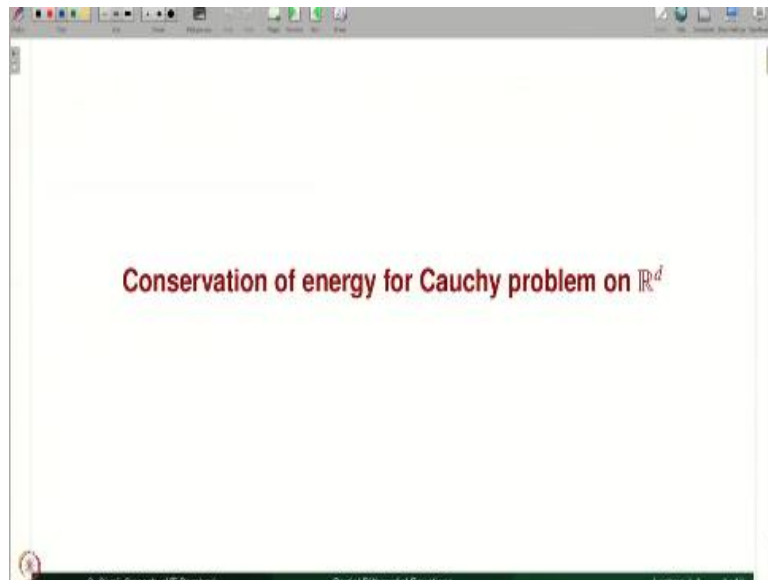


Partial differential Equations
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Lecture – 5.4
Qualitative Analysis of wave Equation
Uniqueness by Energy Method

In our studies of wave equation, we have solved the Cauchy problem for the wave equation in dimensions 1, 2, 3 and also initial boundary value problem when d is equal to 1. We have proved uniqueness of solutions by at least 2 methods so far. In this lecture, we are going to see a third method which is known as energy method. So, that is the topic of this lecture is uniqueness by energy method.

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So, first we look at Cauchy problem on \mathbb{R}^d and we show that energy is conserved. We will define what is energy here.

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Cauchy problem for Homogeneous Wave equation

Given functions $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$, Cauchy problem is to find a solution to

$$\square_d u \equiv u_{tt} - c^2 (u_{x_1 x_1} + u_{x_2 x_2} + \dots + u_{x_d x_d}) = 0, \quad \mathbf{x} \in \mathbb{R}^d, t > 0,$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

$$u_t(\mathbf{x}, 0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

where \mathbf{x} denotes the point $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, and $c > 0$.

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So, Cauchy problem for homogeneous wave equation is; we are in d space dimensions. So, square d'Alembert in d dimensions equal to 0 and this is a Cauchy data phi and psi.

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Energy associated to the Cauchy problem for wave equation is defined by

$$E(t) := \int_{\mathbb{R}^d} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx$$

- When is the integral defining $E(t)$ meaningful?
- The function u needs to satisfy conditions like
 - For each fixed t , the function $\mathbf{x} \mapsto u(\mathbf{x}, t)$ is of compact support. This is guaranteed if φ, ψ themselves are of compact support. (or)
 - For each fixed t , the function $\mathbf{x} \mapsto u(\mathbf{x}, t)$ should have good decay as $\|\mathbf{x}\| \rightarrow \infty$.

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Energy associated to the Cauchy problem for the wave equation is defined by E of t equal to this, where u is the solution to the Cauchy problem. When is this integral meaningful? Because the integral is in \mathbb{R}^d , we have to ask is this integral meaningful. Is it a finite real number for each fix t ? Of course, that would require that u t to be square integrable on \mathbb{R}^d and non-grad u square equivalently all partial derivatives square of the all partial derivatives must be integrable on \mathbb{R}^d .

A simple condition which guarantees this is this. For each fix t , you should have compact support the function \mathbf{x} going to u of \mathbf{x} t is of compact support, then this integral is really on a

bounded set on a compact set and these are continuous functions they in fact, C^2 functions; u is a C^2 function, therefore, u_t is C^1 and u will also be C^1 . Therefore, this integral, there is no problem.

They are definitely continuous functions and we are integrating on a compact set instead of \mathbb{R}^d because of this hypothesis being satisfied. So, then $E(t)$ makes sense. Of course, this in turn guarantee when the Cauchy data itself is a compact support. We have seen this already. Cauchy data compact support solution to the homogeneous wave equation is also of compact support for each fixed t or we need to assume things like this.

For each fix t , this function should have some good decay properties, so, that these integrals will be finite. We will not elaborate more on this point. We are going to deal with these kinds of assumptions in this lecture.

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Theorem

Let the Cauchy data φ and ψ be compactly supported functions defined on \mathbb{R}^d .
 Let u be a solution to the Cauchy problem for the homogeneous wave equation.

Then

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0.$$

In other words, $E(t)$ is a constant function. That is, the energy is conserved.

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So, let the Cauchy data be compactly supported functions on \mathbb{R}^d and let u be solution to the Cauchy problem for the homogeneous wave equation. Then the assertion is d by dt of this quantity which we call it energy is 0 that means energy is constant function of t energy depends on t . It is independent of t and hence, it is actually equal to energy at 0. What is the energy at time 0? What is u_t at $x, 0$? It is ψ .

And what is $\text{grad } u$? That is simply $\text{grad } \varphi$ when time is 0; u is equal to φ . In other words, $E(t)$ is a constant function that is the energy is conserved.

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Proof of Theorem

- As in the proof of causality principle, we multiply the homogeneous wave equation $\square_d u = 0$ with u_t . On re-arranging the terms, we get

$$\sum_{i=1}^d \frac{\partial}{\partial x_i} (-c^2 u_t u_{x_i}) + \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) = 0.$$
- Integrating the last equality over \mathbb{R}^d , we get

$$-\sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} (c^2 u_t u_{x_i}) dx + \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0.$$

Assumptions on the Cauchy data should be such that the above integrals are meaningful.

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Let us prove this theorem. The proofs are by energy method invariably go through the same first step which is to multiply the given equation with the u_t . This was also done exactly in that causality principle proof multiply the equation with u_t , then we rearrange, you get this way. Now, we are planning to integrate this on \mathbb{R}^d . So, exactly the same equation on \mathbb{R}^d . Now, usually this term is actually energy d by dt of the energy.

This term, so everything depends on this term now. How the energy behaves? Here, this is $\text{div}(\text{div} x_i)$ is there. So, we plan to do integration by parts in this term and then conclude things about the energy. As mentioned before, assumptions on the Cauchy data should be such that these integrals are meaningful.

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Proof of Theorem (contd.)

The first term on the LHS in

$$-\sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} (c^2 u_t u_{x_i}) dx + \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0$$

is equal to zero. This is due to

- The functions φ, ψ are of compact support implies that the function $x \mapsto u(x, t)$ is also of compact support for each fixed t .
- In other words, the function $x \mapsto u(x, t)$ is identically equal to zero for sufficiently large values of $\|x\|$.

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The first time on the LHS in this equation, namely this term is 0. This is because the functions phi and psi have compact support that implies that the function x going to u of x t is also of compact support for each fixed t. In other words, the function x going to u of x t is identically equal to 0 for sufficiently large values of norm x.

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Proof of Theorem (contd.)

Let K_t denote the support of the function $x \mapsto u(x, t)$.

Let $R > 0$ be such that $K_t \subset B\left(0, \frac{R}{2}\right) \subset B(0, R)$.

$$-\sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} (c^2 u_t u_{x_i}) dx = -\sum_{i=1}^d \int_{B(0, R)} \frac{\partial}{\partial x_i} (c^2 u_t u_{x_i}) dx$$

$$= -\sum_{i=1}^d \int_{S(0, R)} c^2 u_t u_{x_i} \nu_i d\sigma$$

$\underbrace{\int_{B(0, R)} \dots}_{=0}$

$$= -\int_{S(0, R)} c^2 u_t \frac{\partial u}{\partial \nu} d\sigma$$

$$= 0$$

(Diagram: Sphere of radius R, boundary S(0, R), condition u_{x_i} = 0)

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Let K_t denote the support of the function x going into u of x, t . So, K_t suggests it will depend on T . Of course, this function also depends on t . For each fixed t , K_t denotes the support of this function. Let R positive be such that K_t is contained in this ball of radius R by 2 with centre at the origin because it is a compact set. We can always find such an R . Of course, ball of radius R by 2 is contained ball of radius R having the same centre.

Now, the first term that we want to show is equal to 0, look at this integral; it is an R^d because the support of x going to u of x, t is compact, this integral is really on K_t or let us say integral on this ball or maybe on this ball. So, for convenience, we write it as the integral under bigger ball. Now, we are going to do integration by parts. Integration by parts in this integral will give you 1 domain integral and 1 boundary integral.

The domain integral here will be 0 because it is dou by dou x of c squared $u_t u_{x_i}$ into 1. So, when the dou by dou x shifts to 1, it is 0. So, what we are left with will only be the surface integral or the boundary integral. So, you have this integrand because dou by dou x I, you get the i th component of an outward normal $\nu_i d\sigma$. Now, if you see a summation over $i = 1$ to d of $u_{x_i} \nu_i$ is nothing but the normal derivative dou by dou ν and this is equal to 0.

Why is that? In fact, we need not write this step. I am writing this because for a future user, we are going to use it later on in this lecture. Why is this 0? Because $u \times i$ is 0. Why is $u \times i$ 0? Because see, u is supported in K_t and K_t is contained in the ball of radius R by 2 and this integral that we have is on the sphere s of $0, R$ in particular u is 0 in this annular region. Therefore, all the derivatives will be 0 here.

Here, $u \times i$ will be 0 and hence, this term will be 0 because $u \times i$ on the sphere is 0 and hence, we have this equal.

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Proof of Theorem (contd.)

Thus we get

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0.$$

- Thus the function $E(t)$ is a constant function.
- That is, the energy is conserved.

$$E(t) = E(0) = \int_{\mathbb{R}^n} \left(\frac{1}{2} u_t^2(x, 0) + \frac{c^2}{2} \|\nabla u(x, 0)\|^2 \right) dx$$

$$= \int_{\mathbb{R}^n} \left(\frac{1}{2} \psi^2(x) + \frac{c^2}{2} \|\nabla \varphi(x)\|^2 \right) dx$$

for all $t > 0$.

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So, thus we get the second term which remains in the equation equals 0, but second term is nothing but d by dt of E_t . So, d by dt of $E_t = 0$ that means E is a constant function that means, energy is conserved. So, E of $t = E$ of 0 ; but, what is E of 0 ? It is this expression. I have to put t equal to 0 in the definition of E of t . But, I know what these are that is u_t is ψ and u of 0 is φ therefore, $\text{grad } u$ will be $\text{grad } \varphi$. So, therefore, energy for all times positive is actually equal to the energy when t equals 0 which is given by Cauchy data.

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Uniqueness of solutions to Cauchy problem

Given functions $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}, f : \mathbb{R}^d \times (0, \infty)$, show that the Cauchy problem

$$\square_d u \equiv u_{tt} - c^2(u_{x_1x_1} + u_{x_2x_2} + \dots + u_{x_dx_d}) = f(x, t), \quad \mathbf{x} \in \mathbb{R}^d, t > 0,$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

$$u_t(\mathbf{x}, 0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

has a unique classical solution.

Hint: If u and v are solutions to the above problem, which problem does $w := u - v$ solve?

So, now, uniqueness of solutions to Cauchy problem. So, let us consider a non-homogeneous equation with the Cauchy data. We want to show it as a unique classical solution. So, what is the general strategy for showing any problem has a unique solution is let u and v are solutions, consider the difference and show that the difference is 0. So, we would like to see what is the equation that $u - v$ solves.

u satisfies this equation; v satisfies exactly the same equation; instead of u , I have a v here equal to f , $v(x, 0)$ is $\varphi(x)$, $v_t(x, 0)$ is $\psi(x)$. When I subtract, the operator is linear here, the delamination operator is linear, these conditions are also linear in u and u_t . So, w will satisfy homogeneous wave equation with 0 Cauchy data.

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Uniqueness of solutions to Cauchy problem (contd.)

w solves the Cauchy problem

$$\square_d w = 0, \quad \mathbf{x} \in \mathbb{R}^d, t > 0,$$

$$w(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^d,$$

$$w_t(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^d.$$

We proved that $E(t) = E(0)$. For the above problem

$$E(0) = \int_{\mathbb{R}^d} \left(\frac{1}{2} \psi^2(\mathbf{x}) + \frac{c^2}{2} \|\nabla \varphi(\mathbf{x})\|^2 \right) d\mathbf{x} = 0$$

But, we prove the u of t equal to u of 0 for homogeneous wave equation. Therefore, we have to see what is E of 0 here. E of 0 is given in terms of ψ and ϕ which is 0. Therefore, for this problem, E of 0 is 0.

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Uniqueness of solutions to Cauchy problem (contd.)

Thus, $E(t) = 0$ for all $t > 0$. Since

$$E(t) = \int_{\mathbb{R}^d} \left(\frac{1}{2} w_t^2(x, t) + \frac{c^2}{2} \|\nabla w(x, t)\|^2 \right) dx = 0,$$

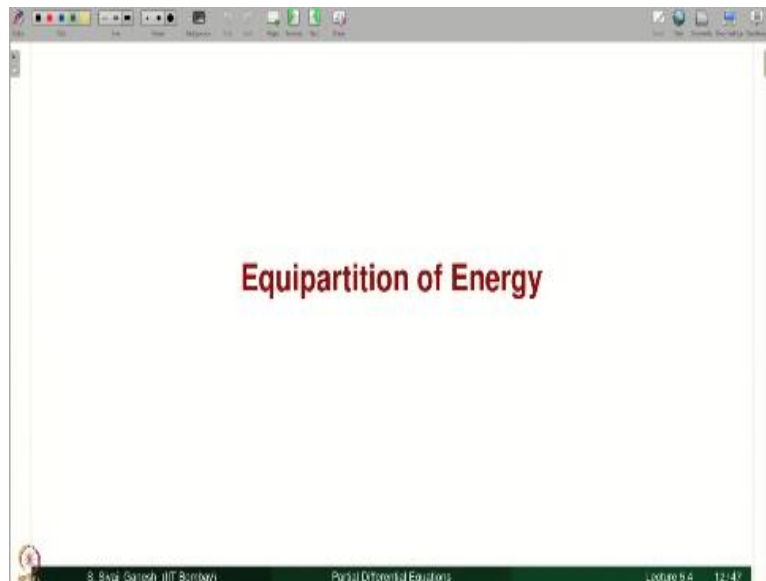
- $w_t(x, t) = 0, \nabla w(x, t) = 0$ for all $(x, t) \in \mathbb{R}^d \times (0, \infty)$.
- Thus $w(x, t) = 0$ for all $(x, t) \in \mathbb{R}^d \times (0, \infty)$.
- In other words, $u(x, t) = v(x, t)$ for all $(x, t) \in \mathbb{R}^d \times (0, \infty)$.
- Thus the Cauchy problem has a unique solution.

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And therefore, E of t is 0 for all t by what is E of T ? It is this expression; this expression equal to 0, integrand is always non-negative, it is sum of 2 non-negative quantities. So, that is 0 if and only if each of the terms is 0 that means w_t is 0 and $\text{grad } w$ is 0; this implies w is a constant function. And w is a constant function and it should be 0 function because it is 0 at time $t = 0$. So, we are assuming the solutions to be classical solutions.

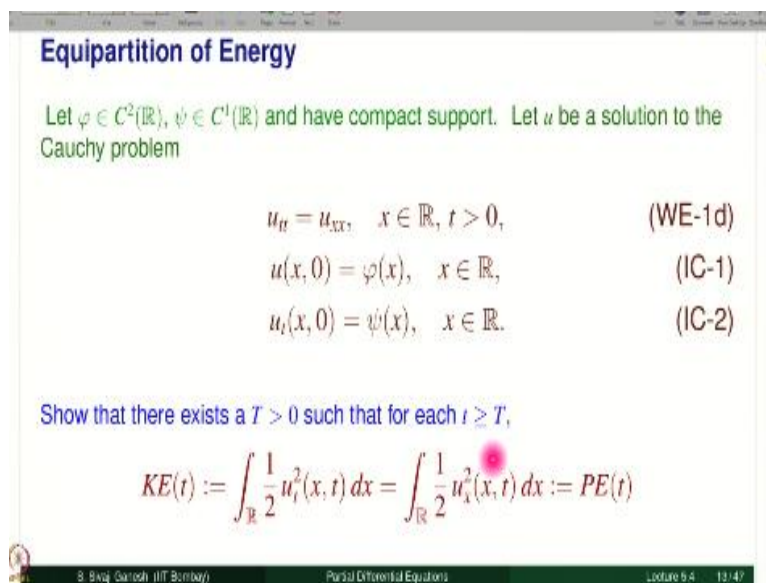
Therefore, w of x, t equal to 0 for all x, t in \mathbb{R}^d cross $(0, \infty)$. In other words, u of x, t equal to v of x, t that means, we are shown uniqueness to the Cauchy problem.

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Let us look at the second problem which is called equipartition of energy. It means energy is partitioned into 2 equal parts. Let us see what are the 2 parts.

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Like phi and psi have a compact support with the usual regularity assumptions and let u be a solution to the homogeneous wave equation with phi and psi as the Cauchy data. show that there exists a T, there is a time t such that for all times after that t greater than or equal to T, we have the kinetic energy which is given by half u t square equal to potential energy which is half u x square.

Energy is sum of kinetic energy and potential energy and what we are showing here is that there is a time T after which, kinetic energy equals potential energy that means, energy is equally partitioned equipartition of energy. How do we show this? We will use. We know the

formula for the solution. We compute u_t ; we compute u_x , substitute here and see what we get.

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Equipartition of Energy

Solution to the Cauchy problem is given by d'Alembert formula:

$$u(x, t) = \frac{\varphi(x-t) + \varphi(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds.$$

Let us compute the derivatives of u .

$$u_t(x, t) = \frac{-\varphi'(x-t) + \varphi'(x+t)}{2} + \frac{\psi(x+t) + \psi(x-t)}{2},$$

$$u_x(x, t) = \frac{\varphi'(x-t) + \varphi'(x+t)}{2} + \frac{\psi(x+t) - \psi(x-t)}{2}$$

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So, this is the formula called a d'Alembert formula which gives solution to the Cauchy problem for the homogeneous wave equation. Let us come to the derivatives of u . Let us compute u_t . First, I had to differentiate φ and then differentiate $x - t$ with respect to T which will give me a minus sign and again differentiate φ , differentiate t with respect to T therefore, you have $1 + 1$ by 2 .

Now, here we need to differentiate, here x and t both are in the limits of integration. So, we have to use what is called a Leibniz rule for differentiation of integrals. So, this is as we discussed earlier, it is a combination of fundamental theorem of calculus and chain rule. So, ψ of $x + t$ into derivative of this with respect to T which is 1 minus ψ at this point $x - t$ into derivative of this quantity $x - t$ with respect to T which is -1 .

Therefore, we get a plus here and this quantity is u_t . Similarly, you can compute u_x . Here, you see that φ' plus ψ of $x + t$ is there, here φ' plus ψ of $x - t$. So, let us separate those term, rearrange.

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Equipartition of Energy (contd.)

On re-arranging the terms, we get

$$u_t(x, t) = \frac{\varphi'(x+t) + \psi(x+t)}{2} + \frac{\psi(x-t) - \varphi'(x-t)}{2},$$

$$u_x(x, t) = \frac{\varphi'(x+t) + \psi(x+t)}{2} - \frac{\psi(x-t) - \varphi'(x-t)}{2}$$

When is $KE(t) := \int_{\mathbb{R}} \frac{1}{2} u_t^2(x, t) dx = \int_{\mathbb{R}} \frac{1}{2} u_x^2(x, t) dx := PE(t)$?

$(A+B)^2 = (A-B)^2$ if and only if $AB = 0$

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When we rearrange u_t has this expression and u_x has this separate this expression. If you notice, the first 2 terms in both expressions are the same; second 2 terms, this second term and this second term are also same, but for sign. So, it looks like a plus b. This looks like $A - B$. It is interesting to keep this analogy in mind, because now we are going to ask when is kinetic energy equal to potential energy.

Half is always equal half, you cancel half; u_t square and u_x square, u_t square as I mentioned $a + b$ whole square $A - B$ whole square. So, $A + B$ whole square equals $A - B$ whole square if and only if $AB = 0$. Therefore, when you substitute u_t inside this and u_x inside this, the A square term, B square term will get cancelled because it is the same on both sides. Here A square is this on the left side.

On the right side in the expansion for u_x square, you get A square here. So, they get cancelled because they are the same. Similarly B is same. So, B square gets cancelled. So, what remains is integral product of this quantity into this quantity equal to 0.

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Equipartition of Energy (contd.)

$$KE(t) := \int_{\mathbb{R}} \frac{1}{2} u_t^2(x, t) dx = \int_{\mathbb{R}} \frac{1}{2} u_x^2(x, t) dx := PE(t) ?$$

if and only if

$$\int_{\mathbb{R}} (\varphi'(x+t) + \psi(x+t)) (\psi(x-t) - \varphi'(x-t)) dx = 0.$$

Let the supports of the functions φ, ψ be contained in the interval $[a, b]$. Such an interval exists as φ, ψ have compact support.

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Please do this computation by yourself. Pause the slide here and then do the computation. Now, we have assumed phi and psi compact supports. So, let them be contained in an interval a b. In a compact set, you can always put it inside some interval a b closed and bounded interval. So, let a b be such that support of phi and psi are contained in a b. It means outside a b, phi and psi are 0.

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Equipartition of Energy (contd.)

- The integrand on the LHS in

$$\int_{\mathbb{R}} (\varphi'(x+t) + \psi(x+t)) (\psi(x-t) - \varphi'(x-t)) dx = 0$$

is identically equal to zero whenever t is such that $x-t \notin [a, b]$ or $x+t \notin [a, b]$.

- Question.** Suppose both $x-t$ and $x+t$ lie in the interval $[a, b]$. Then what must happen?
- Answer.** $2t \leq b-a$.

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Now, the integrand on the LHS is identical equals 0, because, see, one way of assuring that integral is 0 is to assure the integrand is 0. So, that is why we are interested in making the integrand equal to 0. Whenever t such that either $x-t$ is not in a b or $x+t$ is not in a b. Imagine $x-t$ is not in a b, what will happen? $x-t$ is not in the a b, therefore, psi will be 0; phi dash will also be 0.

Therefore, this term is 0. Therefore, this integral is 0. Similarly, if $x + t$ is not in a, b , this first quantity, quantity in the first brackets is 0. It does not matter what this is. Product will be 0 and hence, integral will be 0. So, therefore, the integrand of this integral and hence, the integral itself is 0 whenever t such that $x - t$ or $x + t$, one of them at least is outside the interval a, b . So, therefore, we ask the opposite question.

Suppose, both of them are there in this interval a, b , then what can we say? $x - t, x + t$ interval is lying in interval a, b . So, we should be able to say something. a here, b here, $x - t, x + t$, length of this interval is $2t$. Length of the interval a, b is $b - a$, therefore, what should happen is; $2t$ should be less than or equal to $b - a$. Suppose, t such that $2t$ is bigger than $b - a$, what does it mean?

Both $x - t$ and $x + t$ cannot lie in the interval a, b that means, at least one of them is outside interval a, b and hence, we have this integral equals 0 as a consequence $KE = P$. Now, we asked the question.

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Equipartition of Energy (contd.)

- Choose $T = b - a$. Then $2T > b - a$.
- Thus, both $x - t$ and $x + t$ can NOT lie in the interval $[a, b]$ for $t \geq T$.
- Hence, the integrand on the LHS in

$$\int_{\mathbb{R}} (\varphi'(x+t) + \psi(x+t)) (\psi(x-t) - \varphi'(x-t)) dx = 0$$
 is identically equal to zero.
- As a consequence, we have

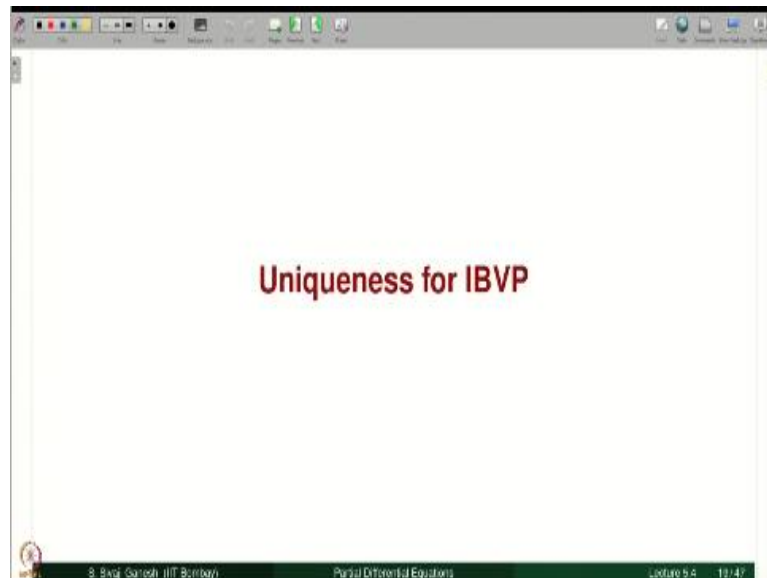
$$KE(t) := \int_{\mathbb{R}} \frac{1}{2} u_t^2(x, t) dx = \int_{\mathbb{R}} \frac{1}{2} u_x^2(x, t) dx := PE(t) \quad \square$$

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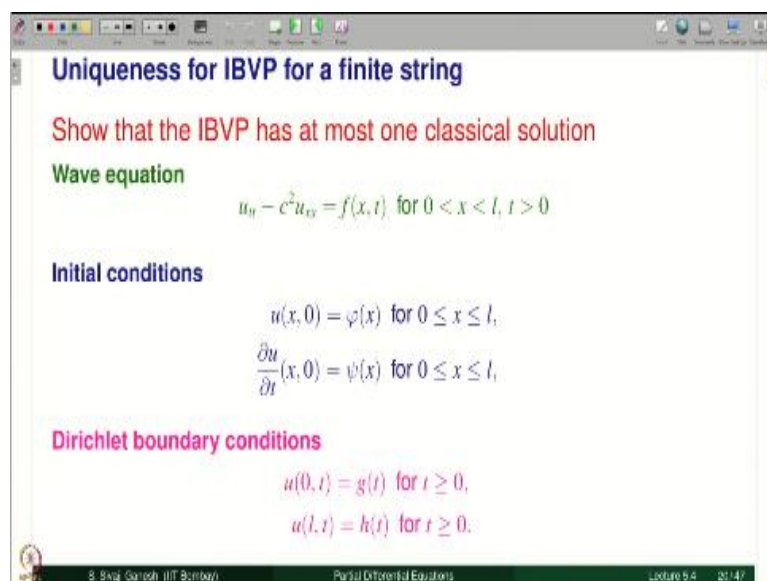
So, therefore, we choose $t = b - a$, then $2t$ will be bigger than $b - a$, therefore, $x - t$ and $x + t$ both of them cannot lie simultaneously in the interval a, b for any t bigger than or equal to T . If one lies, other cannot lie, because both of them lie it means $2t$ is less than or equal to $b - a$, but I have chosen here $2T$ is bigger than $b - a$ therefore, $2t$, 2 times small t is greater than or equal to 2 times T and that is bigger than $b - a$.

So, both of them cannot lie in interval a b whenever t is bigger than equal to T and hence, what we wanted the integrand is identical equal to 0 and hence integral is 0. Therefore, we have $KE = PE$ for all t bigger than or equal to T .

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Now, let us look at another proof of uniqueness for IBVP. Show that IBVP with the non-homogeneous wave equation, non-homogeneous Cauchy data, nonzero Cauchy data or nonzero boundary conditions Dirichlet boundary conditions, this has at most one classical solution. Strategy is same.

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Uniqueness for IBVP (contd.)

Due to the linearity of the wave operator the given IBVP has at most one classical solution if and only if IBVP for homogeneous wave equation along with zero initial-boundary data has **ONLY** the trivial solution.

Homogeneous Wave equation

$$u_{tt} - c^2 u_{xx} = 0 \text{ for } 0 < x < l, t > 0$$

Initial conditions

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 \text{ for } 0 \leq x \leq l.$$

Dirichlet boundary conditions

$$u(0, t) = 0, u(l, t) = 0 \text{ for } t \geq 0.$$

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Let u and v be solutions, take the difference and find out the equation satisfied by the difference. Therefore, due to the linearity of the operator and the conditions that we have u , u_t and u and u_x , therefore, the given problem has unique solution or at most one classical solution, you have to be very careful. Here, we are saying at most one classical solution. We are not saying it has a solution that should be proved separately.

Of course, we have proved it. So, it has at most one classical solution if and only if this problem with homogeneous wave equation and 0 Cauchy data, 0 boundary conditions has only trivial solution. Trivial solution means 0 solution. Of course, we know that $0, u = 0$ is a solution to this that we know but what we are to show is that is the only solution. Then uniqueness for the non-homogeneous initial boundary problem follow-ups.

(Refer Slide Time: 19:26)

Uniqueness for IBVP (contd.)

- We multiply the homogeneous wave equation with u_t . On re-arranging the terms, we get

$$\frac{\partial}{\partial x} (-c^2 u_t u_x) + \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) = 0$$

- Integrating the last equality over $[0, l]$, we get

$$-\int_0^l \frac{\partial}{\partial x} (c^2 u_t u_x) dx + \int_0^l \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx = 0.$$

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As I mentioned earlier, energy method procedures like this by multiplying the given equation, the suitable multiplier in the context of wave equation, it is u_t . So, I am rearranging the terms as before we get this. So, integrating the last equality was 0 L, we get this quantity.

(Refer Slide Time: 19:48)

Uniqueness for IBVP (contd.)

The first term on the LHS in

$$-\int_0^l \frac{\partial}{\partial x} (c^2 u_t u_x) dx + \int_0^l \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx = 0$$

is equal to zero. This is due to

- $u_t(0,t)u_x(0,t) = u_t(l,t)u_x(l,t) = 0$ whenever $u(0,t) = u(l,t) = 0$ (Dirichlet BCs) or $u_x(0,t) = u_x(l,t) = 0$ (Neumann BCs).

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Now, the first term is equal to 0. That is because u_t of 0 t into u_x of 0 t that is what will come once you do integration by parts in one variable, we do not call it; it is simply fundamental theorem of calculus so, maybe the second form. So, whenever you have derivative with respect to x and dx integrand evaluated the upper limit minus integrand evaluated the lower limit that is what will be the answer of this integral.

Of course, there is a minus sign that we will take care later. So, $u_t u_x$ values are at l and at 0 that is what we have written here, u_t of 0 t , u_x of 0 t , both of them are 0, because we know that u of 0 t and u of l t are 0 because of the Dirichlet boundary conditions. Therefore, u_t of 0 t is 0 and u_t of l t is 0. Therefore, this product is 0. Therefore, this term will reduce to 0 or in fact, in the same proof, you can observe that this will be 0.

Then this term will not be there, this term will be 0, but you do not require that further for this thing to happen, you do not require this Dirichlet boundary conditions. Even if you have Neumann boundary conditions, you have u_x 0 t and u_x l t is given to be 0. Therefore, this product is 0 even in that case. It means we are now actually showing the uniqueness of solutions even to the Neumann boundary value problem.

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Uniqueness for IBVP (contd.) Thus we have

$$\frac{\partial}{\partial t} \int_0^l \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx = \int_0^l \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx = 0.$$

- $u_t(x, t) = 0, u_x(x, t) = 0$ for all $(x, t) \in (0, l) \times (0, \infty)$.
- Thus $u(x, t) = 0$ for all $(x, t) \in (0, l) \times (0, \infty)$.
- Hence, we conclude that the given IBVP has a unique solution. □

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So, what we have is that the time derivative of this quantity is equal to 0 and we have shown that this quantity is equal to 0. As before now, this is the time derivative of this quantity is 0, therefore, this quantity is constant and this quantity is constant and that constant has to be 0, because when time $t = 0$, the initial energy is 0 because we are working with 0 Cauchy data. Therefore, this will be 0. As a consequence, u_t of x, t is 0; u_x of x, t is 0 for all x and t and hence, u is identically equal to 0.

It means u is constant under constant has to be 0 because u is already 0 on at $t = 0$, as well as some the boundaries if you are dealing with Dirichlet a problem. So, u of $x, t = 0$. Hence, we conclude that the given IBVP has a unique solution.

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An IBVP in d dimensions

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IBVP for Homogeneous Wave equation

Let Ω be a bounded domain in \mathbb{R}^d with smooth boundary $\partial\Omega$.

$$\square_d u \equiv u_{tt} - c^2 (u_{x_1 x_1} + u_{x_2 x_2} + \dots + u_{x_d x_d}) = 0, \quad \mathbf{x} \in \Omega, t > 0,$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

$$\frac{\partial u}{\partial \mathbf{n}} + b \frac{\partial u}{\partial t} = 0, \quad \mathbf{x} \in \partial\Omega.$$

where \mathbf{n} denotes the unit outward normal to $\partial\Omega$, $b > 0$, and $c > 0$.

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Let us look at an IBVP in d dimensions. So, homogeneous wave equation, this is the Cauchy data, but with some mixed condition like this on the boundary of omega.

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Let u solve the IBVP.

Show that energy defined by the formula

$$E(t) := \int_{\Omega} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx$$

decreases.

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So, let u be a solution to IBVP. Show that the energy defined by this formula decreases. How do you show? u of t is a decreasing function. Actually what we mean is E of t is a decreasing function. We show the d by dt of E t is less than or equal to 0.

(Refer Slide Time: 22:53)

IBVP in d dimensions (contd.)

On integrating by parts in the first integral on the LHS in

$$-\sum_{i=1}^d \int_{\Omega} \frac{\partial}{\partial x_i} (c^2 u_i u_{x_i}) dx + \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0,$$

we get

$$-c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial \mathbf{n}} d\sigma + \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0.$$

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So, as before multiply the equation with u_t , rearranging the terms will give you this integrate or omega. This is what we like d by dt of E_t . So, always we are doing anything is with this term, do integration by parts. This becomes $c^2 u_t$ and we already saw u_{x_i} into u_{x_i} summation i equal to 1 to d , is du by du . In one of the earlier problems, we have demonstrate how this comes. So, this is what it is plus that term.

(Refer Slide Time: 23:21)

IBVP in d dimensions (contd.)

From the equation

$$-c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial \mathbf{n}} d\sigma + \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx = 0,$$

we get

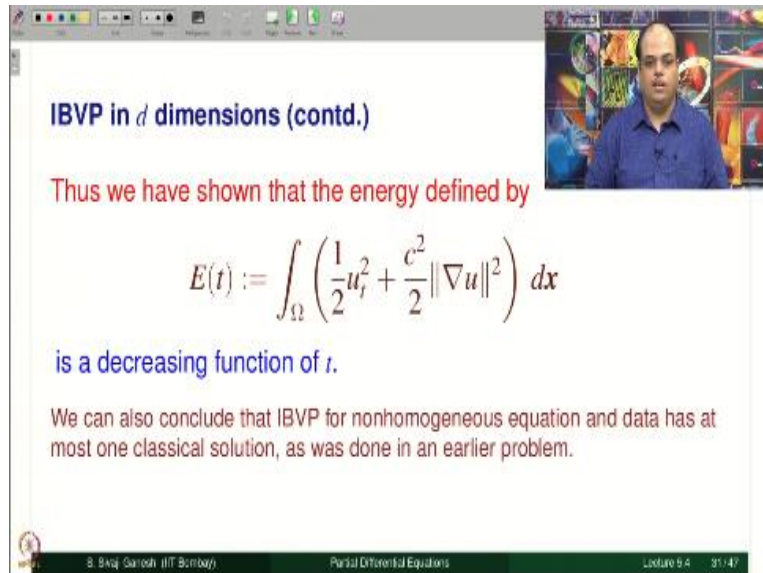
$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx &= c^2 \int_{\partial\Omega} u_t \frac{\partial u}{\partial \mathbf{n}} d\sigma \\ &= -bc^2 \int_{\partial\Omega} u_t^2 d\sigma \leq 0 \end{aligned}$$

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So, from this equation, what we get keep d by dt of v_t in one side, take the other thing to other side. Now, here we will use the boundary condition u_n and u_t , there is a boundary condition which connects them, after substituting that expression, we get this. Now, if you notice this integrand, integrand is greater than or equal to 0, $d\sigma$ has this property that it integrates non-negative functions, it gives a non-negative number, c^2 is positive, b is positive.

Therefore the quantity on RHS is negative less than or equal to 0. That means we are shown $\frac{d}{dt} E(t)$ is less than or equal to 0.

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IBVP in d dimensions (contd.)

Thus we have shown that the energy defined by

$$E(t) := \int_{\Omega} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} \|\nabla u\|^2 \right) dx$$

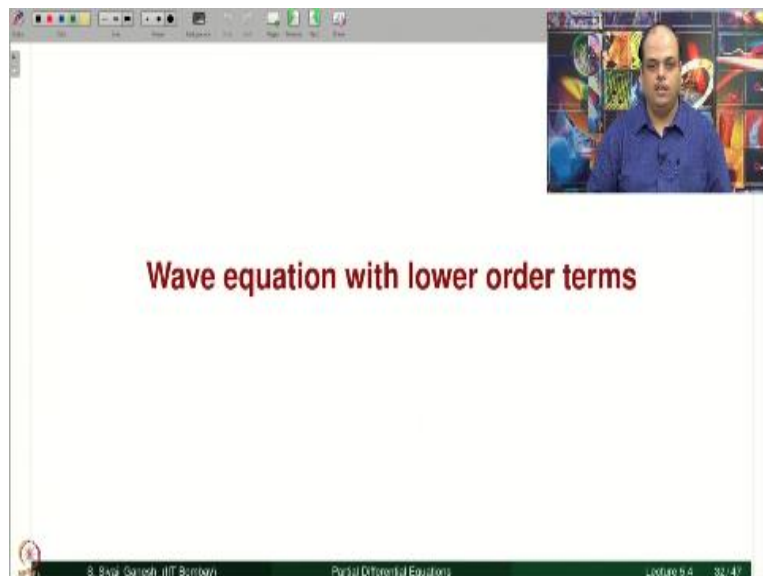
is a decreasing function of t .

We can also conclude that IBVP for nonhomogeneous equation and data has at most one classical solution, as was done in an earlier problem.

8. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 5.4 31:47

So, $E(t)$ is a decreasing function of t . We can also conclude uniqueness of the IBVP in this case that is left to you as an exercise.

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Wave equation with lower order terms

8. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 5.4 32:47

Now, let us look at another equation. Now, we have not just wave equation, but we have some extra terms in the equation like this plus u .

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Let φ, ψ be twice continuously differentiable functions. Let u be a solution of the following IBVP:

$$u_{tt} - u_{xx} + u_t = 0, \quad 0 < x < l, t > 0,$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0$$

Prove that the energy defined by

$$E(t) := \frac{1}{2} \int_0^l (u_t^2 + u_x^2) dx$$

is a decreasing function of t .

8. Raju Sarathi, IIT Bombay | Partial Differential Equations | Lecture 9.4 | 32/41

Same Cauchy problem and these are the boundary conditions. So, initial boundary value problem for this equation. Here, energy is a decreasing function of t .

(Refer Slide Time: 24:49)

Wave equation with lower order terms

- As before, we multiply the given equation with u_t . On re-arranging the terms, we get

$$\frac{\partial}{\partial x} (-c^2 u_t u_x) + \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) + u_t^2 = 0.$$
- Integrating the last equality over $[0, l]$, we get

$$-\int_0^l \frac{\partial}{\partial x} (c^2 u_t u_x) dx + \int_0^l \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx + \int_0^l u_t^2 dx = 0.$$

The first term on the LHS is zero due to Dirichlet BCs.

8. Raju Sarathi, IIT Bombay | Partial Differential Equations | Lecture 9.4 | 34/41

As before, multiply the given equation with u_t , rearranging the terms, I am indicating the red colour the new terms that we never had earlier or new in this problem, integrate and settle with this term, see what happens. Once again, it is 0 because Dirichlet boundary conditions. I am not explaining more because we have gained enough experience in deciding when this is integral 0.

So, therefore, what I have is $dE/dt = - \int_0^l u_t^2 dx$ which is less than or equal to 0. Therefore, $E(t)$ is a decreasing function of t .

(Refer Slide Time: 25:27)

Wave equation with lower order terms (contd.)

Thus we have

$$\int_0^l \frac{\partial}{\partial t} \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx + \int_0^l u_t^2 dx = 0.$$

From the last equation, we get

$$\frac{\partial}{\partial t} \int_0^l \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx = - \int_0^l u_t^2 dx \leq 0.$$

8. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 5.4 38/47

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Uniqueness for a semilinear wave equation

8. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 5.4 38/47

Now, let us look at a semi linear wave equation. Further also, we showed that there can be only at most one classical solution.

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Show that the following IBVP

$$u_{tt} - u_{xx} = u_t - u^3, \quad 0 < x < l, t > 0,$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0$$

has at most one classical solution.

Let u, v be solutions. Denote $w := u - v$

8. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 9.4 37/47

So, this is the d'Alembert. Right hand side is $u_t - u^3$. If you observe, u_t in the previous problem, it was on the other side. Now, u_t is on this side. You try to follow the same proof. Forget about CQ; let this be removed, remove your cube term and solve as before and see what you can show that is an exercise to you whether energy still decreases or not. Let us discuss this problem now.

We will show, it has at most one classical solution. Starting point is the same. Let u and v be solutions, consider the difference and see what is the problem that w satisfies. Whenever there is linear terms here, after subtracting the 2 equations for u and v , you get w_t , w_{xx} here, w_t here, here you will get $-u^3 + v^3$; here, you get w_x , w_t , $w(0, t)$, $w(l, t)$, everything is fine except for this term.

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Uniqueness for semilinear wave equation

w solves the following IBVP

$$w_{tt} - w_{xx} = w_t - (u^3 - v^3), \quad 0 < x < l, t > 0,$$

$$w(x, 0) = 0, \quad w_t(x, 0) = 0, \quad 0 \leq x \leq l,$$

$$w(0, t) = 0, \quad w(l, t) = 0, \quad t \geq 0.$$

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These are equation satisfied by w. Please pause the video, make sure you get the computations, correct computations.

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Uniqueness for semilinear wave equation (contd.)

- We multiply the equation $w_t - w_{xx} = w_t - (u^3 - v^3)$ with w . On re-arranging the terms, we get

$$\frac{\partial}{\partial x} (-w_t w_x) + \frac{\partial}{\partial t} \left(\frac{1}{2} w_t^2 + \frac{1}{2} w_x^2 \right) = w_t^2 - w_t w (u^2 + v^2 + uv).$$
- Integrating the last equality over $[0, l]$, we get

$$-\int_0^l \frac{\partial}{\partial x} (w_t w_x) dx + \int_0^l \frac{\partial}{\partial t} \left(\frac{1}{2} w_t^2 + \frac{1}{2} w_x^2 \right) dx = \int_0^l w_t^2 dx - \int_0^l w_t w (u^2 + v^2 + uv) dx.$$

The first term on the LHS is zero due to Dirichlet BCs.

S. Bag, Sarathi IIT Bombay, Partial Differential Equations, Lecture 5.4, 39/47

So, multiply with the w t, equation might have changed but since, we are in the context of wave equation, you always multiply with w t that will give you, this is the wave equation part; this is the right hand side part, integrate. So, this you deal with the condition that you have on w and these are the new terms, let us see how to handle. This will be your d by dt of E of t. First term again is 0 because Dirichlet boundary conditions.

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Uniqueness for semilinear wave equation (contd.)

Thus, we have

$$\frac{\partial}{\partial t} \int_0^l \left(\frac{1}{2} w_t^2 + \frac{1}{2} w_x^2 \right) dx = \int_0^l w_t^2 dx - \int_0^l w_t w (u^2 + v^2 + uv) dx.$$

Observe that

$$\left| \int_0^l w_t w (u^2 + v^2 + uv) dx \right| \leq \int_0^l |w_t| |w| |u^2 + v^2 + uv| dx \leq K \int_0^l |w_t| |w| dx \leq \frac{K}{2} \left(\int_0^l w_t^2 dx + \int_0^l w^2 dx \right)$$

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So, what we have is this; let us do some estimation on the right hand side term which is here. The neu cube minus v cube put as u - v into u square + v square + u v; so that u - v became w that is why you have a w here. This is a very simple estimation. Here, the K will depend on

explicitly u and v are known depends on u and v . So, you take a bound for u and v on $0 \leq t \leq 1$ cross $0 \leq x \leq 1$ maybe; so you have these terms. Now, here a is less than or equal to $a^2 + b^2$ by 2 that is what I have used and then if you separate, you get this.

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Uniqueness for semilinear wave equation (contd.)

Summarizing the last slide, we have

$$\frac{\partial}{\partial t} \int_0^1 \left(\frac{1}{2} w_t^2 + \frac{1}{2} w_x^2 \right) dx \leq \frac{K}{2} \left(\int_0^1 w_t^2 dx + \int_0^1 w^2 dx \right)$$

- At last, we have an inequality featuring only the function w .
- But it would have been nicer if the RHS also features only w_x and not w as there is no w explicitly on the LHS.
- or the w on RHS could be 'converted' to a w_x !

3.84g. Ganesh (IIT Bombay) Partial Differential Equations Lecture 9.4 41:47

Now, summarizing the last slide, this is LHS. The RHS one term was this; second term we gave rise to a w_t and w_t square and w square; w_t square, I have mixed with this w_t square. So, w square will be the new term that you will see now. At last, we have an inequality featuring the function w and its derivatives. But, the problem here is that left hand side, you have $w_t w_x$; right hand side, you have w_t and w .

So, this w is not there on the LHS because I would like to see it as d by dt of some quantity is less than or equal to some constant times the same quantity. If I want to do that, it asked me that maybe I have a w here; I have a w_x instead of w here. So, w on RHS maybe could be converted into a w_x .

(Refer Slide Time: 29:21)

Uniqueness for semilinear wave equation (contd.)
 Observe that

$$w(x, t) = \int_0^x w_y(y, t) dy.$$

$$|w(x, t)| \leq \left(\int_0^x |w_y(y, t)|^2 dy \right)^{1/2} \sqrt{x}$$

$$|w(x, t)|^2 \leq \left(\int_0^x |w_y(y, t)|^2 dy \right) x \leq l \left(\int_0^l |w_y(y, t)|^2 dy \right)$$

$$\int_0^l |w(x, t)|^2 dx \leq l^2 \left(\int_0^l |w_y(y, t)|^2 dy \right)$$

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We will let us go ahead with that $w(x, t)$ can be written as this because this is a very fundamental theorem. It is $w(x, t) - w(0, t)$ which is 0; $w(0, t)$ is 0, therefore, this is simply $w(x, t)$. Therefore, modulus is less than or equal to integral modulus and then that can be further written as this into this. This is what is called Cauchy Schwarz inequality. $\int F G$, $\int \text{mod } G$ is less than or equal to $\int \text{mod } F$ square power half into $\int \text{mod } g$ square.

So, here F is this G is 1, so you get this. Now, we know x is less than or equal to l , therefore, let us take the square first. So, this $1/2$ will go away, root x becomes x , because x is less than or equal to l , you get this and this is a non-negative integral 0 to x and 0 to l , the relation is 0 to x is always less than or equal to the integral on 0 to l . Therefore, this inequality we have. And integrate both sides with respect to 0 on the interval 0 to l with respect to x and you get this.

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Uniqueness for semilinear wave equation (contd.)

Thus the inequality

$$\frac{\partial}{\partial t} \int_0^l \left(\frac{1}{2} w_t^2 + \frac{c^2}{2} w_x^2 \right) dx \leq \frac{K}{2} \left(\int_0^l w_t^2 dx + \int_0^l w_x^2 dx \right)$$

becomes

$$\frac{\partial}{\partial t} \int_0^l \left(\frac{1}{2} w_t^2 + \frac{c^2}{2} w_x^2 \right) dx \leq \tilde{K} \left(\int_0^l w_t^2 dx + \int_0^l w_x^2 dx \right)$$

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So, therefore, this inequality which you have now becomes this. Now, we are happy because the same w_t , w_x on both sides.

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Uniqueness for semilinear wave equation (contd.)

Denoting

$$E(t) := \int_0^l \left(\frac{1}{2} w_t^2 + \frac{c^2}{2} w_x^2 \right) dx,$$

the inequality

$$\frac{\partial}{\partial t} \int_0^l \left(\frac{1}{2} w_t^2 + \frac{c^2}{2} w_x^2 \right) dx \leq \tilde{K} \left(\int_0^l w_t^2 dx + \int_0^l w_x^2 dx \right)$$

takes the form

$$\frac{dE}{dt}(t) \leq M E(t)$$

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Denoting E of t equal to this, what we have is $\frac{dE}{dt}$ of E of t is less than or equal to \tilde{K} times this is like E of t , maybe I think there is a c^2 missing but that can be absorbed into this \tilde{K} like and gives it c^2 here. It is not a problem. So, it looks like this $\frac{dE}{dt}$ less than or equal to M into E of t .

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Uniqueness for semilinear wave equation (contd.)

On applying Gronwall's inequality, the inequality

$$\frac{dE}{dt}(t) \leq M E(t)$$

yields

$$E(t) \leq E(0) e^{Mt}.$$

Since $E(0) = 0$, for every $t > 0$ we get $E(t) = 0$.
 It follows that $w \equiv 0$, and thus solution to the given IBVP is unique.

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Now, we will use Gronwall's inequality, which says if $\frac{dE}{dt}$ is less than or equal to M times E , then the solution E of t is less than or equal to $E(0) e^{Mt}$. $E(0) e^{Mt}$ is a solution of $\frac{dE}{dt} = M E$, this is a solution, but because of the inequality, we get the inequality here. That is what is Gronwall's inequality says. Now, $E(0) = 0$.

Therefore, $E(t) \leq 0$, but by definition $E(t)$ is always greater than or equal to 0. Therefore, $E(t) = 0$. Now, it follows that $w = 0$ therefore, solution is unique to the IBVP.

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Summary

- 1 Yet another proof of uniqueness of solutions to the Cauchy problem in full space \mathbb{R}^d was presented.
- 2 Proof of uniqueness of solutions to IBVPs with a variety of boundary conditions was presented.
- 3 Clearly, the most complicated of all the problems presented in this lecture is the one for Semilinear wave equation.
 - We had to employ Cauchy-Schwarz inequality, Gronwall's inequality, and an inequality connecting square integrals of w and w_x .

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Let us summarize. Yet another proof of uniqueness of solutions to the Cauchy problem in full space \mathbb{R}^d was presented; proof of uniqueness of solutions to IBVP with a variety of boundary

conditions was presented in fact Dirichlet (0) (32:18). Clearly, the most complicated of all the problems presented in this lecture is the one for semi linear wave equation.

We had to imply Cauchy Schwarz inequality, Gronwall's inequality and inequality connecting square integrals of w and w_x , which is known as Poincare inequality in advanced a theory of PDEs. Thank you.

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