Partial Differential Equations

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Lecture – 5.3
Qualitative Analysis of Wave Equation
Causality Principle, Finite Speed of Propagation

Let us continue our qualitative study of the wave equation. And today, we are going to look at what is called a causality principle and finite speed of propagation, a property which is exclusive to hyperbolic equations. We will look at them 2 examples.

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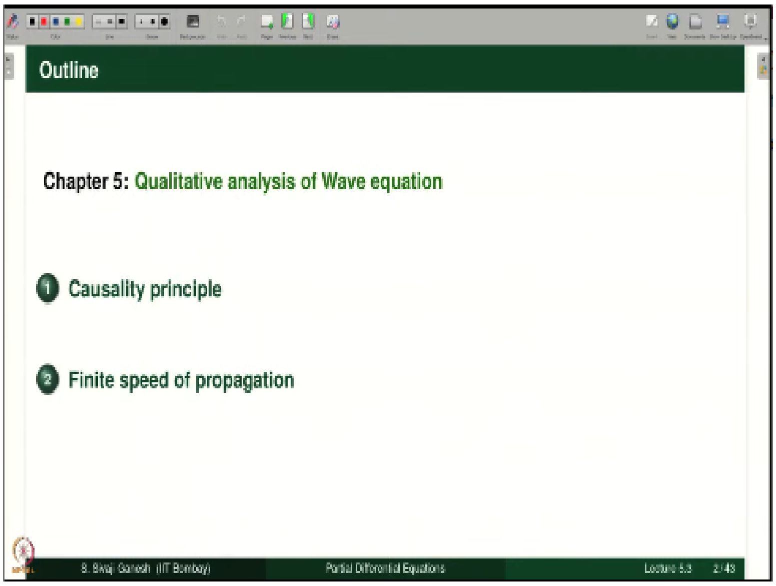

These are the 2 sides of domain of dependence and domain of influence of that concepts. So, causality principle, finite speed of propagation.

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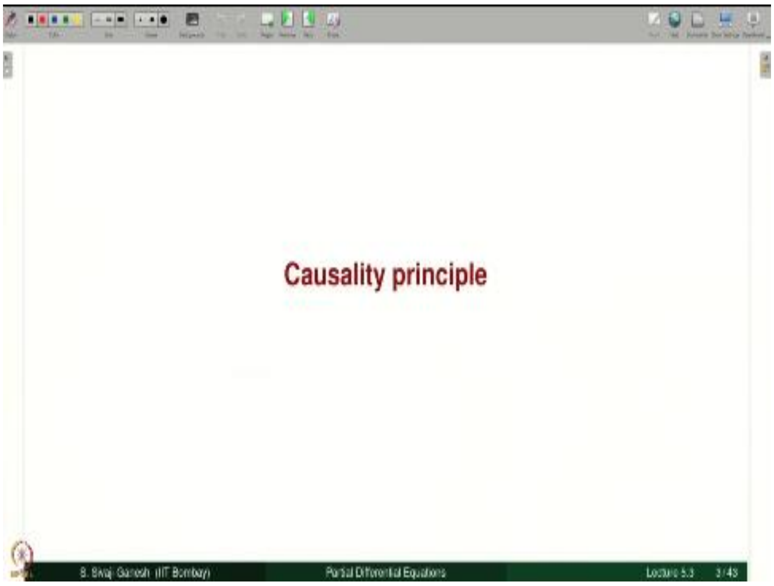

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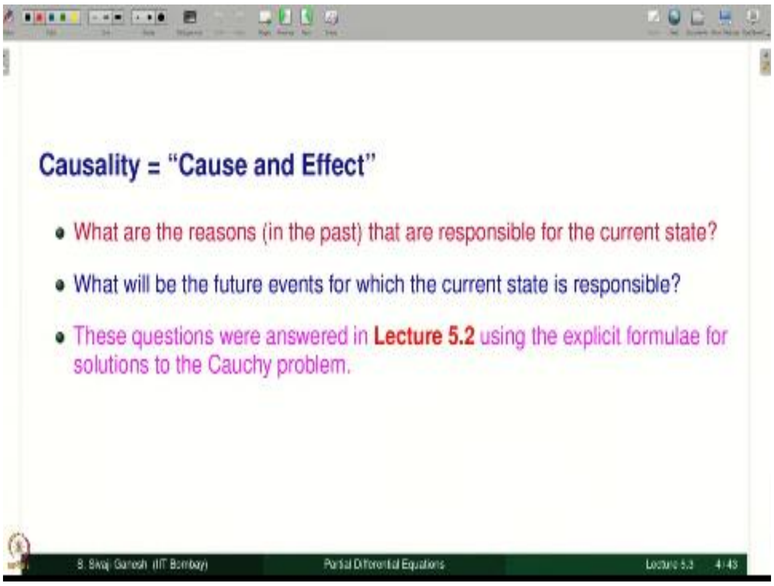

Causality means cause and effect. What are the reasons in the past that are responsible for the current state? What will be the future events for which the current state is responsible for our influences? These questions were answered in lecture 5.2 using the explicit formulae for solutions to the Cauchy problem.

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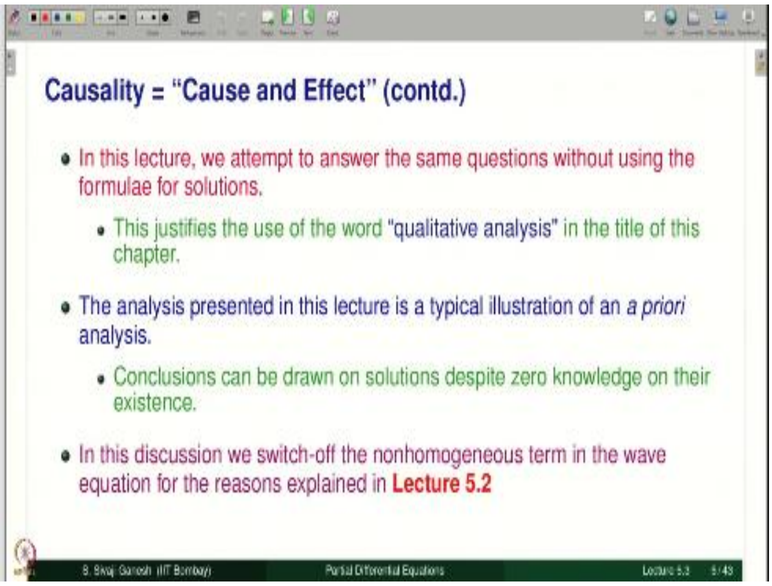

In this lecture, we attempt to answer the same questions without using the formula for solutions. This kind of justifies the use of the word qualitative analysis in the title of this chapter, because we are not using any quantitative formula for the solutions. The analysis presented in this lecture is a typical illustration of an a priori analysis. A priori means done before. Conclusions can be drawn on solutions. Despite 0 knowledge on their existence.

We may not be even knowing whether solution exists or not, still we can conclude certain things about the solution of course, if they exist that we do not know. So, in this discussion, once again like in lecture 5.2, we are going to switch off the nonhomogeneous term and we have already explained the reasons for that in lecture 5.2.

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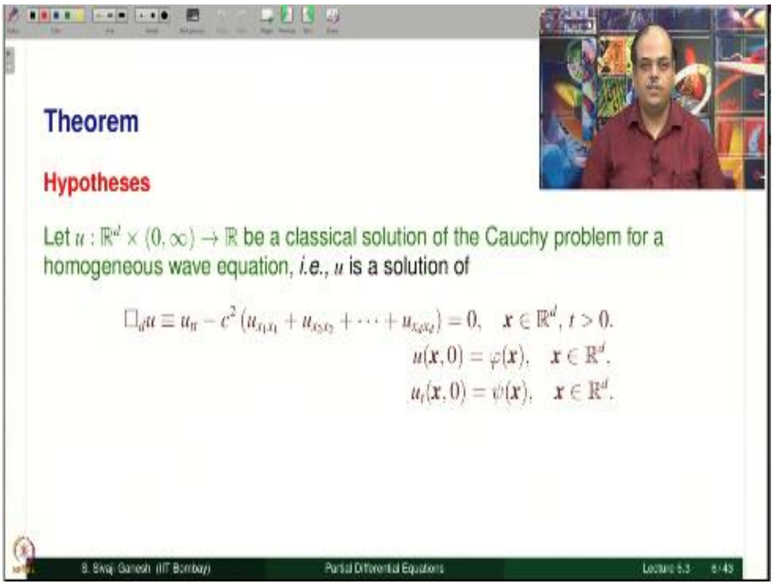

So, let us state in the form of a theorem, the hypothesis is let you from R d cross 0 infinity to R be a classical solution or the Cauchy problem for homogeneous wave equation that is this is a problem where phi and psi are given of course, in the Cauchy problem and u is a solution to this Cauchy.

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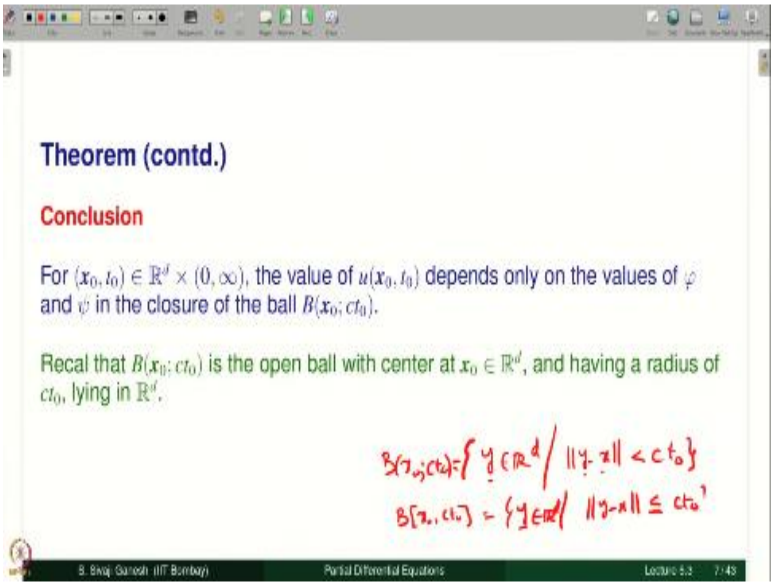

Conclusion: for x 0 t 0 a point in the space time, the value of u of x 0 t 0 depends only on the values of phi and psi in the closure of this open ball with centre x 0 and radius ct 0. Closure of this open ball is nothing but the closed ball with the same radius and centre. Recall that B of x 0, ct 0 is the open ball with centre at x 0 in R d and having a radius of ct 0 lying in R d. So, maybe, we just briefly write what that is.

So, those elements in R d whose distance the Euclidean distance from the point x is less than ct 0. This is the open ball and the closed unit ball, we do not use but then let me introduce good notation is that those elements in R d such that y – x is less than or equal to ct 0.

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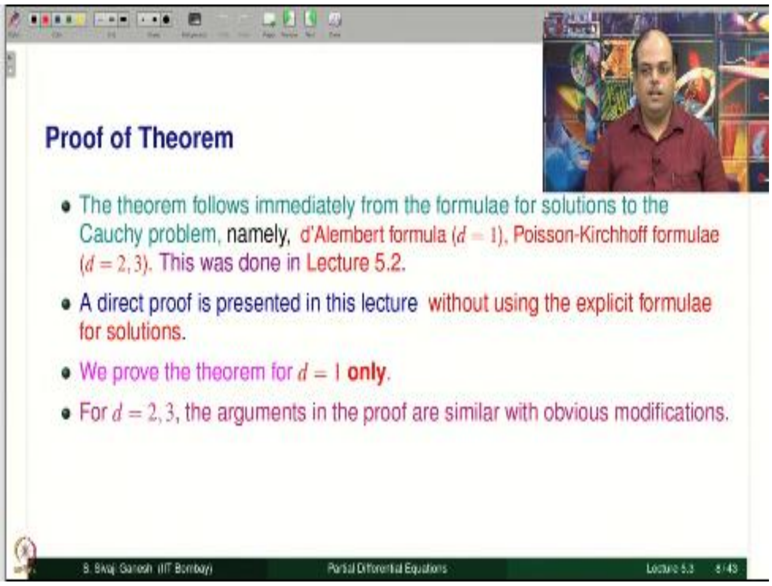

Proof of this theorem: it follows immediately from the formula for a solution to the Cauchy problem namely, d'Alembert formula for d = 1, Poisson-Kirchhoff formula for d = 2 and 3. This was done in lecture 5.2. A direct proof is presented in this lecture without using the explicit formula for solutions. Of course, we are to use something that maybe some experience. We prove the theorem for d = 1 only.

Why is that? Because a proof for d = 2 and 3 are similar, but for the obvious modification that needs to be done to the proof of d = 1 to avoid a lot of repetition. We do not do the proof for d = 2, 3, but mentioned one important inequality there and how that can be derived, just the idea we will present.

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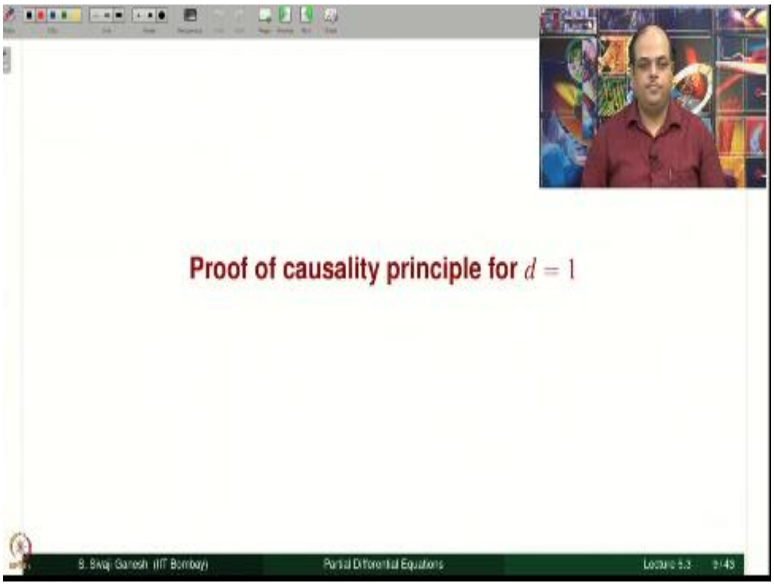

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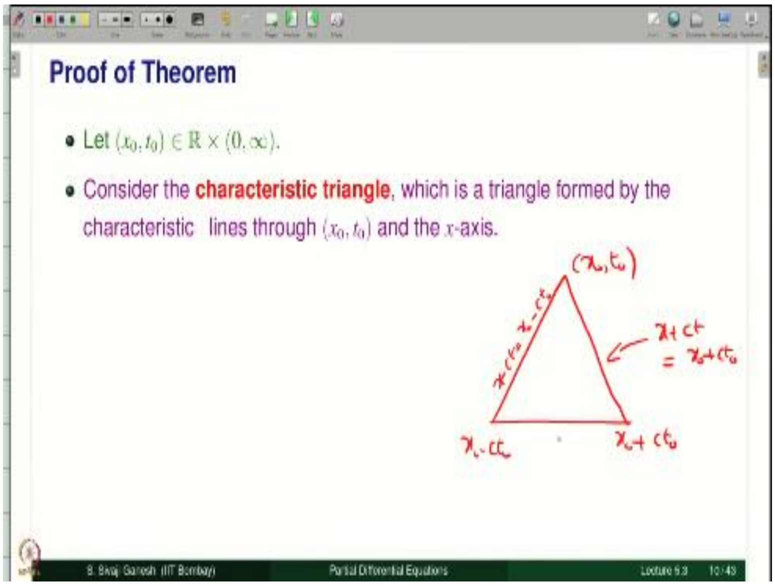

So, proof of causality principle for d = 1. Let x 0 t 0 be a point in R cross 0 infinity. Consider the characteristic triangle which is a triangle formed by the characteristics lines through x 0 t 0 on the x axis that is this is the point x 0 t 0. This point is x 0 – ct 0; is the point x 0 + ct 0; this is called the characteristic triangle. What are these lines? This is x – ct = x 0 – ct 0 and this line is x + ct = x 0 + ct 0. Of course, this line is nothing but t = 0 x axis.

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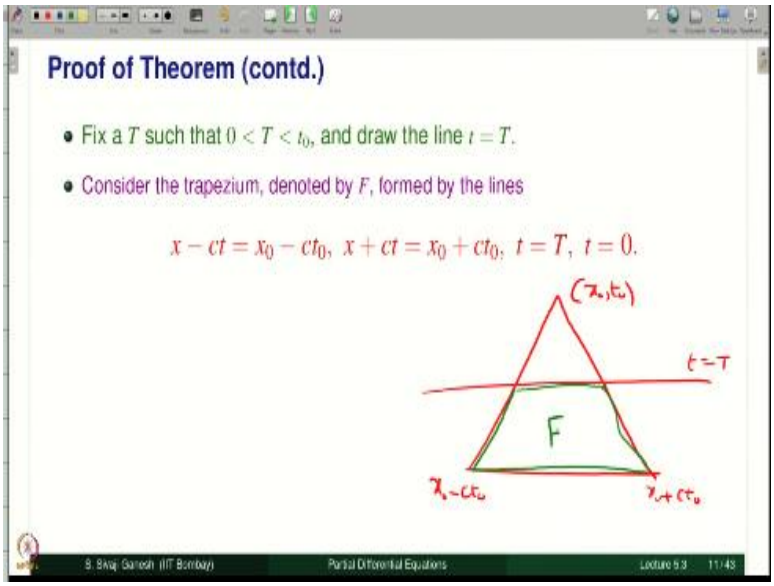

So, fix a T such that T lies between 0 and t 0 and draw the line t = T, we will get a trapezium. Recall point x 0 t 0; x 0 – ct 0; x 0 + ct 0, draw this line t = T. So, we get a trapezium which is far more by the intersection of all these lines that is this part, let us call it F.

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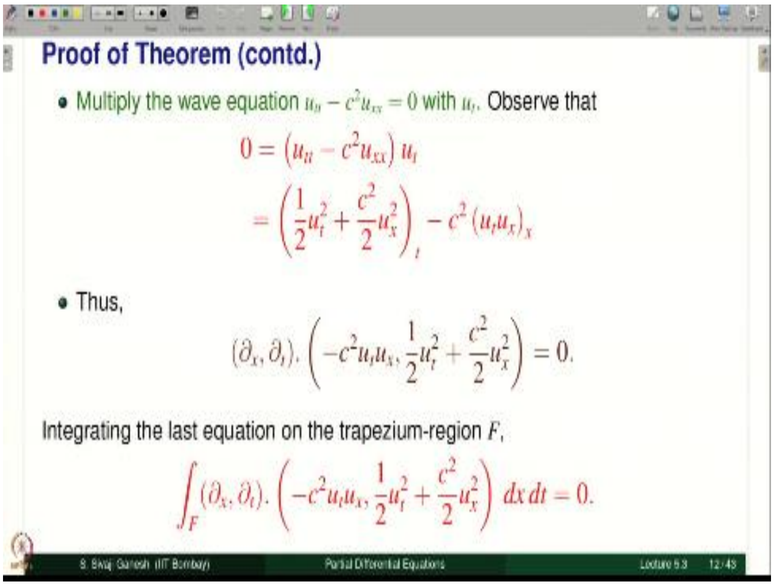

So, multiply the wave equation, this is a trick; the wave equation, you have to multiply with u t, you get this and reorganize the terms, you get this. If you expand this, it will reduce to this. Now, this is in a good shape because here derivatives u t t into u t is there; u x into u t is there whereas here some t derivative of some quantity, x derivative of some quantities is there, It is like the divergence theorem; we are in a good shape to apply divergence theorem.

Therefore, this is a good arrangement. So, in fact, when you see the divergence with x and t here of this quantity is 0. That is precisely this equation; this equals 0 means this. Now, integrate this equation on the trapezium region that we have indicated on the previous slide, you get this. Now, we are ready to do integration by parts or apply Greens theorem and divergence will be converted to something else.

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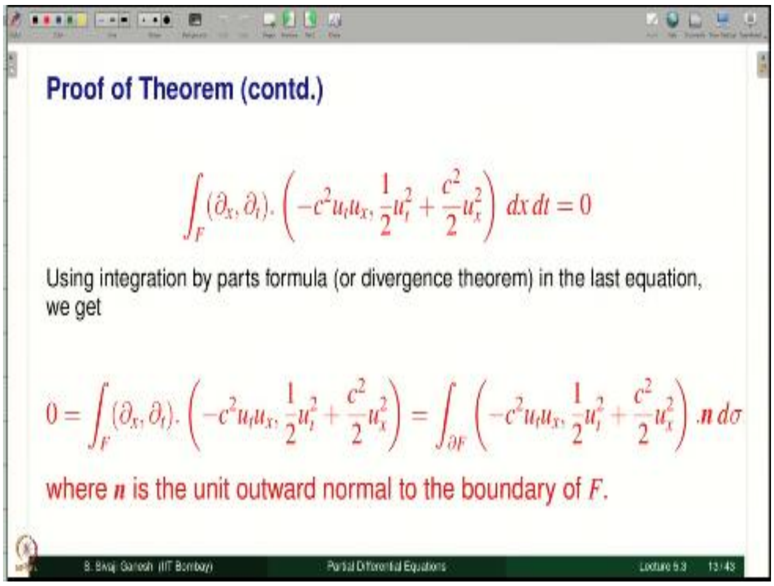

So, what we get is that this divergence of course is equal to 0 is now convert into an integral on the boundary of F. F is the triangle trapezium region; boundary of F is actually a trapezium and that this becomes the integrand dot n d sigma; n is the normal outward unit normal to the points of boundary of F. Of course, that varies from point to point on the boundary of F.

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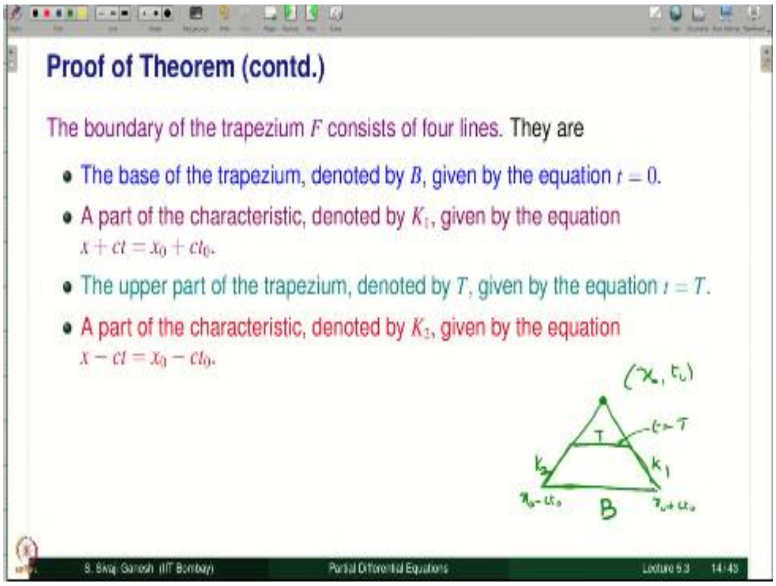

So, the boundary of the trapezium consists of 4 lines actually the boundary of the trapezium region is a trapezium itself, it consists of 4 lines; they are the base of the trapezium denoted by B given by the equation t = 0; a part of the characteristic denoted by K 1 given by the equation x + ct = x 0 + ct 0; upper part of the trapezium denoted by T for top given by the equation t = T; and a part of the characteristic denoted by K 2.

Remember, this is a point x 0 t 0; then we have drawn this. So, this is x – x 0 – ct 0, x 0 + ct 0, then we took this part. So, this equation for this is t = T. So, this, we are calling a B for base; this is K 1; T for top on K 2. The integral is now on this, these lines. So, we need to determine what is the normal to each of these sides.

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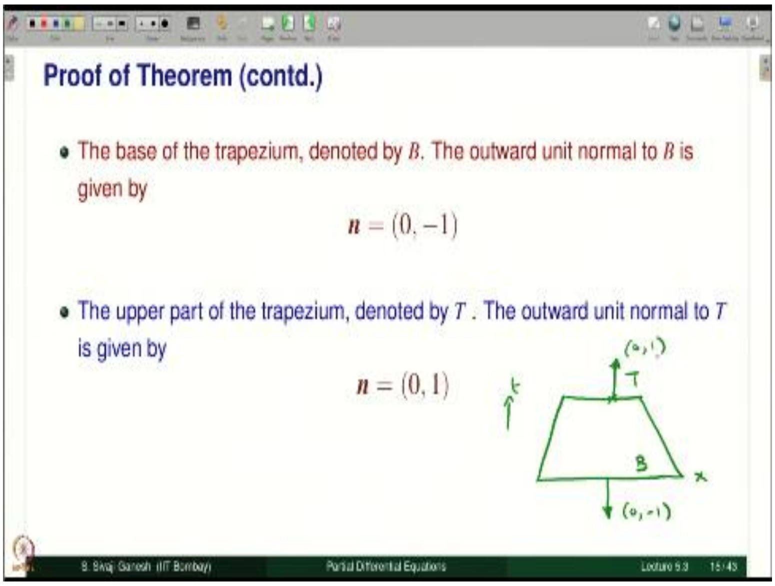

So, base of a trapezium outward unit normal is 0 – 1; top unit outward normal is 0, 1. So, this is the base; this is in the direction of negativity T axis, because this is x axis; this is t axis positive t axis direction, so this is in the negative direction so, 0 – 1 and here, the top normal is in this direction, outward unit normal. It is in the direction of the positive t axis, so 0,1.

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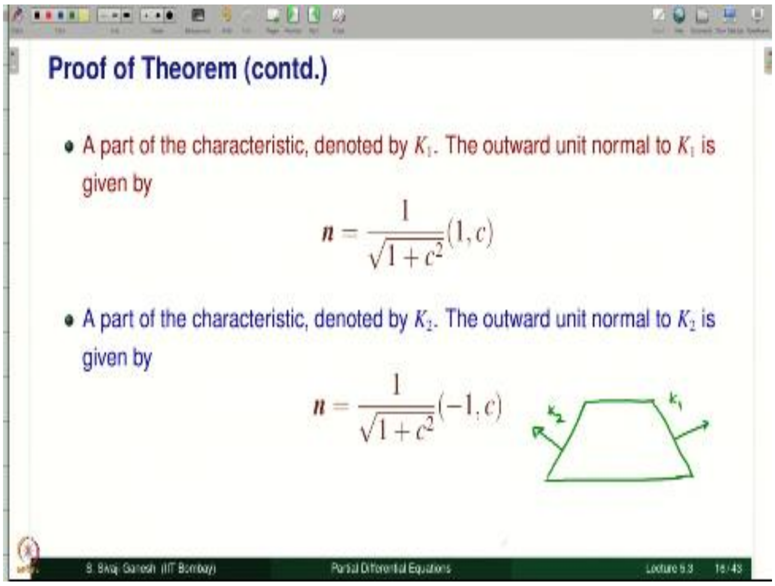

Then we have the sides of the trapezium which is K 1 and K 2 at this point because K 1 itself is a straight line; it is very easy; it is constant on the direction is the same on all the points on K 1. Similarly, the outward unit normal is in this direction and that is constant for all points on K 2, therefore, the life is simpler.

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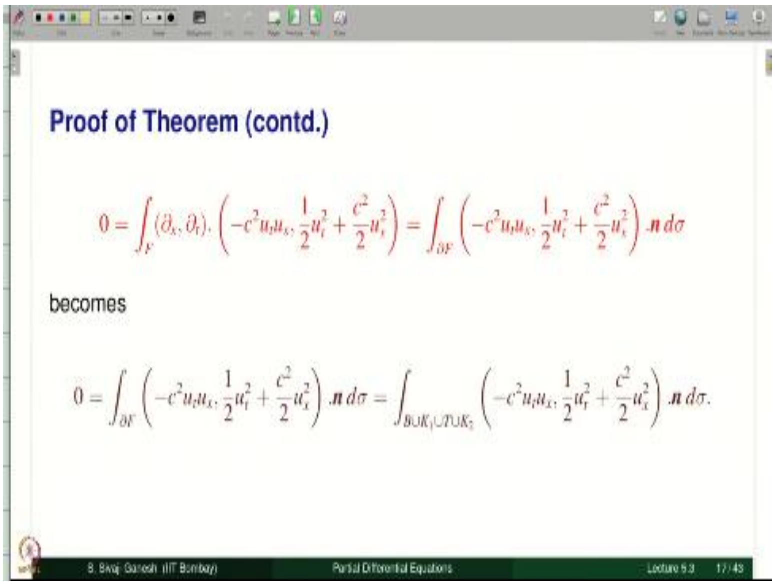

So, this integral on the boundary now, we can split into 4 parts which is B union K 1 union t union K 2.

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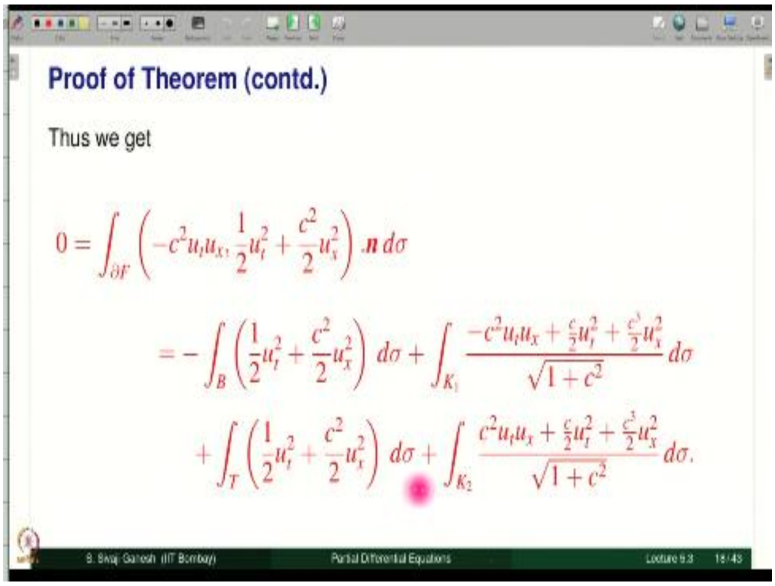

So, on B, we have applied what is a normal similarly on K 1, K 2 on t, we have used the formula for the normal that we written down on the previous slide we get this expression. So, some of these 4 terms is 0.

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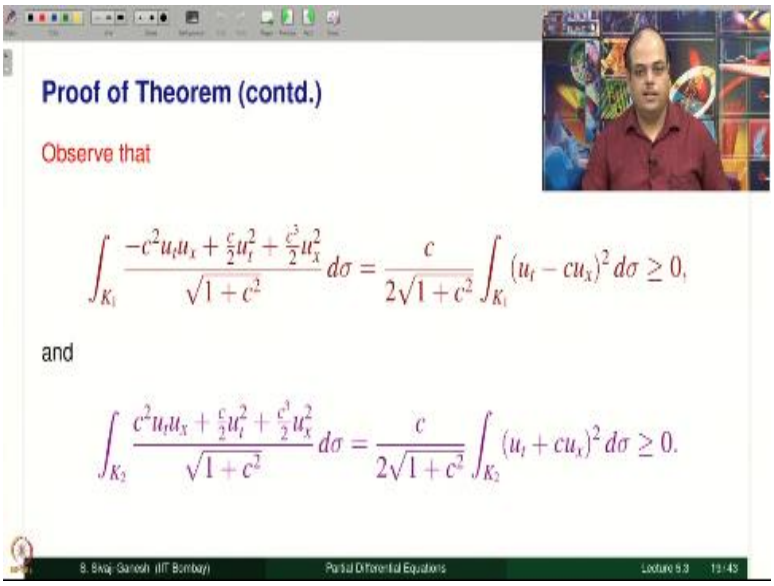

Now, we do a trick; the integral and K 1 can be expressed like this and these an integral on K 1 of some non-negative quantity because of the presence of the square. These always greater than or equal to 0 and hence, this integral is greater than or equal to 0 that is the property of d sigma. Non-negative functions, integral will be non-negative, similarly, here. Remember, this d sigma is nothing but the measure on the boundary which is coming from the domain trapezium.

So, this has all the nice properties. If you integrate a non-negative function, integral will be non-negative. So, the integral or K 1 and K 2 both of them are non-negative numbers.

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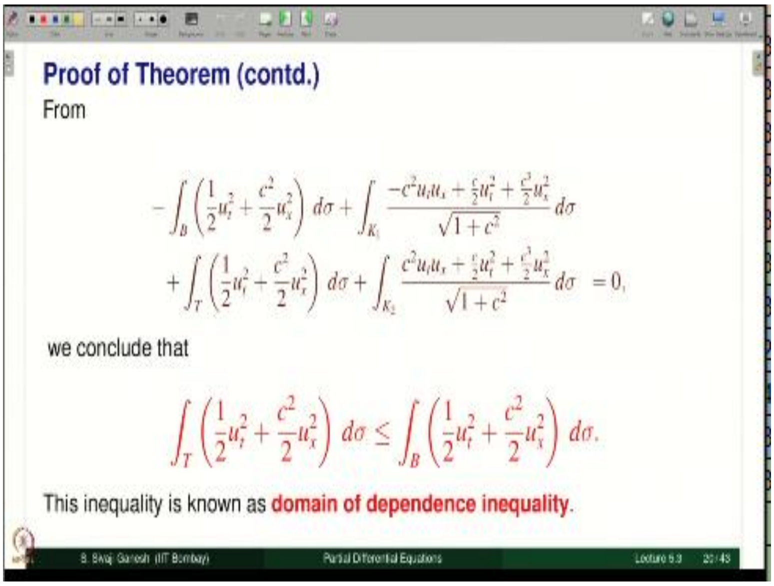

So, here, I have some 4 quantities in that sum of 2 of them is non-negative. So, what can I say about the sum of the other 2? It should be non-positive that means that these 2 terms together is less than or equal to 0 which means, I have the integral of this quantity on the top is less than or equal to integral of this quantity on the bottom. If you notice from this inequality, if u is 0 on the bottom, u and u t are 0 on the bottom, then the zero, then there will be 0 on the top also.

This is the one which gives us uniqueness of solutions as we are going to see on the next slides. So, this inequality is called domain of dependence inequality that means, on the trapezium that on the top portion the integral is less than or equal to integral on the bottom portion.

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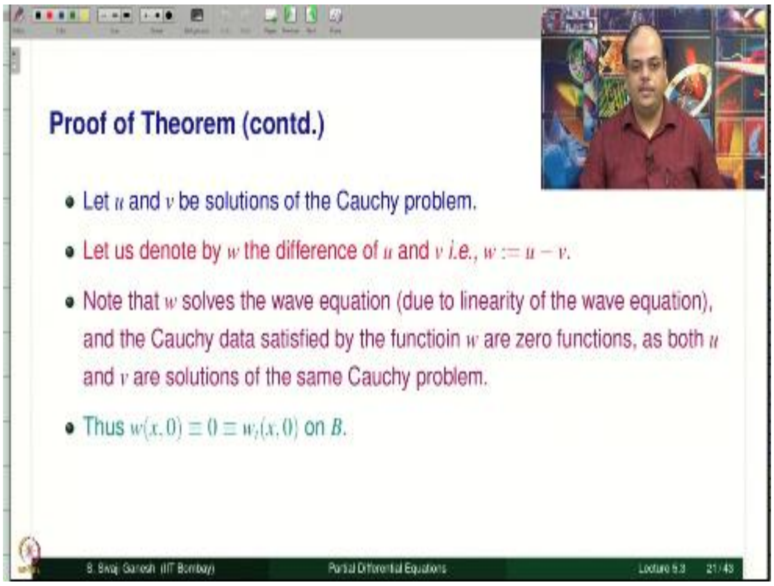

So, let u and v be solutions of the Cauchy problem; define w by the difference u and v that is w equal u – v. Note that that w solves a wave equation due to linearity of the wave equation, of course and the Cauchy data will be 0 because both u and v are solutions to the same Cauchy problem. Therefore, both u and v will satisfy the same Cauchy data and hence, the difference will be satisfying the 0 Cauchy data. Therefore, w of x 0 is 0 and w t of x 0 is 0 on the bottom.

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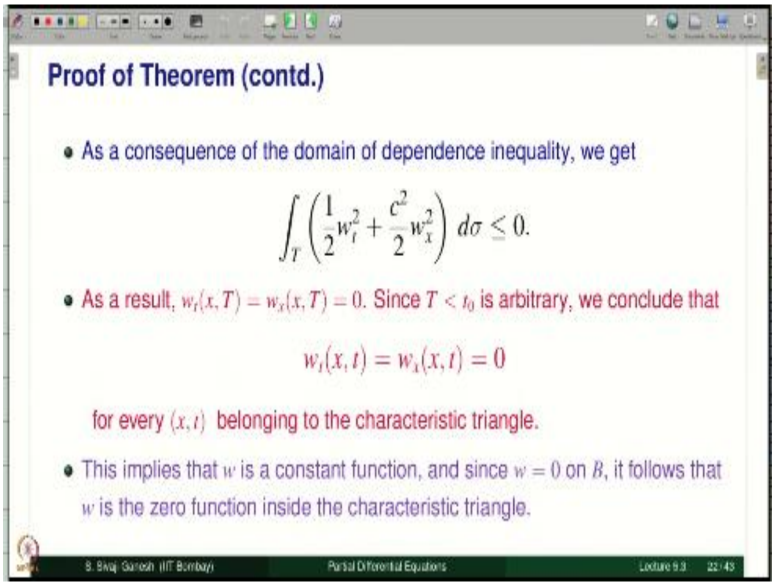

Now, we can apply the domain of dependency inequality and conclude that we have this; right hand side has become 0; right hand side was the same integral on B that is 0 because on B, both w t and w x are 0. Therefore, this is what we have, but if you look at this already non-negative quantity and we are saying there is less than or equal to 0, so, this is always greater than or equal to 0, therefore, the only possibility is that the integrand is 0 which means wt at what point, x, T and w x at the point x, T t 0.

Now, T is arbitrary chosen T less than t 0. Therefore, we get wt of x t and w x of x t is 0 for every x t belonging to the characteristic triangle. This implies that w is a constant function but w is already 0 on B, therefore, it must be 0 everywhere. w is 0 everywhere same as saying u is equal to v everywhere in the characteristic triangle.

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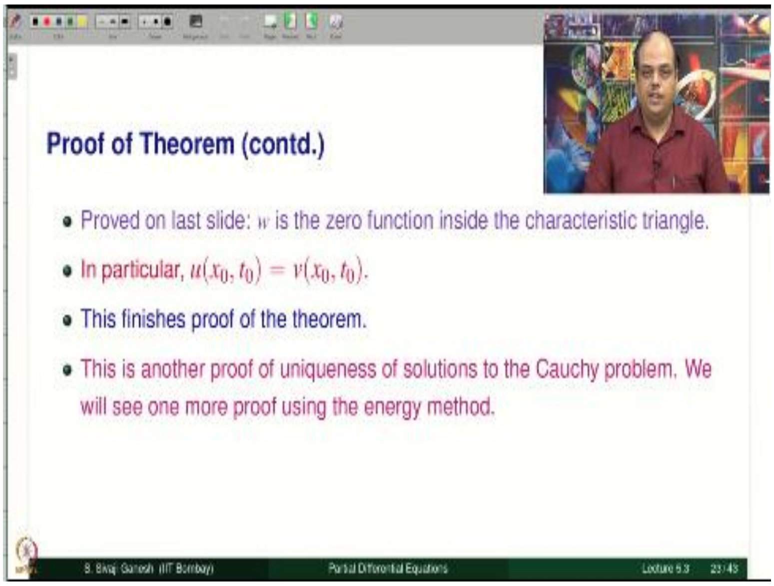

Therefore, the solution is the same; u at x 0 t 0 will be same as v at x 0 t 0. So, proved on the last slide w is a 0 function inside the characteristic triangle. In particular, u of x 0 = v of x 0 t 0. This finishes the proof of the theorem. This is another proof of uniqueness of solutions to the Cauchy problem. We already gave one proof of uniqueness earlier. Now, this is another proof of uniqueness of solutions.

One more proof, we are going to see using the energy method; pretty much the actors in that energy method, we have already seen in the domain of dependence inequality. We will do that in the forthcoming lectures.

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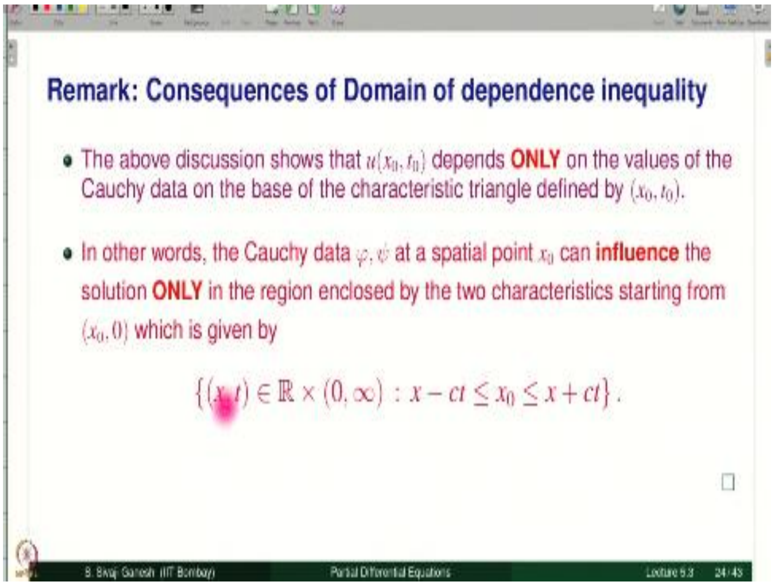

Consequences of domain of dependency inequality: the above discussion shows that u of x 0 t 0 depends only on the values of the Cauchy data on the base of the characteristic triangle defined by x 0 t 0 namely, the interval on the x axis x 0 – ct 0, x 0 + ct 0. In other words, the Cauchy data phi, psi at a spatial point x 0 can influence the solution only in the region enclosed by the 2 characteristics starting from x 0, 0 on the x axis which is given by x t in the space time domain such that x – ct is less than or equal to x 0 and x + ct is greater than or equal to x 0.

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Let us look at the causality principle for d greater than or equal to 2 on characteristic cone. What is the characteristic cone?

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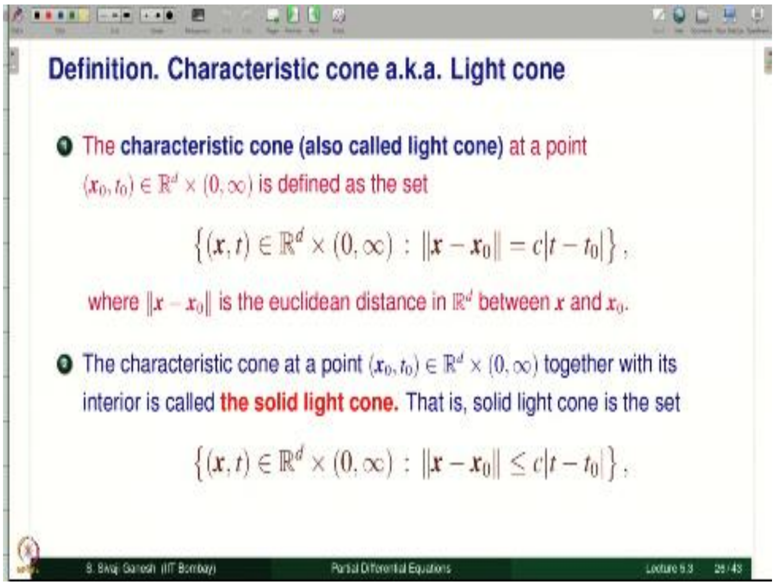

The characteristic cone also called light cone at a point x 0 t 0 in space time is defined as the set x t in R d cross 0 infinity such that norm x – x 0 equals t times mod t – T 0 where norm x – x 0 is a Euclidean distance in R d between x and x 0. The characteristic cone at the point x 0 t 0 together with its interior, this is only the surface. This is with the interior. We are considering now that is called solid light cone that is the solid light cone is this such.

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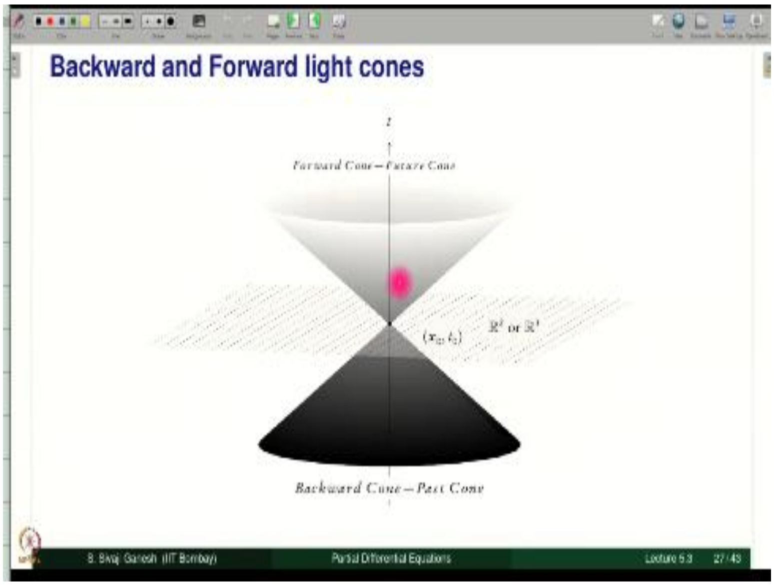

So, this is the picture. This, you can imagine R 2 or R 3. It is easy to imagine R 2 or R3; you cannot really imagine because the picture will be in 4 dimensions, but imagine this is R 2, then this is what is called forward cone or future cone. This is the backward cone or past cone. This is what x which is x 0 t 0.

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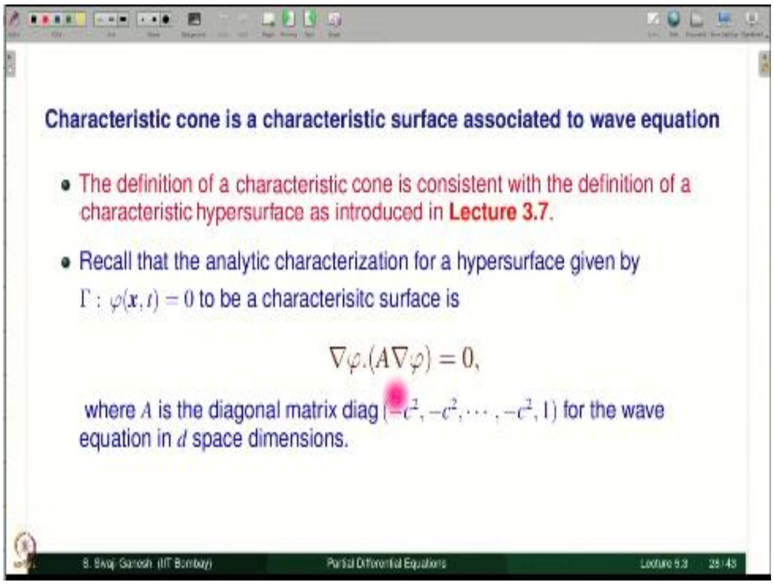

So, the definition of a characteristic cone is consistent with the definition of a characteristic hypersurface which we have introduced in lecture 3.7. Recall that the analytic characterization for a hypersurface given by phi of x t = 0 to be a characteristic surface is a grad phi dot A grad phi = 0, where A is the diagonal matrix; – c square, – c square, – c square, t times and 1, further wave equation and d space dimensions.

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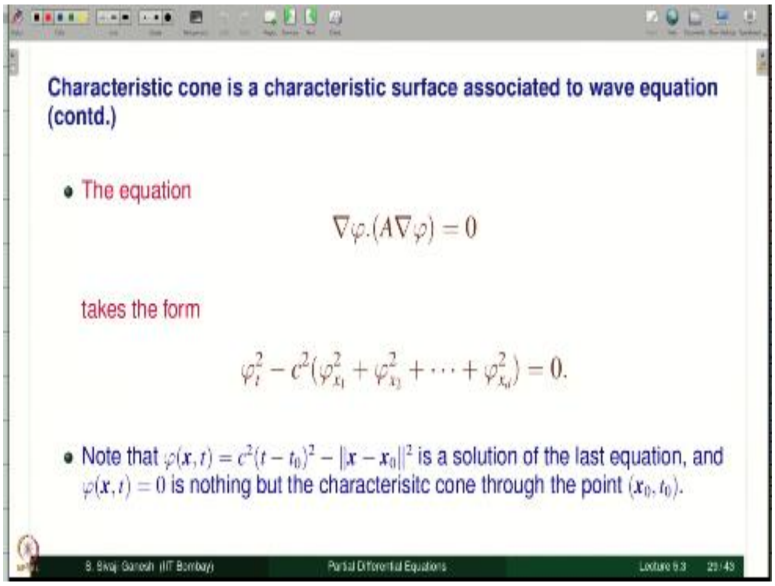

So, this equation takes this form when we expand this gradient is in x t. And what is A? It is a diagonal matrix described on the last slide. So, when we do that, we get this expression. It is phi t square – c square into norm grad phi square = 0; so, this equation is nothing but phi t square – c square mod grad phi square = 0 or we can put norm grad c square if you want or phi t is equal to plus or minus c mod grad phi.

Note that this function which is here is a solution to the last equation. You can substitute and check. Now, what is phi of x t = 0 represent? It is nothing but the characteristic cone to the point x 0 t 0.

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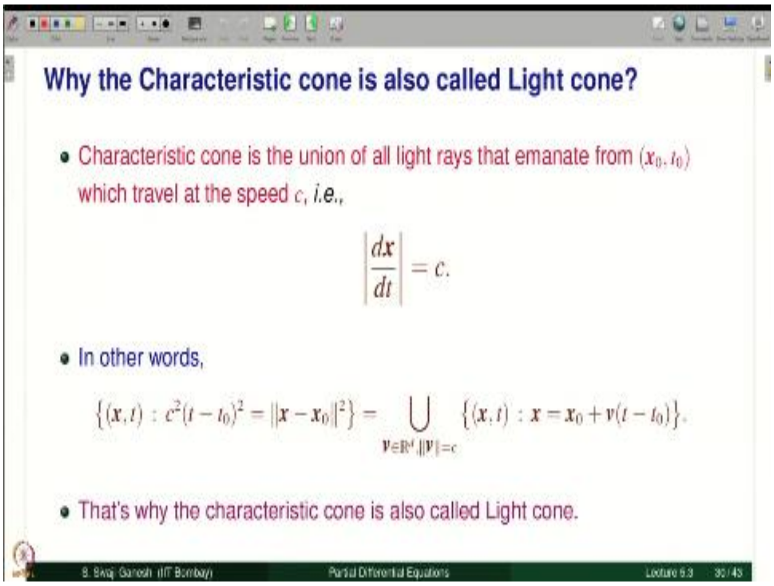

So, why the characteristic cone is also called light cone? Characteristic cone is a union of all light rays that emanate from the point x 0 t 0 which is travel at the speed c that is mod dx by dt = c, this is the speed, expression for the speed. In other words, this set which we have here is nothing but union of these sets. What is this? These are line passing through the point x 0 at t 0 because when t = t 0 in the direction v.

So, take all this direction or the vectors with this length c and then this is precisely that. So, both the sets are same. That is why the characteristic cone is also called light cone.

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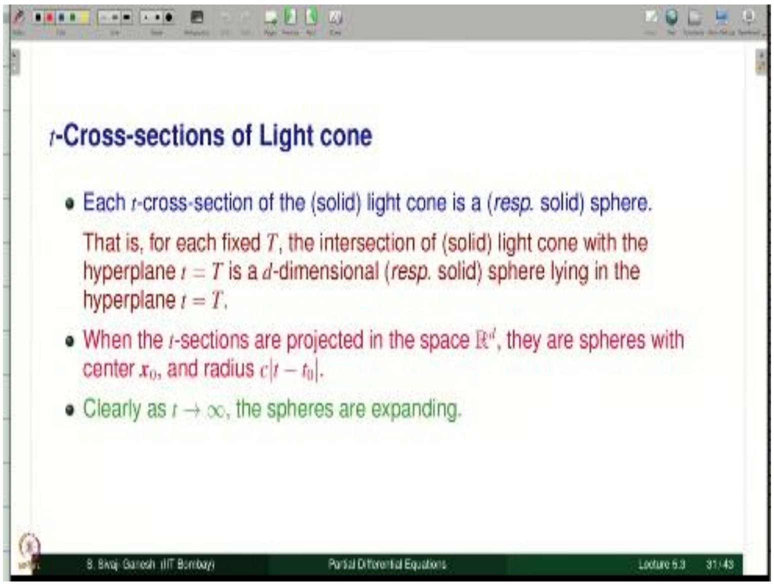

Speed is c. t cross sections of light cone, what they are? Each t cross section of the solid light cone will be a solid sphere. If you omit solid here, light cone will be sphere. Solid sphere means the interior is included that is for each fix T, the intersection of light cone with a hyperplane t = T is a d dimensional sphere lying in the hyperplane t = T that the sphere lying in d dimensions, not d dimensional sphere.

Sphere will be one dimension less. If you consider a solid sphere Yeah, so, let us not discuss that. It is the sphere lying in R d that is what the sentence means. When the t sections are projected in the space R d, they are the spheres with center x 0 and radius c times mod t – t 0. Clearly, as t goes to infinity, the spheres are expanding.

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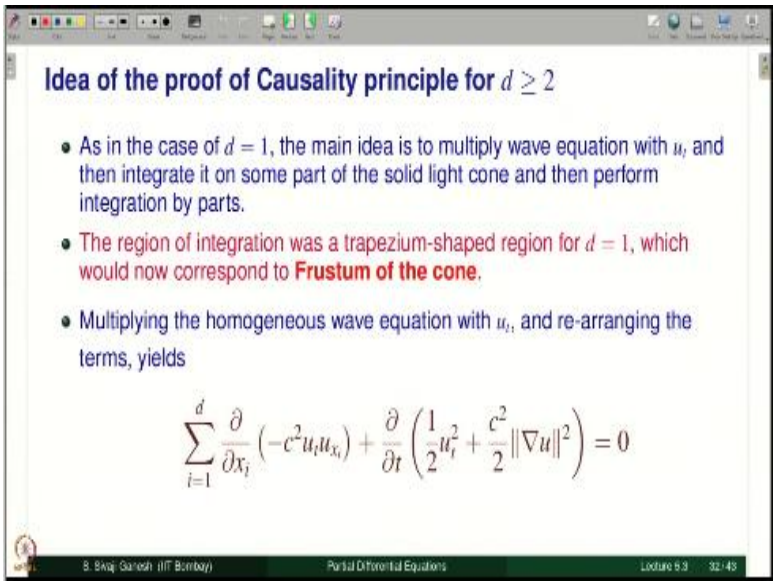

So, what is the idea of the proof of causality principle in d greater than or equal to 2? As in the case of d = 1, the main idea is to multiply the wave equation with u t and then integrate it on some part of the solid light cone and then perform integration by parts. The region of integration was a trapezium shaped region for d = 1 which would now correspond to frustum of the cone.

That is the reason why we use the notation F to denote the trapezium shaped region in d = 1, frustum of the cone. So, imagine, this kind of cone so, you cut this, so you have this. So, this is the frustum of the cone. So, we have a bottom portion; you have a top portion and you have a lateral portion. So, multiply the homogeneous wave equation with u t and rearrange exactly in one dimension we have done. So, we get this.

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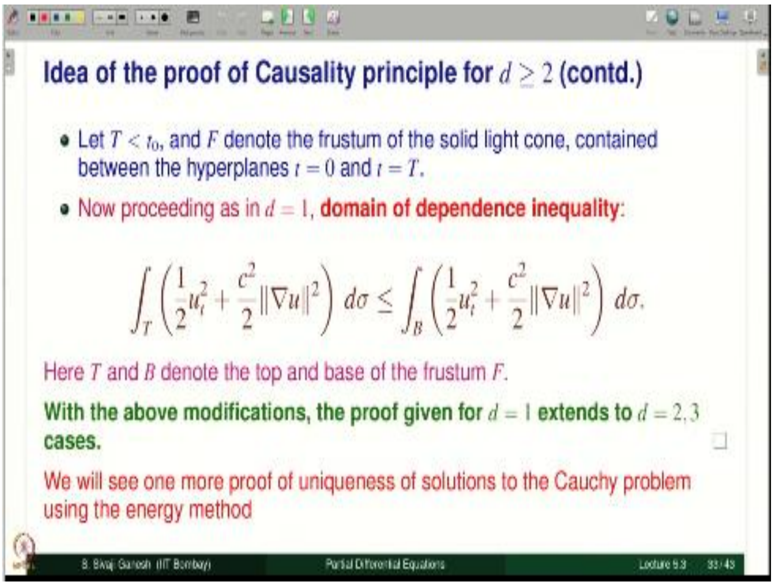

And then integrate on the first mod of cone defined by t = 0, this is the bottom portion; t = T is the top portion. Now, preceding exactly as in d = 1, the domain of dependency inequality we obtain that is exactly the same integral over T is less than or integral or B. From here, the uniqueness of solutions to Cauchy problem follows again. You take u and B to be solutions.

Even for the non-homogeneous Cauchy problem, subtract u – v as we call it as w that will satisfy the homogeneous wave equation with the homogeneous Cauchy data which means that integral on v 0 on bottom that function and the derivative with respect to t will be 0. Therefore, we have this is 0 and this is true for every arbitrary T and therefore, in the first term, both of them coincide and hence, even at the point x 0 t 0, same proof.

So, with these modifications, the proof given for d = 1, it goes through for d = 2, 3 also. We will see one more proof of uniqueness using energy method later on.

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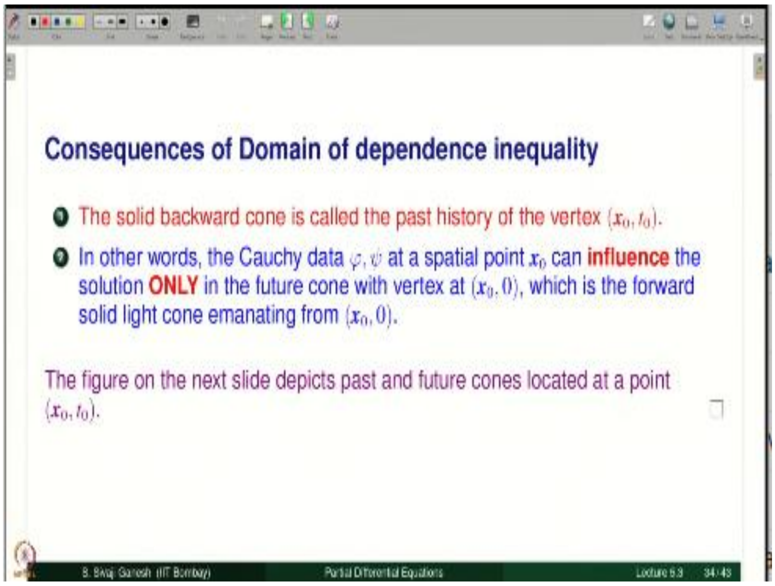

So, what are the consequences of domain of dependency inequality? The solid backward cone is called the past history of the vertex x 0 t 0. So, if this is x 0 t 0, the past cone we said, is less. So, this is the past and this will be the future of t 0. In other words, the Cauchy data at a spatial point x 0 can influence the solution only in the future cone with vertex at x, 0 which is forward solid light cone emanating from x 0, 0.

The figure on the next slide depicts past and future cones located at a point x 0 t 0. So, we have already seen this picture.

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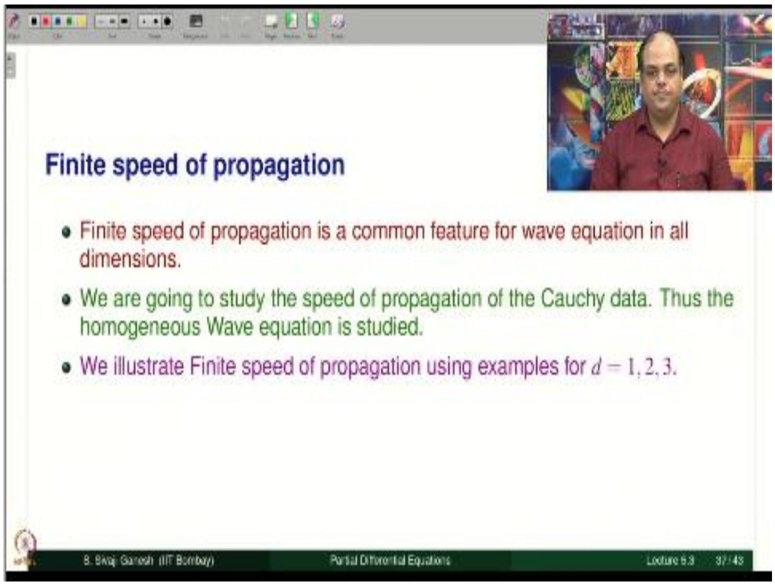

So, finite speed of propagation. Finite speed of propagation is a common feature for wave equation in all dimensions. We are going to study the speed of propagation of the Cauchy data. Thus, the homogeneous wave equation is studied. We illustrate finite propagation using examples for d = 1, 2, 3.

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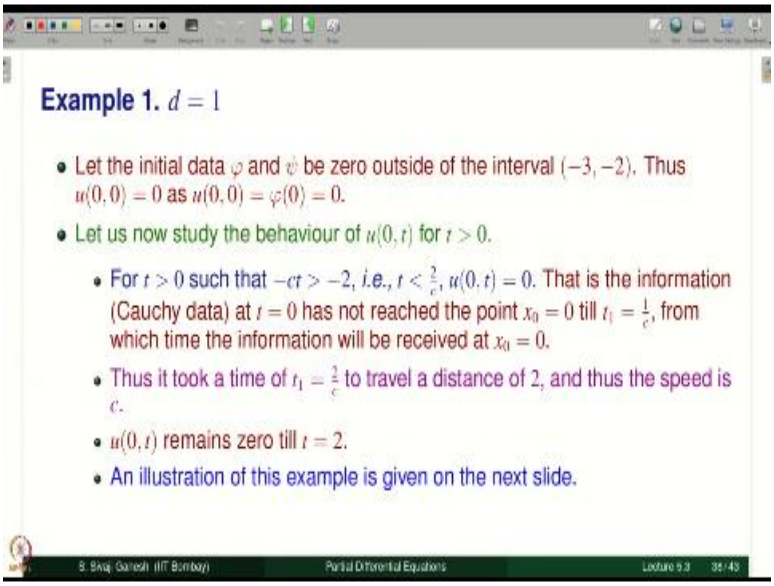

So, let us consider d = 1; let the initial data phi and psi be 0 outside this interval – 3, – 2. Therefore, u of 0,0 is 0; x = 0 t = 0 that is actually phi of 0. Zero is not in this interval therefore phi is 0. Let us now study the behaviour of u of 0 t that means I am standing at x 0 = 0, I want to study what happens for t positive. For t positive such that – ct is bigger than – 2 that is t less than 2 by c, u of 0 t will be 0.

We will see these in a picture. It will be very easy. So, that is the information at t = 0 has not reached a point x 0 = 0 till this time t 1 = 1 by c, from which time the information will be received at this point. Thus, it took a time of 2 by c to travel a distance of 2. Why the distance of 2? This interval is at a distance of 2 from the point x 0 = 0 and thus, the speed is c. u of 0 t remains 0 till t = 2. An illustration of this example is given on the next slide.

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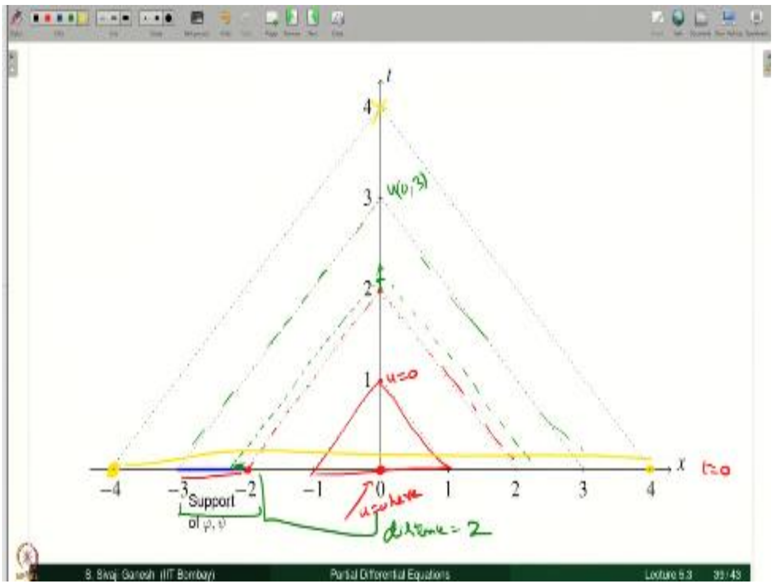

So, here, we are standing at this point x 0 = 0. Right now, at time t = 0 because this is t = 0; the information is only here in this interval – 3, 2 outside that phi and psi of 0. Therefore, u is 0 here. Let us consider this instant time instant one, then also if you see from our formula, this does not intersect the interval – 3 to 2, therefore, u is still 0 here. When you go to 2 that is when you pick up some information from here from the interval, it has reached this point 0 at time t = 2 because it is hitting this point.

Possibly, phi is nonzero here who knows but because the support of phi and psi 0, it will be 0 only. Here, phi of 0, phi of – 2, psi of – 2 will be 0. So, till time 2, you will not reach. The moment you cause time 2, 2 plus something this time, then definitely you are intersecting this piece, this side you will get nothing; this side, a anyway phi and psi 0. So, you may pick up some information from here that means information from this interval – 3,– 2, where the support of phi lies is reaching the point x 0 = 0 which is at a distance of 2; distance is 2.

It takes time 2 time 2 units. So, this example is with c = 1. So, speed is 1, distance is 2, so, you take 2 units of time to reach information. So after 2, the information starts coming. For example, if you are at 3 as you see here, the domain dependence for u of 0, 3 contains – 3, – 2 intersection on empty In fact, in this example, it contains. Now, if you are at this point 4, time 4 at this point, you see that this is interval; definitely in this, you have the support of phi and psi. So, information has reached there.

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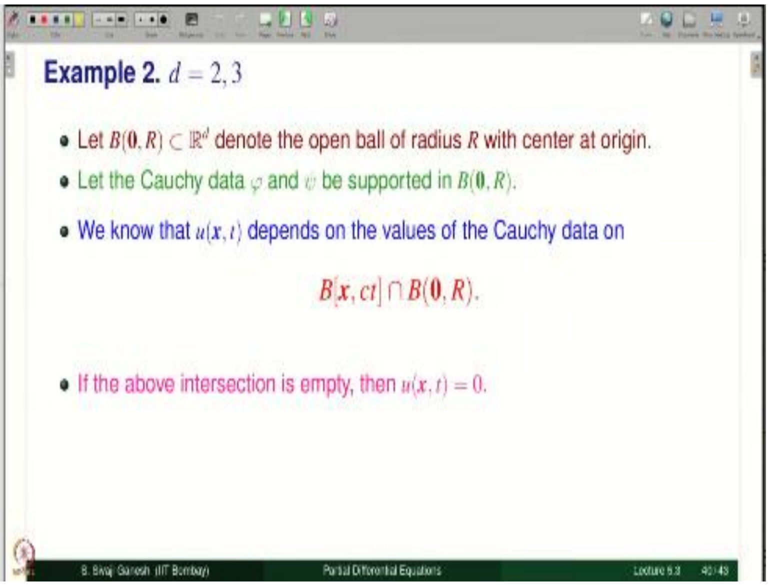

So in d = 2, 3, let us look at. Consider a ball B of 0 R, centre 0, at the origin and radius R, let it do not open ball of radius R with central origin, suppose that the Cauchy data is supported inside this ball. We know that u of x t depends on the value so, the Cauchy data on this B, this is a closed ball x t intersection B 0 R, only this is non-empty, we have nonzero solution. Otherwise, it will be 0. So if the above intersection is empty, u will be 0.

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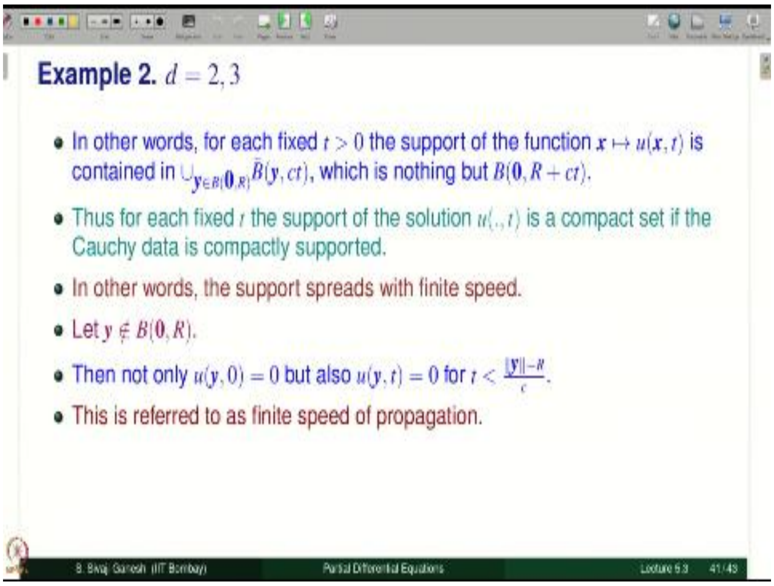

In other words, for each fix t positive, the support of the function, x going to u of x t is contained in union over y in this ball of radius R centre origin of the closed ball with centre as y and radius ct which is nothing but this ball, ball centre 0 radius R plus ct. Thus, for each fixed t, the support of the solution is a compact set if the Cauchy data is compactly supported. We have observed this in d = 1 and illustrated with the picture also.

In other words, the support spreads with finite speed, let y be 0 in this ball of radius R with centre 0, what will happen? Then not only u of y, 0 is 0, this is the initial time, phi and psi are concentrated inside v 0 R; outside that phi and psi are 0. Why is the point outside that? Therefore, u of y 0 is what? It is phi of y and y is not in this ball, therefore, this is 0. But also u of y t will continue to be 0 for all times up to norm y – R by c. This is referred to as finite speed of propagation.

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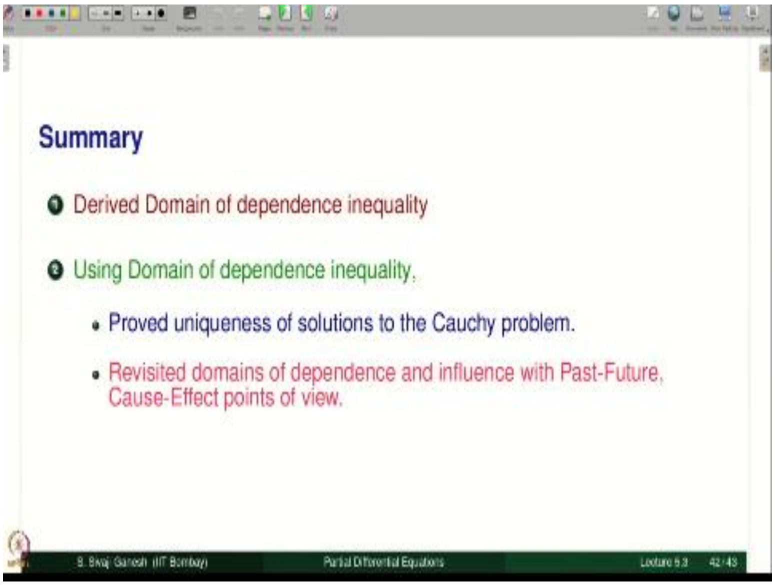

So, what we have done in this lecture is we have derived the domain of dependency inequality for d = 1. For the d = 3, we just gave the idea. So, using domain of dependency inequality, we proved uniqueness of solutions to the Cauchy problem. The second proof of uniqueness and we revisited domains of dependence and influence with past-future, cause effect points of view. Thank you.

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