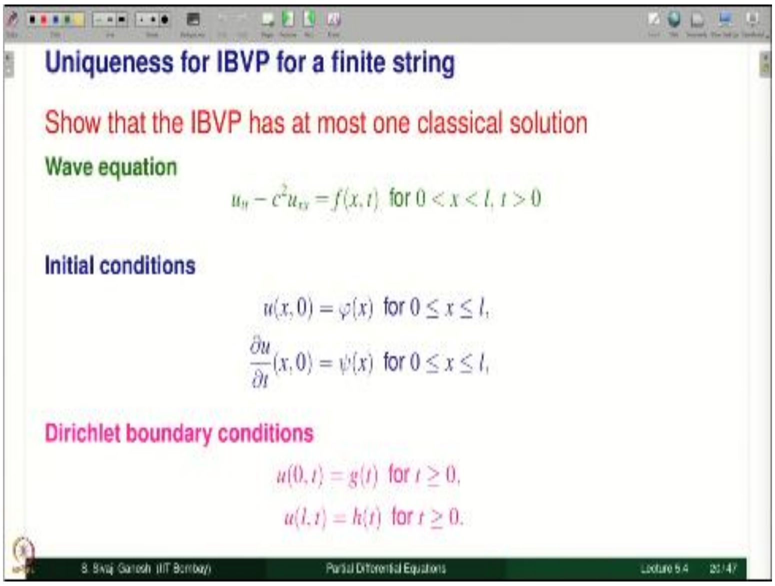

Now, let us look at another proof of uniqueness for IBVP. Show that IBVP with the non-homogeneous wave equation, non-homogeneous Cauchy data, nonzero Cauchy data or nonzero boundary conditions Dirichlet boundary conditions, this has at most one classical solution. Strategy is same.

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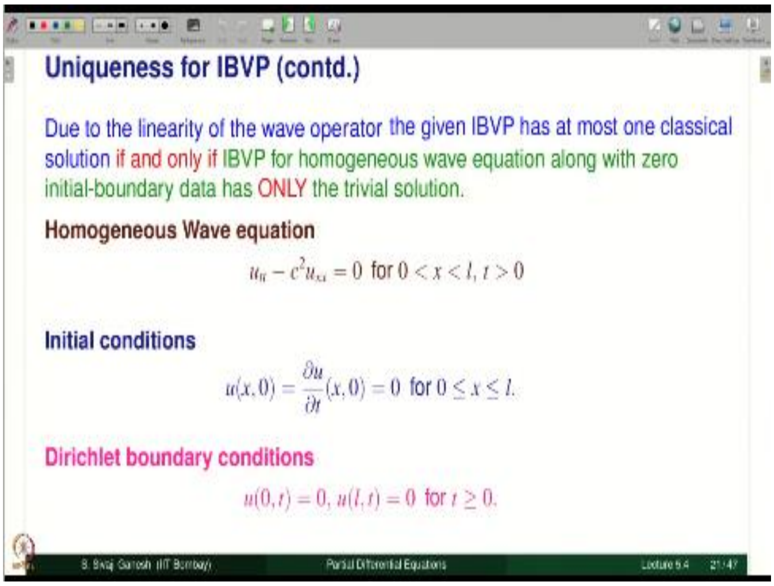

Let u and v be solutions, take the difference and find out the equation satisfied by the difference. Therefore, due to the linearity of the operator and the conditions that we have u, u t and u and u, therefore, the given problem has unique solution or at most one classical solution, you have to be very careful. Here, we are saying at most one classical solution. We are not saying it has a solution that should be proved separately.

Of course, we have proved it. So, it has at most one classical solution if and only if this problem with homogeneous wave equation and 0 Cauchy data, 0 boundary conditions has only trivial solution. Trivial solution means 0 solution. Of course, we know that 0, u = 0 is a solution to this that we know but what we are to show is that is the only solution. Then uniqueness for the non-homogeneous initial boundary problem follow-ups.

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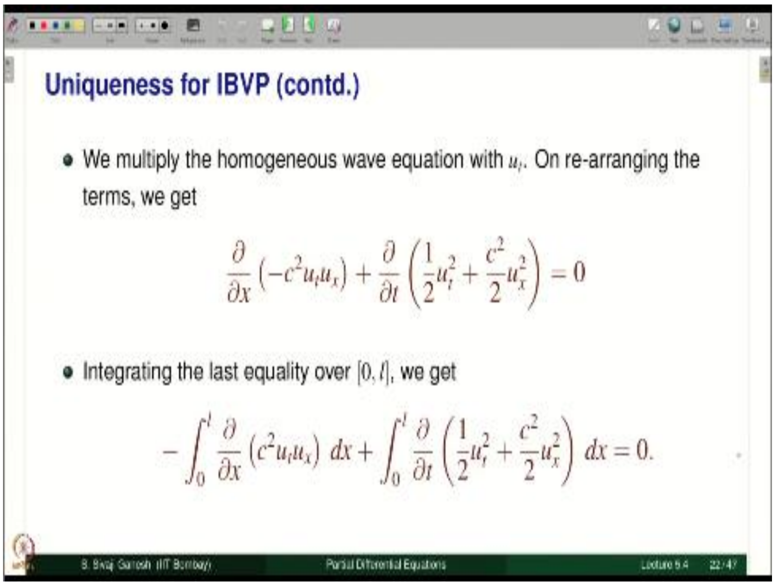

As I mentioned earlier, energy method procedures like this by multiplying the given equation, the suitable multiplayer in the context of wave equation, it is u t. So, I am rearranging the terms as before we get this. So, integrating the last equality was 0 L, we get this quantity.

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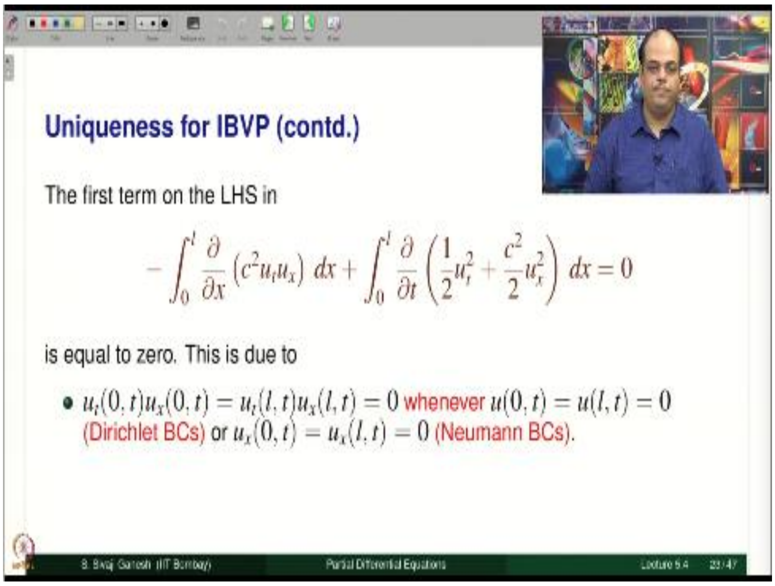

Now, the first term is equal to 0. That is because u t of 0 t into u x of 0 t that is what will come once you do integration by parts in one variable, we do not call it; it is simply fundamental theorem of calculus so, maybe the second form. So, whenever you have derivative with respect to x and dx integrand evaluated the upper limit minus integrand evaluated the lower limit that is what will be the answer of this integral.

Of course, there is a minus sign that we will take care later. So, u t u x values are at l and at 0 that is what we have written here, u t of 0 t, u x of 0 t, both of them are 0, because we know that u of 0 t and u of l t are 0 because of the Dirichlet boundary conditions. Therefore, u t of 0 t is 0 and u t of l t is 0. Therefore, this product is 0. Therefore, this term will reduce to 0 or in fact, in the same proof, you can observe that this will be 0.

Then this term will not be there, this term will be 0, but you do not require that further for this thing to happen, you do not require this Dirichlet boundary conditions. Even if you have Neumann boundary conditions, you have u x 0 t and u x l t is given to be 0. Therefore, this product is 0 even in that case. It means we are now actually showing the uniqueness of solutions even to the Neumann boundary value problem.

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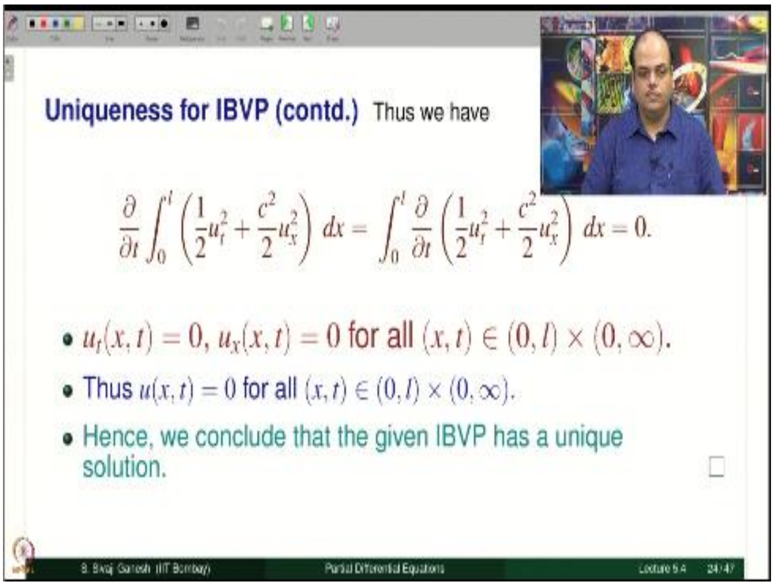

So, what we have is dou by dou t of this equal to this and we have shown that equal to 0 and as before now, this is dou by dou t of this quantity is 0, therefore, this quantity is constant and this quantity is constant and that constant has to be 0, because when time t = 0, the initial energy is 0 because we are working with 0 Cauchy data. Therefore, this will be 0. As a consequence, u t of x t is 0; u x of x t is 0 for all x and t and hence, u is identically equal to 0.

It means u is constant under constant has to be 0 because u is already 0 on at t = 0, as well as some the boundaries if you are dealing with Dirichlet a problem. So, u of x t = 0. Hence, we conclude that the given IBVP has a unique solution.

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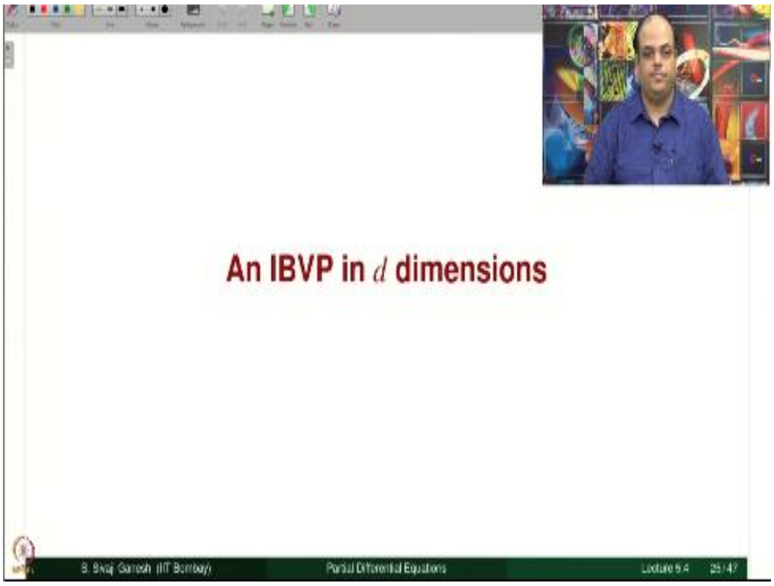

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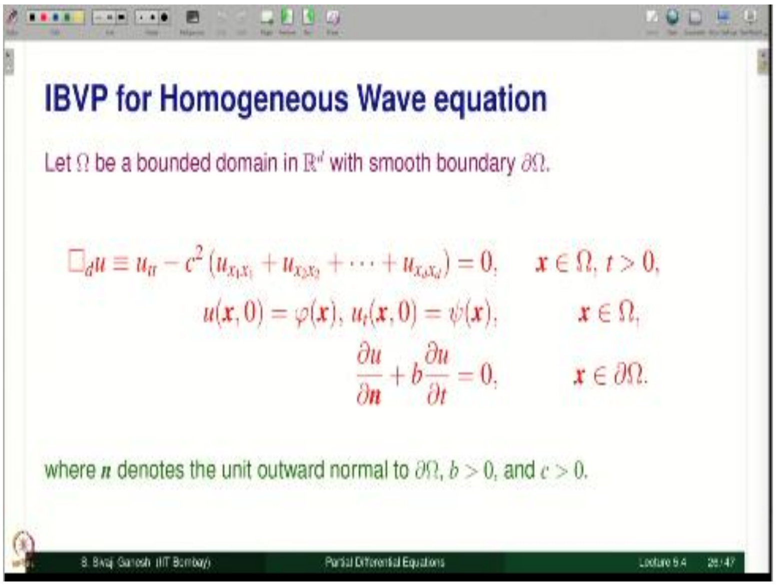

Let us look at an IBVP in d dimensions. So, homogeneous wave equation, this is the Cauchy data, but with some mixed condition like this on the boundary of omega.

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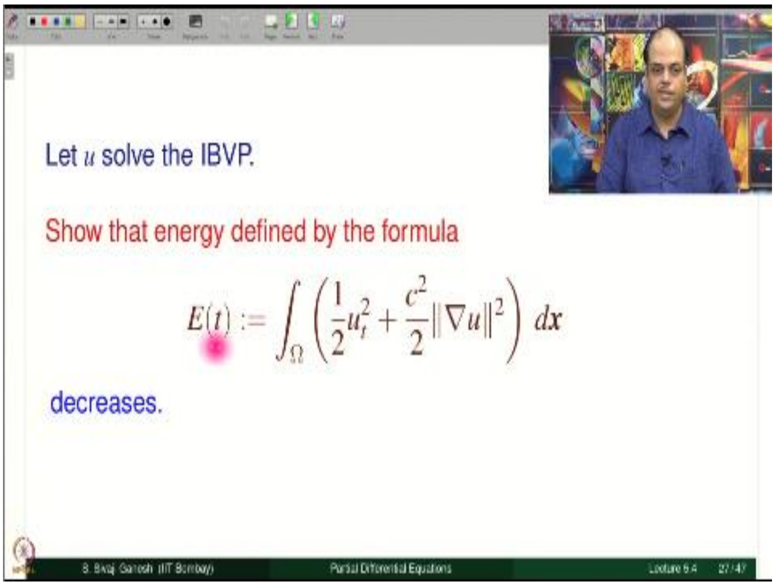

So, let u be a solution to IBVP. Show that the energy defined by this formula decreases. How do you show? u of t is a decreasing function. Actually what we mean is E of t is a decreasing function. We show the d by dt of E t is less than or equal to 0.

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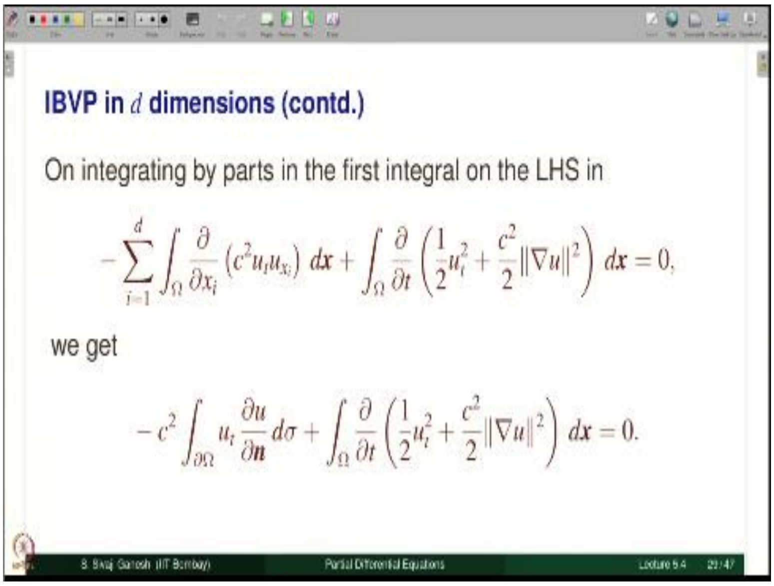

So, as before multiply the equation with u t, rearranging the terms will give you this integrate or omega. This is what we like d by dt of E t. So, always we are doing anything is with this term, do integration by parts. This becomes c square u t and we already saw u x i into neu i summation i equal to 1 to d, is dou u by dou n. In one of the earlier problems, we have demonstrate how this comes. So, this is what it is plus that term.

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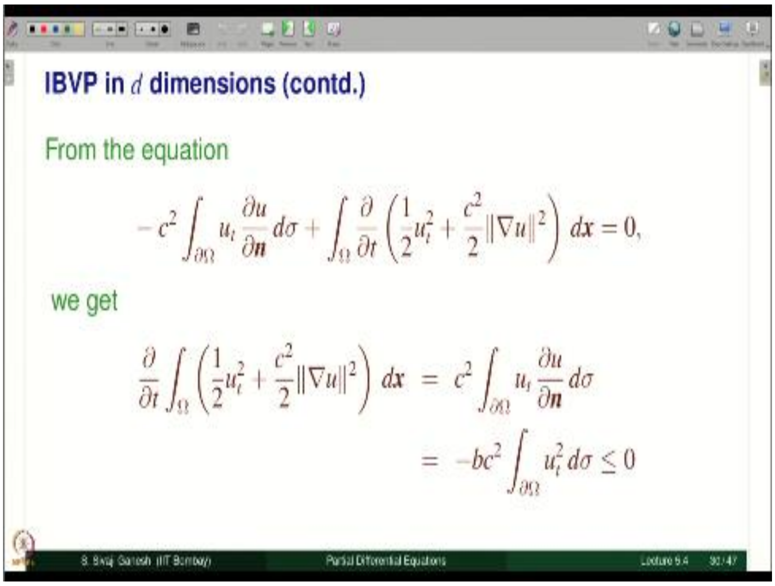

So, from this equation, what we get keep d by dt of v t in one side, take the other thing to other side. Now, here we will use the boundary condition u n and u t, there is a boundary condition which connects them, after substituting that expression, we get this. Now, if you notice this integrand, integrand is greater than or equal to 0, d sigma has this property that it integrates non-negative functions, it gives a non-negative number, c square is positive, b is positive.

Therefore the quantity on RHS is negative less than or equal to 0. That means we are shown d by dt of E t is less than or equal to 0.

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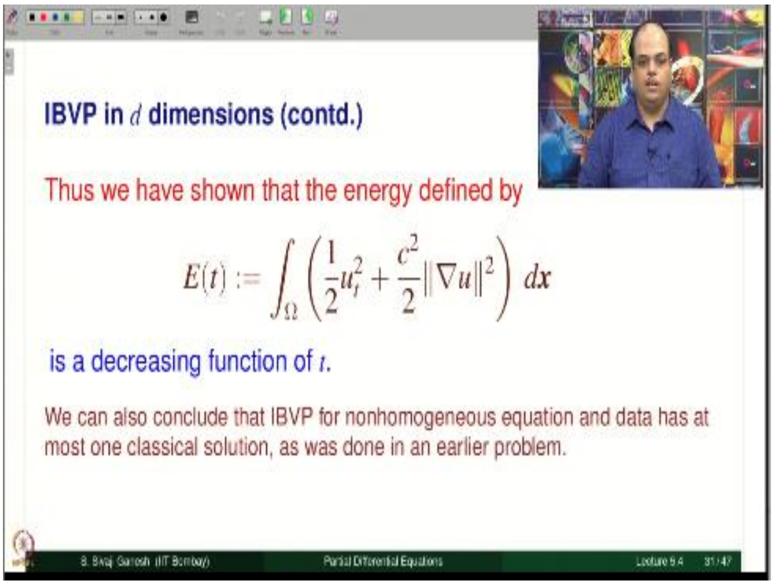

So, E of t is a decreasing function of t. We can also conclude uniqueness of the IBVP in this case that is left to you as an exercise.

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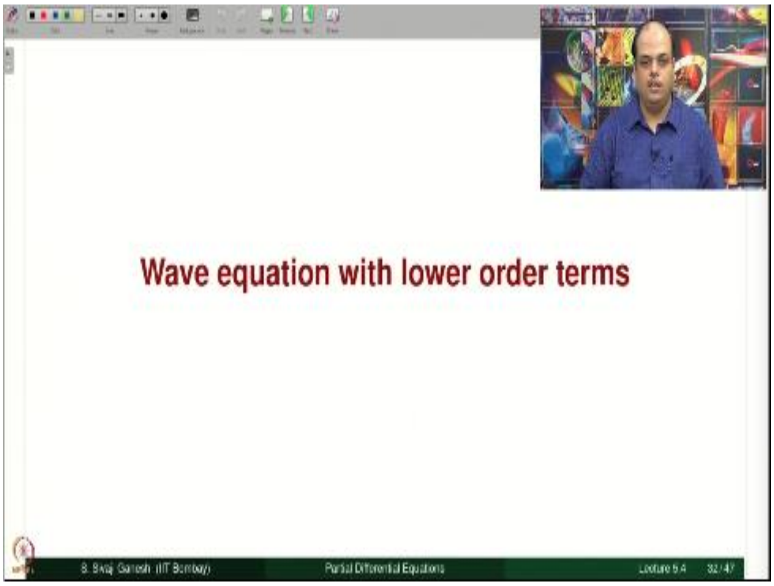

Now, let us look at another equation. Now, we have not just wave equation, but we have some extra terms in the equation like this plus u t.

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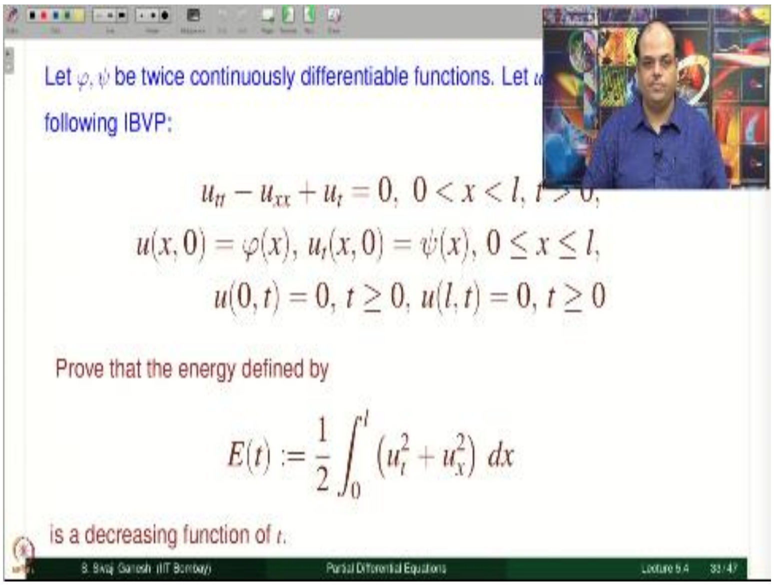

Same Cauchy problem and these are the boundary conditions. So, initial boundary value problem for this equation. Here, energy is a decreasing function of t.

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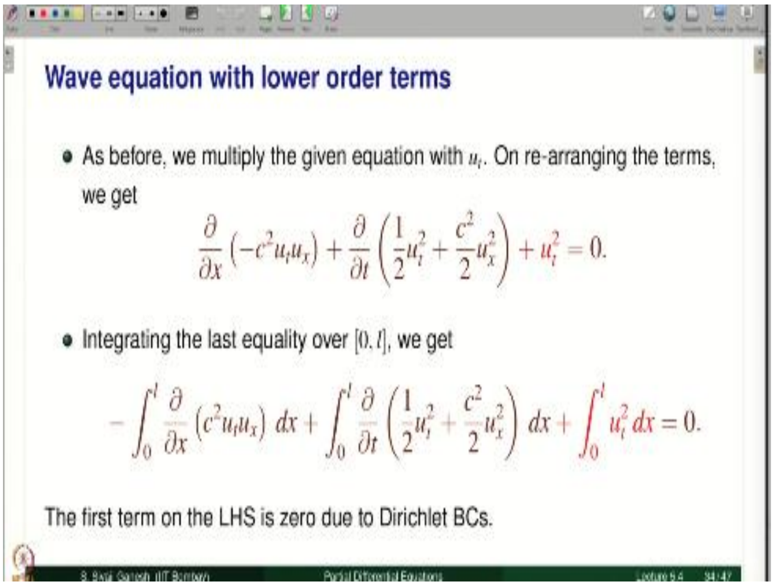

As before, multiply the given equation with u t, rearranging the terms, I am indicating the red colour the new terms that we never had earlier or new in this problem, integrate and settle with this term, see what happens. Once again, it is 0 because Dirichlet boundary conditions. I am not explaining more because we have gained enough experience in deciding when this is integral 0.

So, therefore, what I have is d by dt of E t equal to – 0 to l u t square which is less than or equal to 0. Therefore, E of t is a decreasing function of t.

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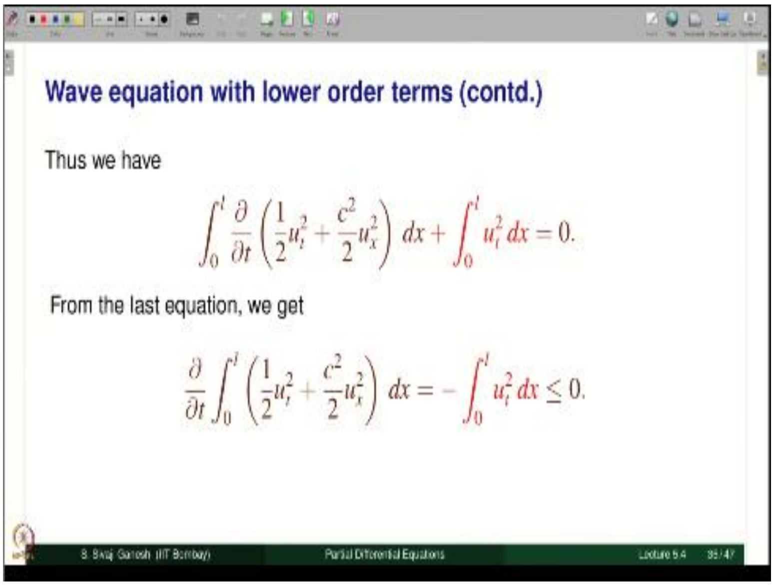

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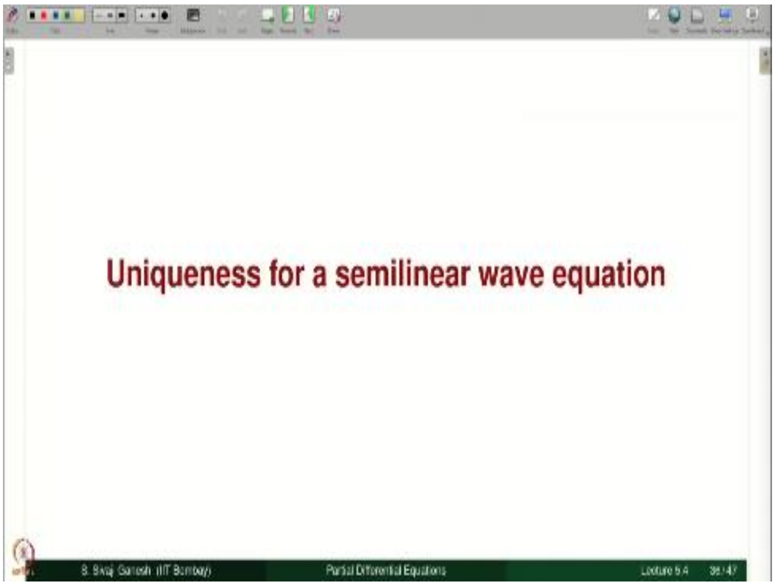

Now, let us look at a semi linear wave equation. Further also, we showed that there can be only at most one classical solution.

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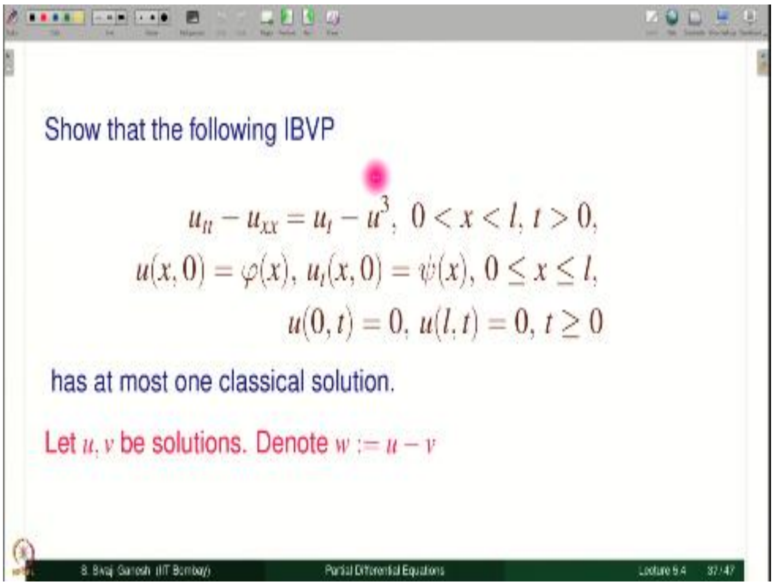

So, this is the d'Alembert. Right hand side is u t – u cube. If you observe, u t in the previous problem, it was on the other side. Now, u t is on this side. You try to follow the same proof. Forget about CQ; let this be remove, remove your cube term and solve as before and see what you can show that is an exercise to you whether energy still decreases or not. Let us discuss this problem now.

We will show, it has at most one classical solution. Starting point is the same. Let u and v be solutions, consider the difference and see what is the problem that w satisfies. Whenever there is linear terms here, after subtracting the 2 equations for u and v, you get w t t, w x x here, w t here, here you will get – u cube + v cube; here, you get w x 0, w t, w 0 t, w l t, everything is fine except for this term.

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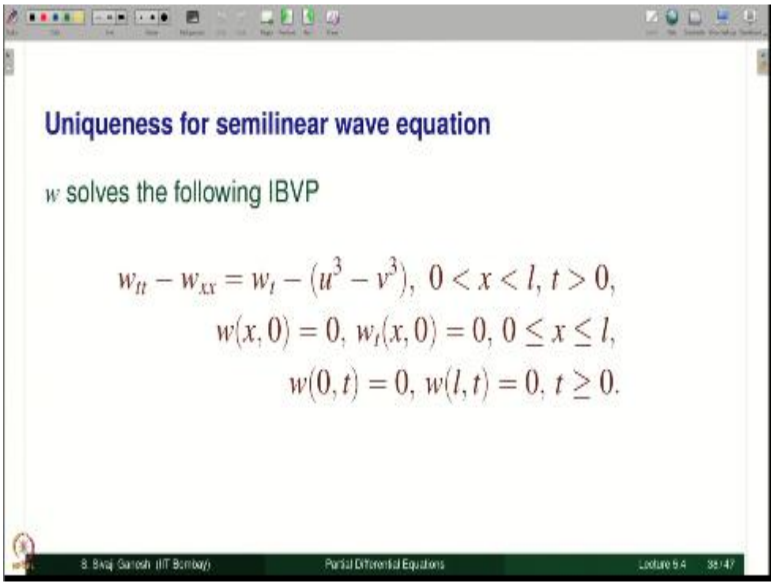

These are equation satisfied by w. Please pause the video, make sure you get the computations, correct computations.

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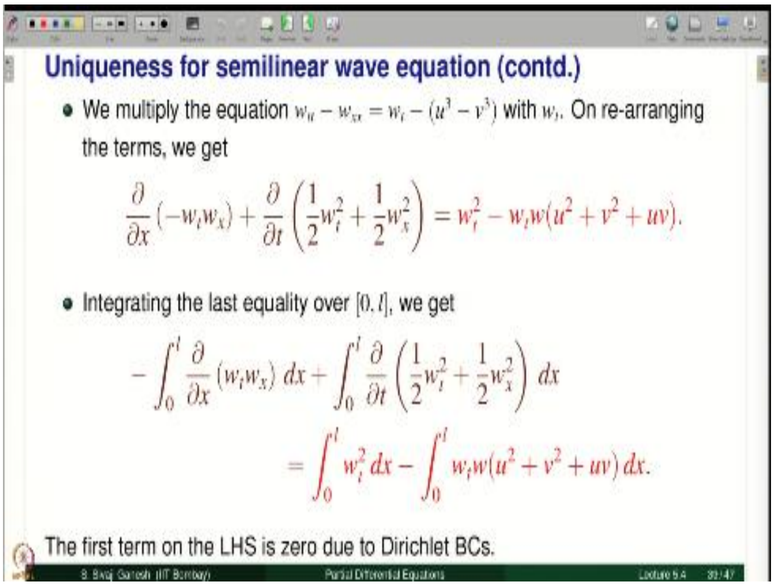

So, multiply with the w t, equation might have changed but since, we are in the context of wave equation, you always multiply with w t that will give you, this is the wave equation part; this is the right hand side part, integrate. So, this you deal with the condition that you have on w and these are the new terms, let us see how to handle. This will be your d by dt of E of t. First term again is 0 because Dirichlet boundary conditions.

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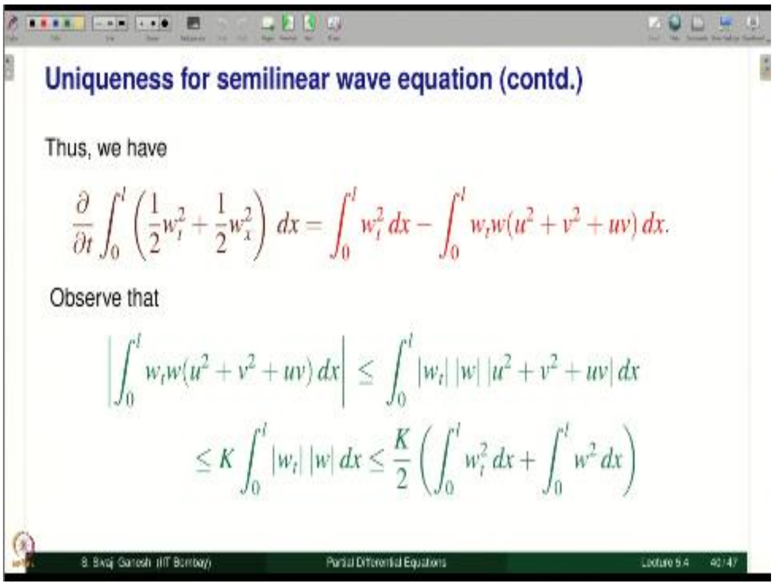

So, what we have is this; let us do some estimation on the right hand side term which is here. The neu cube minus v cube put as u – v into u square + v square + u v; so that u – v became w that is why you have a w here. This is a very simple estimation. Here, the K will depend on explicitly u and v are known depends on u and v. So, you take a bound for u and v on 0 l cross 0 t maybe; so you have these terms. Now, here a b is less than or equal to a square + b square by 2 that is what I have used and then if you separate, you get this.

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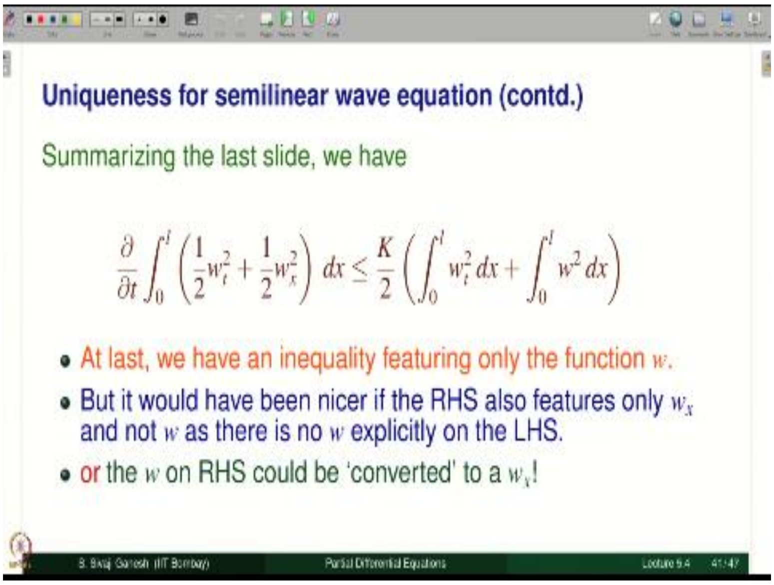

Now, summarizing the last slide, this is LHS. The RHS one term was this; second term we gave rise to a w t and w t square and w square; w t square, I have mixed with this w t square. So, w square will be the new term that you will see now. At last, we have an inequality featuring the function w and its derivatives. But, the problem here is that left hand side, you have w t w x; right hand side, you have w t and w.

So, this w is not there on the LHS because I would like to see it as d by dt of some quantity is less than or equal to some constant times the same quantity. If I want to do that, it asked me that maybe I have a w here; I have a w x instead of w here. So, w on RHS maybe could be converted into a w x.

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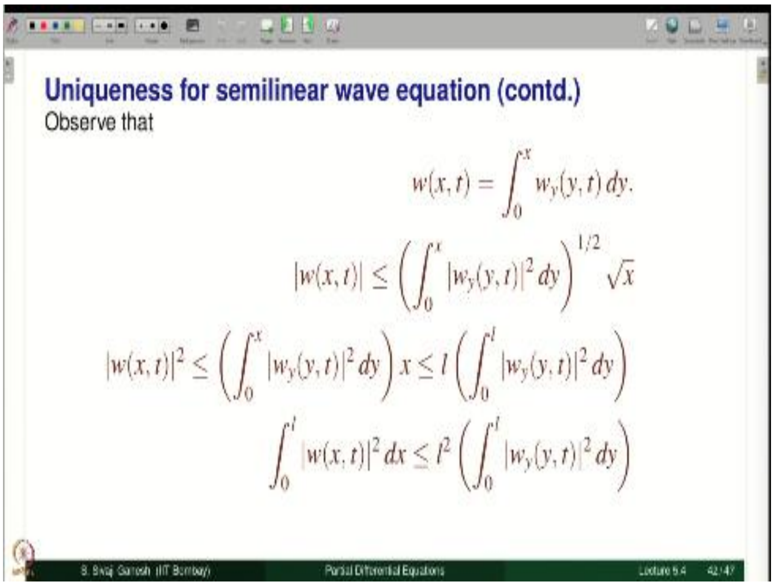

We will let us go ahead with that w x t can be written as this because this is a very fundamental theorem. It is w of x t – w of 0 t which is 0; w 0 t is 0, therefore, this is simply w x t. Therefore, modulus is less than or equal to integral modulus and then that can be further written as this into this. This is what is called Cauchy Schwarz inequality. Integral F G, integral mod G is less than or equal to integral mod F square power half into integral mod g square.

So, here F is this G is 1, so you get this. Now, we know x is less than or equal to l, therefore, let us take the square first. So, this 1 by 2 will go away, root x becomes x, because x is less than or equal to l, you get this and this is a non-negative integral 0 to x and 0 to l, the relation is 0 to x is always less than or equal to the integral on 0 l. Therefore, this inequality we have. And integrate both sides with respect to 0 on the interval 0 l with respect to x and you get this.

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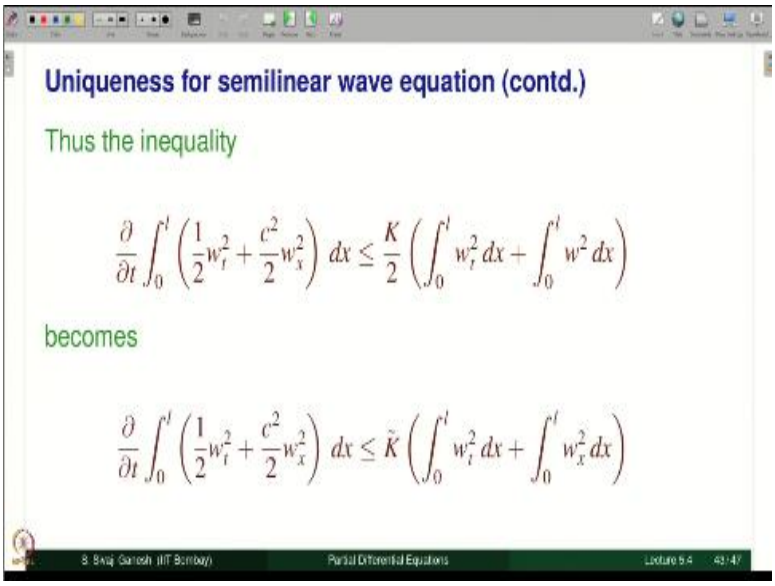

So, therefore, this inequality which you have now becomes this. Now, we are happy because the same w t, w x on both sides.

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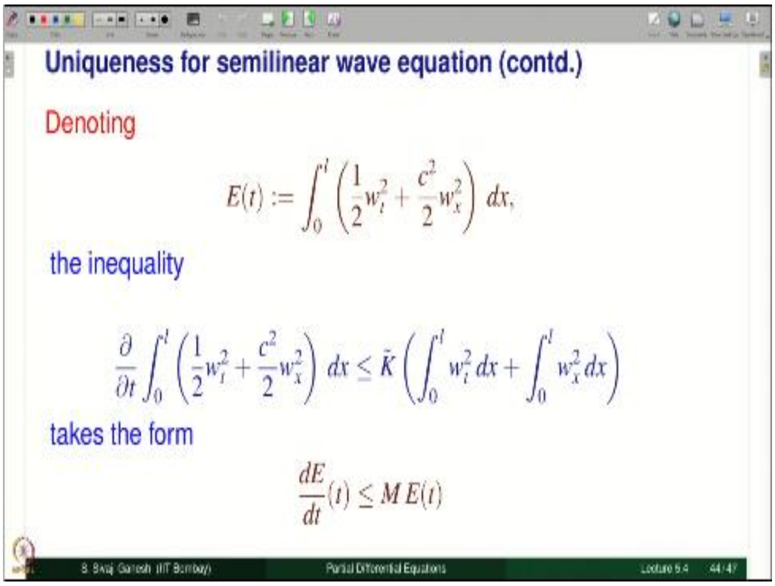

Denoting E of t equal to this, what we have is dou by dou t of E of t is less than or equal to K tilde times this is like E of t, maybe I think there is a c square missing but that can be absorbed into this K tilde like and gives it c square here. It is not a problem. So, it looks like this d by dt less than or equal to M into E t.

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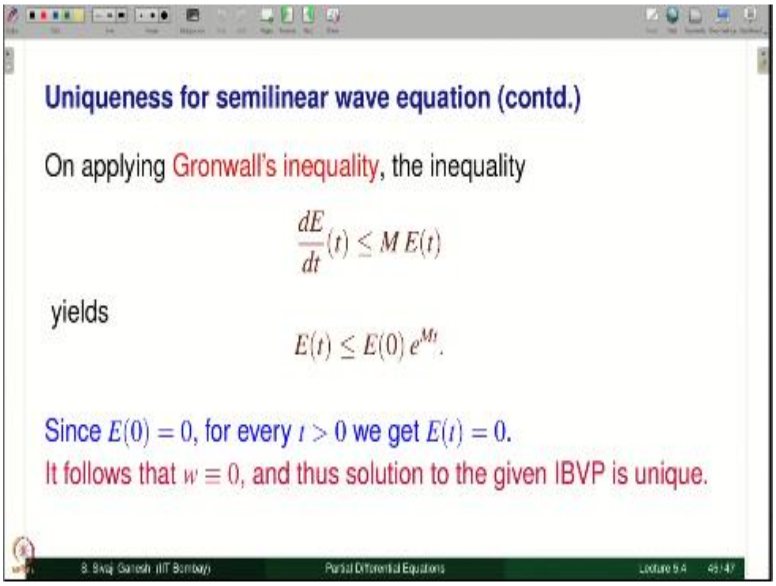

Now, we will use Gronwall's inequality, which says if d by dt is less than or equal to M times E, then the solution E of t is less than or equal to E 0 into e power M t. E of 0 into e power M t is a solution of d by dt equal to M into E power t; d by dt = M E t, this is a solution, but because of the inequality, we get the inequality here. That is what is Gronwall's inequality says. Now, E of 0 is 0.

Therefore, E of t is less than or equal to 0, but by definition E of t is always greater than or equal to 0. Therefore, E of t is 0. Now, it follows that w is 0 therefore, solution is unique to the IBVP.

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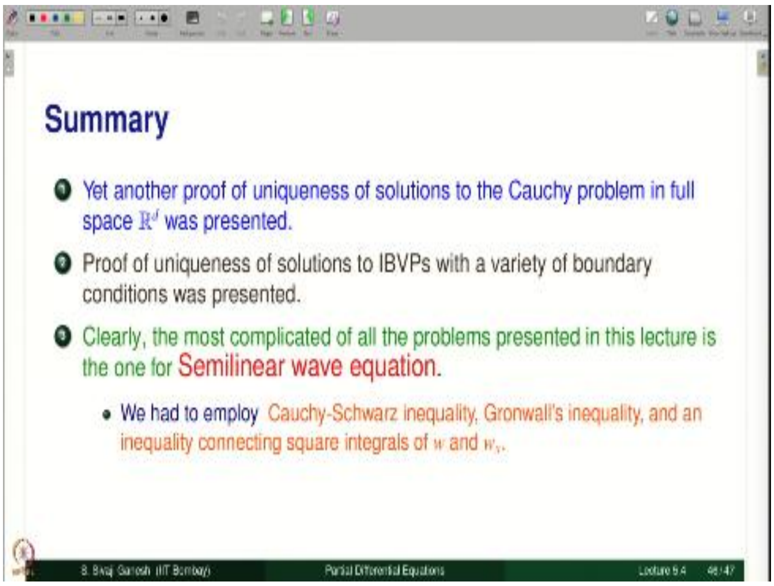

Let us summarize. Yet another proof of uniqueness of solutions to the Cauchy problem in full space R d was presented; proof of uniqueness of solutions to IBPV with a variety of boundary conditions was presented in fact Dirichlet **(()) (32:18)**. Clearly, the most complicated of all the problems presented in this lecture is the one for semi linear wave equation.

We had to imply Cauchy Schwarz inequality, Gronwall's inequality and inequality connecting square integrals of w and w x, which is known as Poincare inequality in advanced a theory of PDEs. Thank you.

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