Partial Differential Equations
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Lecture – 5.2
Qualitative Analysis of Wave equation - Domain of Dependence, Domain of Influence

Continuing with our qualitative study of the wave equation, today we are going to introduce 2 properties which we may say are exclusive to hyperbolic equations and hence in the second order equation that we are going to study namely wave equation heat equation Laplace equation. So, these properties are exclusive to wave equation and they are known as domain of dependence and domain of influence we will discuss more on that in this lecture.

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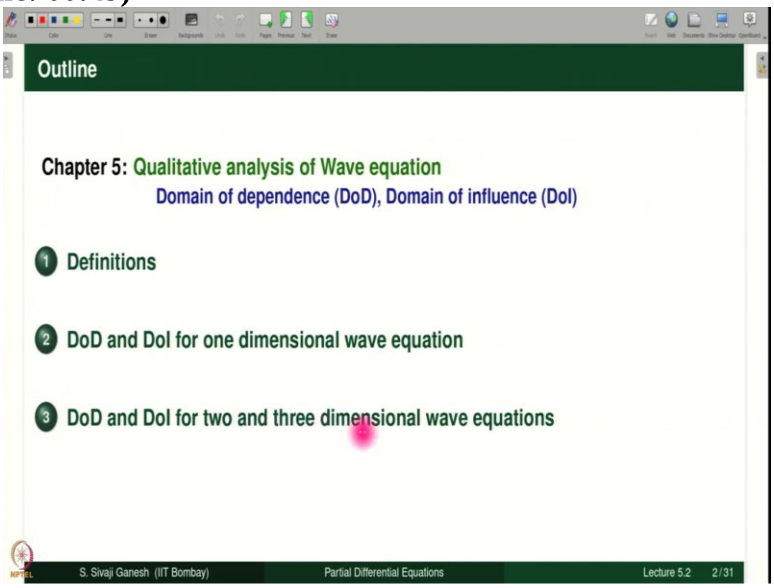

So, the outline for todays lecture is first we start with defining what is domain of dependence and domain of influence and then in short I write DoD and DoI. So, we find out what they are for one dimensional wave equation and then what they are for 2 and 3 dimensional wave equations.

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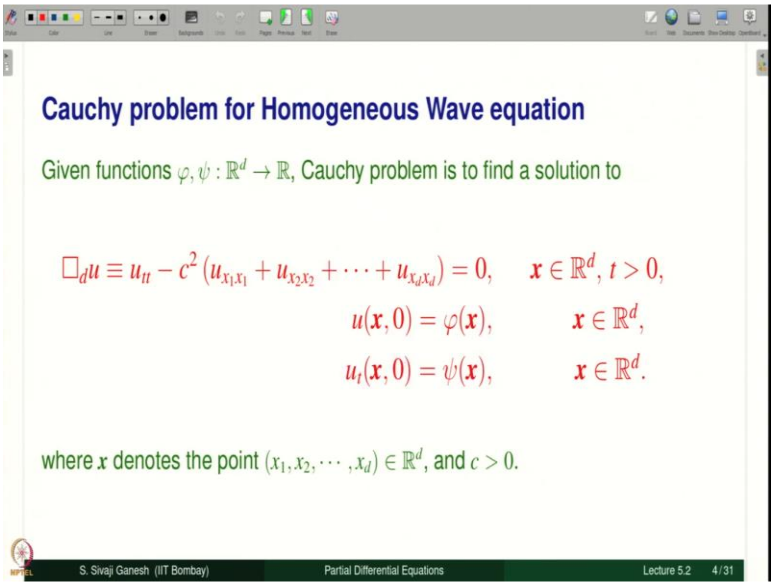

So, Cauchy problem for homogeneous wave equation is given by this the wave operator or D'Alembert's u = 0 that means, we are considering homogeneous wave equation and this is a Cauchy data u x 0 = phi x u t x = phi x. So, we have already solved this Cauchy problem in earlier lectures.

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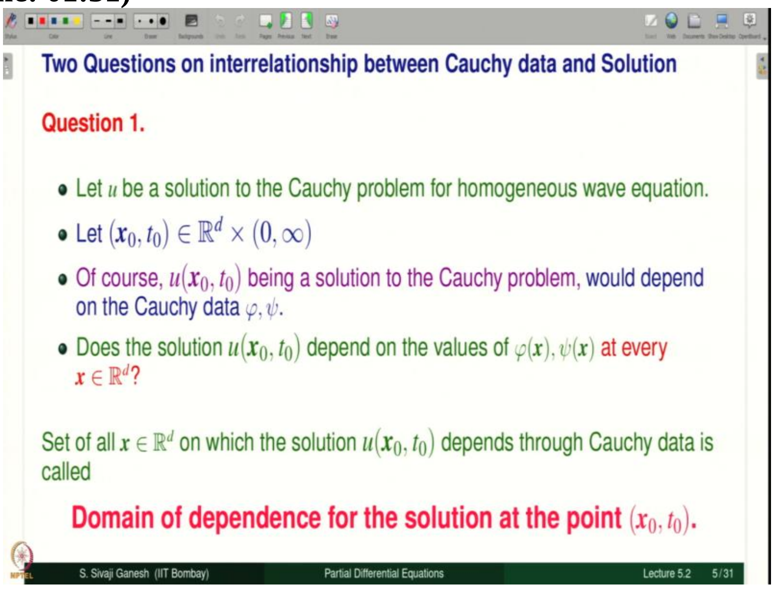

So, there are 2 questions on interrelationship between the Cauchy data and solution. So, we are going to post those questions now, question 1 let u be a solution to the Cauchy problem for homogeneous wave equation let x naught t naught be a point in R d cross 0 infinity sometimes we say x naught t naught is a space time point in the space time. Of course, u of x naught t naught being the value of the solution at the point x naught t naught to the Cauchy problem it would depend on the Cauchy data that is not a surprise.

So, now, the question is does the solution depend on the values of phi x and psi x at every x in R d. That is a question if the answer was yes or maybe then we will not be devoting a lecture to this topic. Therefore, answer is not going to be all x in R d there will be a specific domain in R d we will soon see that so, set of all x in R d on which the solution u of x naught t naught depends through Cauchy data is called domain of dependence for the solution at the point x naught t naught.

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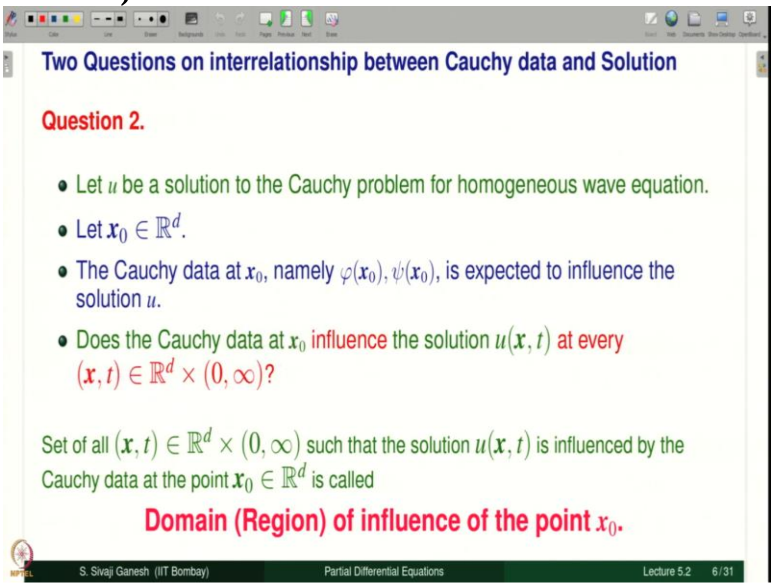

Let us move on to the second question let u be a solution to the Cauchy problem for homogeneous wave equation let x naught be a point R d the Cauchy data at x naught recall the Cauchy data of phi and psi are defined for x in R d. So, the Cauchy data at x naught namely phi x naught and psi x naught is expected to influence the solution. Now, does the Cauchy data at x naught influence the solution u of x t at every x t in R d cross 0 infinity?

Answer is set of all x t in R D cross 0 infinity such that the solution u of x t is influenced by the Cauchy data at the point x naught in R d is called a domain of influence or region of influence of the point x 0. So, here once again answer is going to be not every x t R d cross 0 infinity; the Cauchy data x naught is going to influence.

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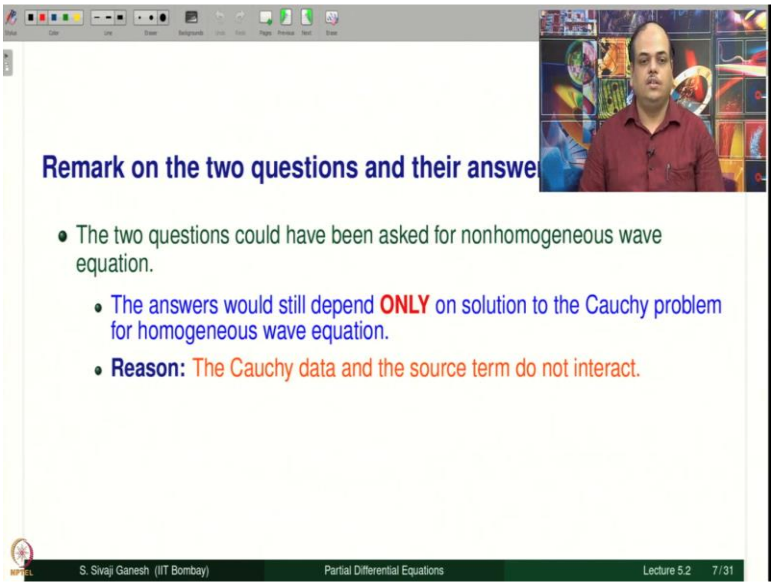

Remark on the 2 questions and their answers. The 2 questions could have been asked for non-homogeneous wave equation also answer would still depend only on the solution on the Cauchy problem for homogeneous wave equation, why is that the Cauchy data the source term do not interact recall the formula that we have derived for the solution of Cauchy problem to a non-homogeneous wave equation.

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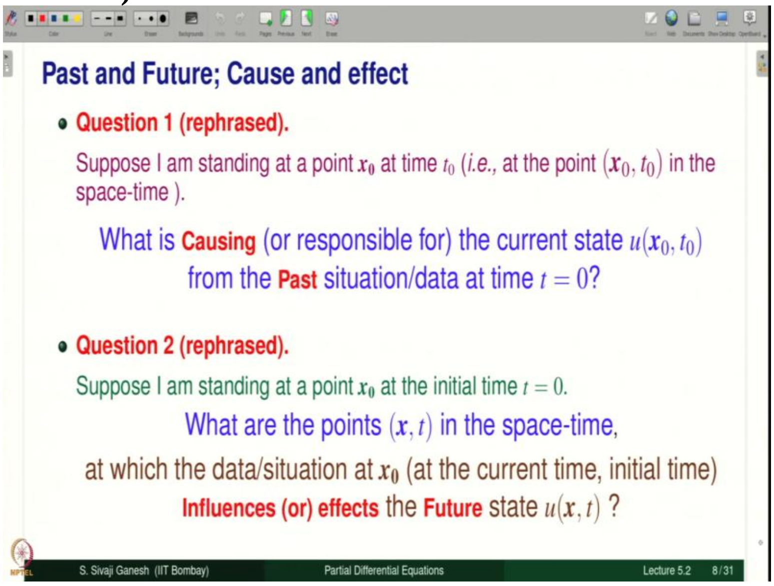

So, past and future cause and effect. These are the 2 points of view that we can present domain of dependence and influence question 1 let us rephrase it. Suppose I am standing at a point at x 0 at time t 0 that is at the point x naught t naught in the space time what is causing or responsible for the current state u of x naught t naught from the past situation or data or time t = 0 questions 2 rephrased. Suppose I am standing at a point x 0 in R d at the initial time t = 0

What are the points x t in the space time at which the data R situation at x 0 a time t = 0 that is at the current time or initial time influences or effects the future state u of x t. So, cause and effect, past and future, dependence and influence.

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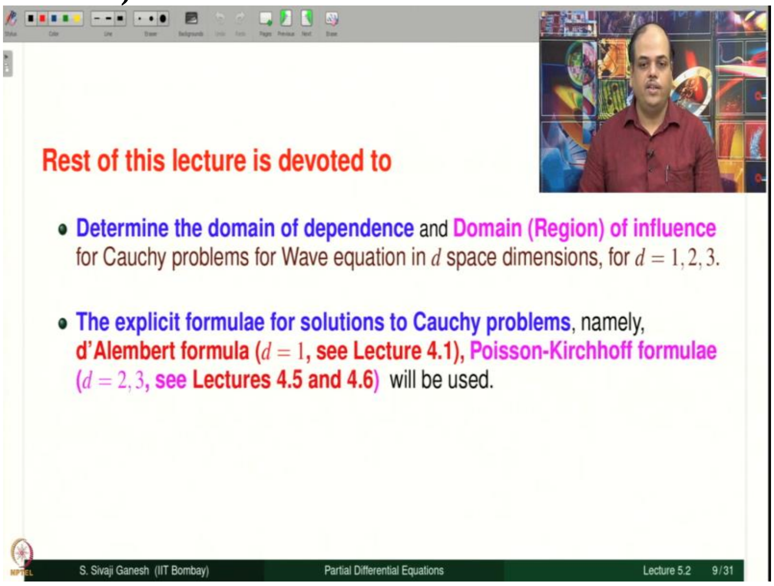

The rest of this lecture is devoted to determine the domain of dependence and domain or region of influence for Cauchy problems for the wave equation in d space dimensions d = 1, 2, 3 the explicit formulae for solutions to Cauchy problems namely D'Alembert's formula for d = 1 for Poisson Kirchhoff formula for d = 2 and 3 they will be used the formulae will be used. So, let us move on to 1 dimensional wave equation and find out what is the domain of dependence.

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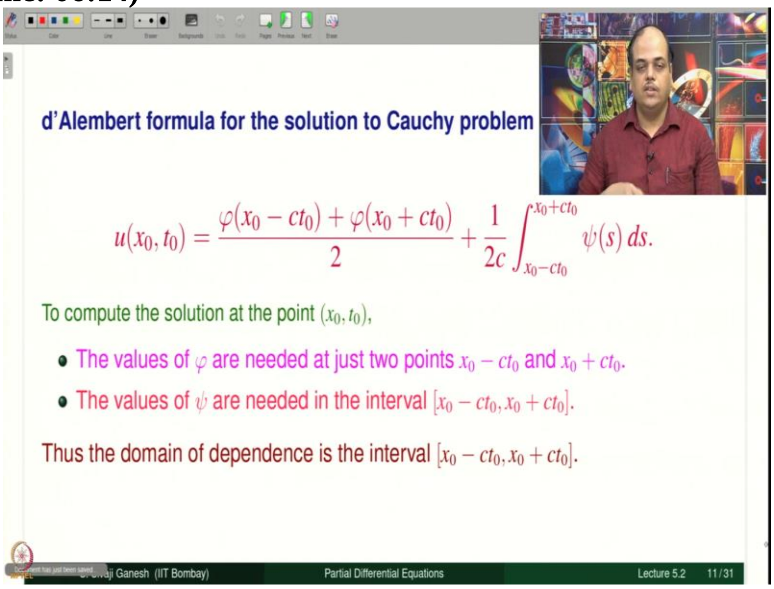

D' Alembert's formula for the solution to a Cauchy problem is given by u of x naught t naught = phi of x naught - c t naught + phi of x naught + ct naught / 2 + 1 / 2 c integral x naught - ct naught to x naught + ct naught psi of s ds. So, to compute the solution at the point x 0, t 0 what we need is the values of phi are needed exactly at 2 points x naught - ct naught x naught + ct naught and the value of psi or needed psi appears here it is an integral on the interval x naught - ct naught x naught + ct naught.

Therefore the domain of dependence is the interval x naught - c t naught, x naught + ct naught this is the interval which is the domain of dependence for the solution at the point x naught t naught.

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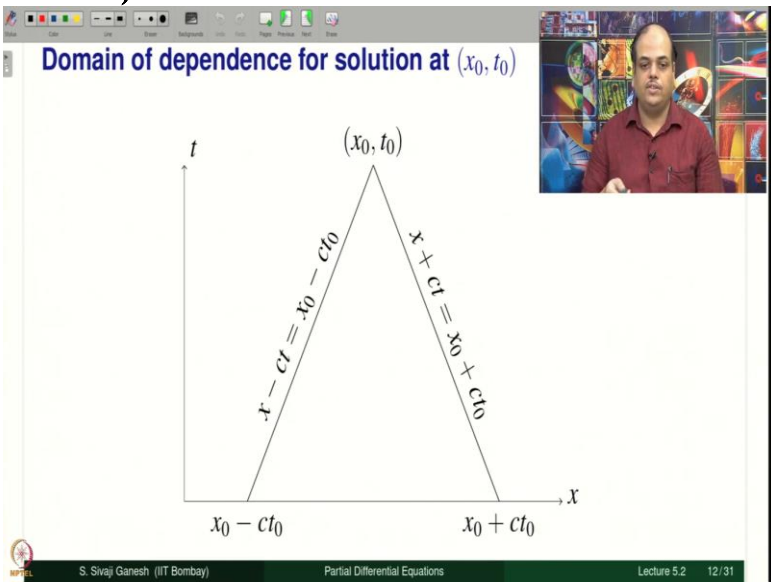

So, this is the picture here we have the point x naught t naught this is a interval on the x axis x naught - ct naught x naught + ct naught if you notice x naught - ct naught is nothing but the line through x naught t naught the characteristic line given by x - ct = x naught - ct naught where it touches the x axis is precisely x naught - ct naught, 0 we are not writing that we just write x naught - ct naught similarly, x naught + ct naught is the point of intersection of this x axis and this characteristic x + ct = x naught + ct naught. So, this is the interval on the x axis on which the value of the solution at x naught t naught depends.

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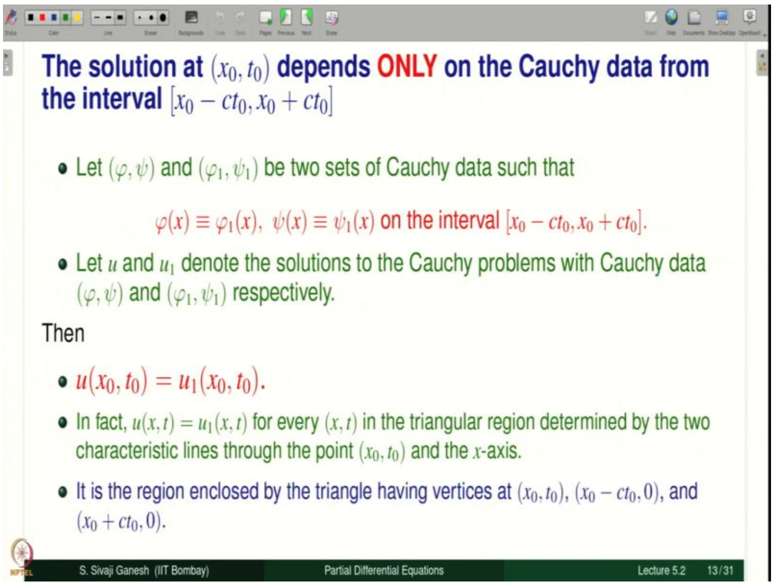

So, the solution at x naught t naught depends only on the Cauchy data from the interval x naught - ct naught, x naught + ct naught what do we mean by this suppose I take 2 sets of Cauchy data phi, psi and phi 1 psi 1 so, that they agree on this interval x naught - ct naught, x naught + ct naught that is phi is identically = phi 1 psi is identically equals psi 1 on this interval let u and u 1 denote the solutions to the Cauchy problems for the wave equation with this Cauchy data phi psi and phi 1 and psi 1 respectively.

Then you have x of x naught t naught = u 1 of x naught t naught in fact you have x t = u of x t for every x t in the triangular region determined by the 2 characteristic lines through the point x naught t naught on the x axis example, this is the point x naught t naught this is x naught - ct naught is x naught + ct naught suppose, I take a point which is inside somewhere here of course, we know the value at this point will depend on this intervals value but on this interval phi and psi are equal. Therefore, this holds for any arbitrary point that you take the solution will be the same with both Cauchy data it is not going to change.

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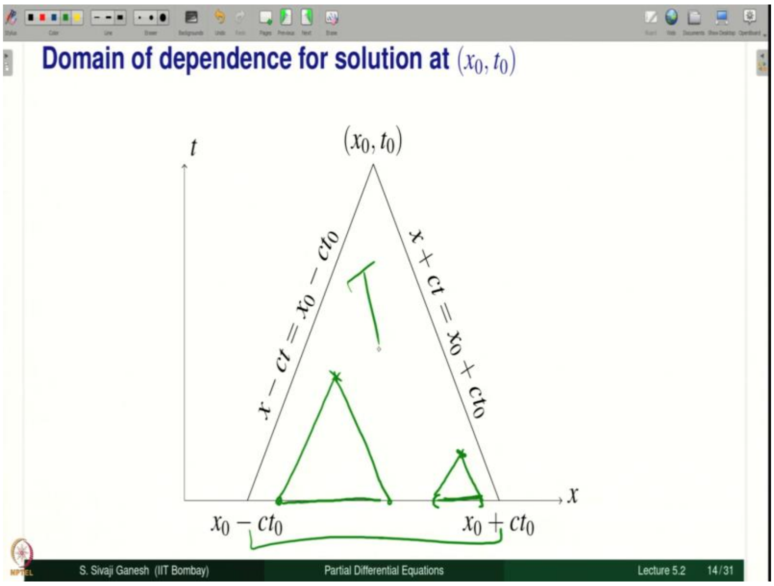

So, this is the domain of dependence picture once again x naught - ct naught x naught + ct naught. So, I have already demonstrated suppose I take a point here then the solution at that point depends on the values of the Cauchy data on this interval. So, if you take a point here these are the characteristics passing through this point and very touches on that is all this. So therefore if the Cauchy data phi psi and phi 1 psi 1 are coinciding on this interval, then the solution will be same for both the Cauchy data that is u and u 1 coincides on this triangular region.

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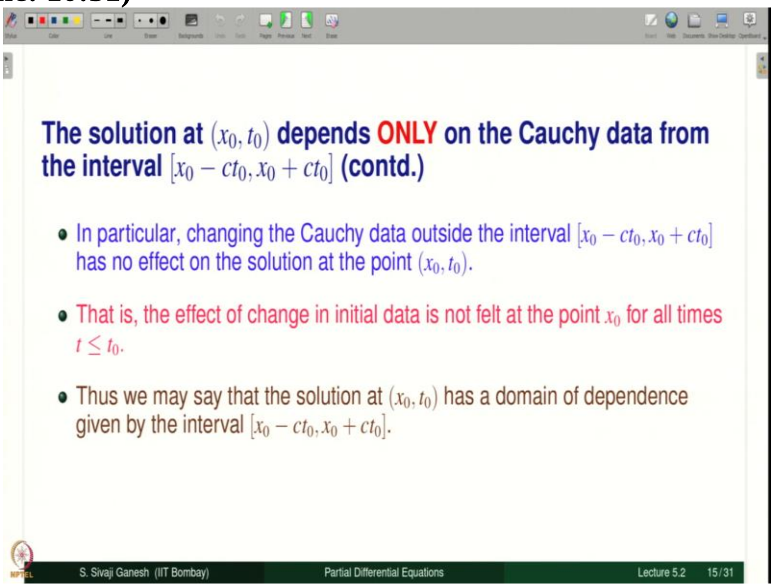

In particular, changing the Cauchy data outside the interval x naught - ct naught x naught + ct naught has no effect on the solution at the point x naught t naught because the solution at x naught t naught depends only on the values of phi and psi in this interval therefore, if you change it outside does not matter that was what was proved by considering the 2 Cauchy data that we considered phi psi and phi psi 1 which are agreeing on the interval x naught - ct naught x naught + ct naught.

Therefore, solution is the same at x naught t naught outside this interval phi psi may not be the same as phi 1 and psi 1 that does not play any role at all. So, that is the effect of change in initial data is not felt at the point x naught for all times t less than or equal to t naught let us have a look at it again. So, this is the point x naught t naught what, does it mean this a point 0 is x naught - c t naught x naught + ct naught suppose I am standing at the point x naught.

This is a t direction t direction suppose I am standing at some time t = capital T then I am at this point at this point solution is here it depends only on this interval. So, up to this time it will depend only on the values here suppose you cross this time on stand here then yes this part will be new write this piece this this part will be new and here phi = phi 1 psi = psi 1 but here phi may not be equal to phi 1 may not be true.

So, therefore the solution at this point u of x let us call this point as x naught, t 1. So, u of x naught, t 1 may not be same as u 1 of x naught, t 1 because in this piece and in this piece phi and psi we have no information whether they coincide or not. So, that we may say that the solution at x naught t naught has a domain of dependence given by this interval.

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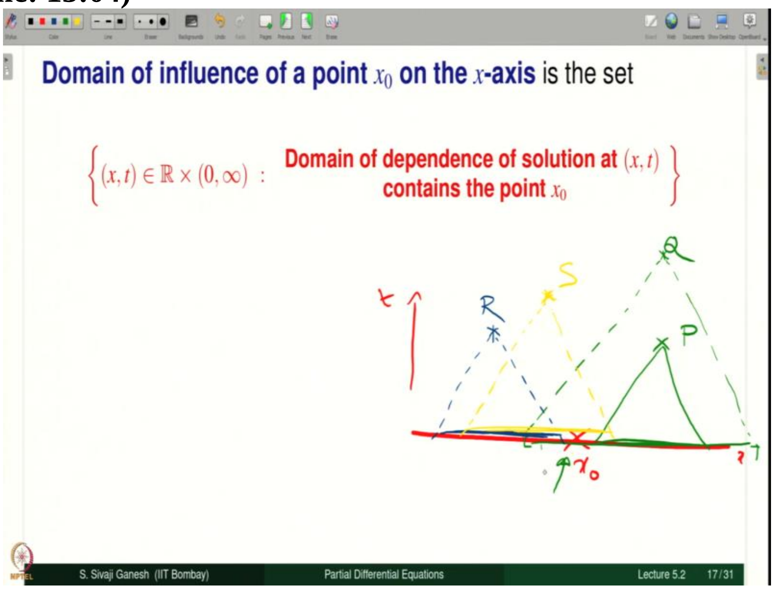

So, let us look at the domain of influence of a point x naught on the x axis it is this side x t in R cross 0 infinity such that the domain of dependence of solution at x t contains the point x naught. So, what is that suppose this is my x these t directions suppose I am a point x 0 now, what is the domain of influence of x 0 it contains those points x t such that the its domain of dependence contains the point x naught.

Let us consider a few points and see, let us take a point P the solution at this point will depend on this interval and this is the domain of dependence for P it does not contain x 0 therefore, this P does not belong to domain of influence of x 0 for example, I am at this point in first time. Now, the domain of dependence for this new point Q is this interval and x naught falls inside that.

Therefore Q belongs to the domain of influence of x 0. Let us consider one more point. Suppose I am here R the solution at this point this is the domain of dependence for R of course x naught is not in it. Therefore, R does not belong to domain of influence of x 0. Suppose I take another point here at this point if you notice in the domain of dependence for this point here is x naught belongs to. So, therefore, x naught influences the solution at S so, S belongs to the domain of influence of x 0.

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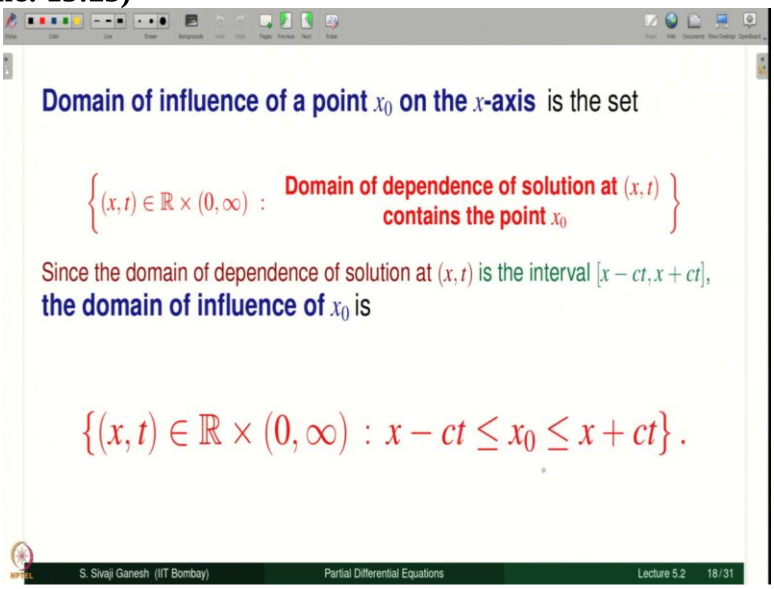

So since the domain of dependence of solution at x t is this interval x – ct, x + ct the domain of influence of x 0 is this set.

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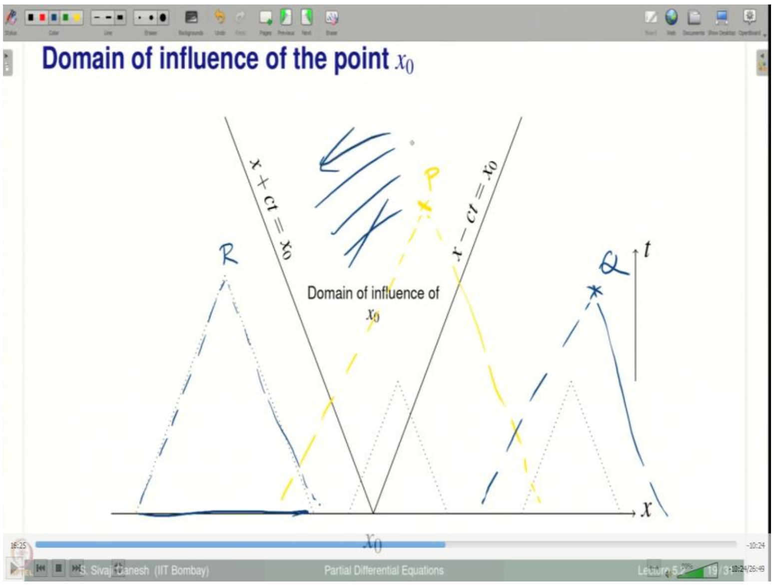

Let us look at this picture here if I take a point here of course x naught will lie in the domain of dependence for this point whereas if I take a point outside this reshaped region definitely not similarly imagine this is R exactly same problem have already written dots here the domain of dependence for R is this interval and x naught is not in that interval. So, any point is outside the V shaped region x naught will not belong to its u domain of dependence of this region and another hand any point inside in this region x naught will belong to the domain of dependence for solution at the points in this region.

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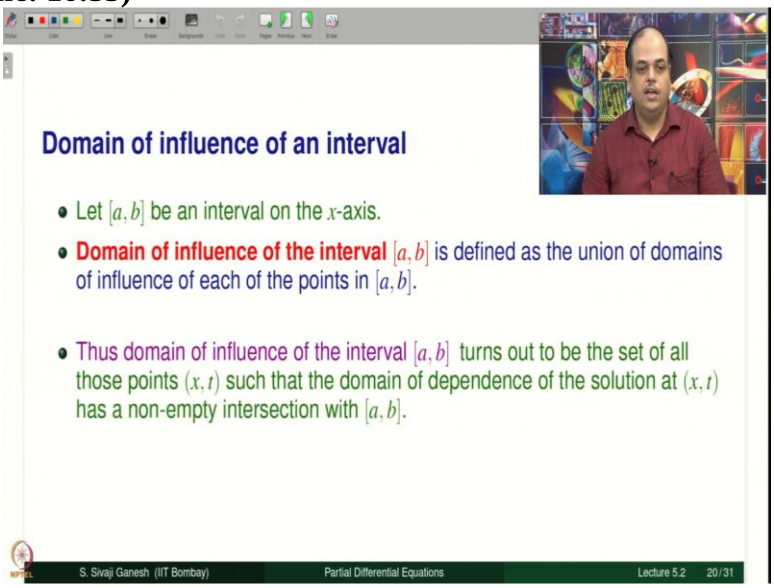

Domain of influence of an interval we have considered domain of dependence of a point. Now, we are going to consider domain of influence for the interval, how do we define that, let us take an interval on the x axis domain of influence of this interval should be the union of domain of influences of the points of interval a, b. Thus domain of influence of the interval a b turns out to be the set of all those points x t, such that the domain of dependence or the solution at x t has a non-empty intersection with a b.

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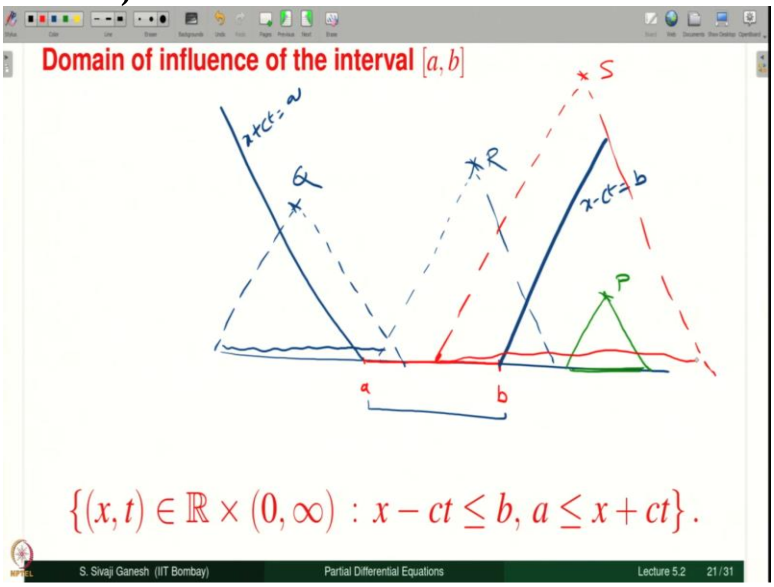

So, let us draw this line and take a F is here, a b is our interval. Now, let us find out certain things for example, I am at a point here. So, at this point, this is the domain of dependence for this point P and it does not intersect with a b. So, therefore, P does not belong to domain of influence of the interval of a b for example, I take another point at this place let us call it Q here this is the domain of dependence for Q.

And it intersects this interval a b. Therefore, the point Q belongs to the domain of influence of interval a b. Now, the region which is given here is nothing but this. This is x - ct = b these x + ct = a. So, if you take any point in this tub shaped region, let us say here, then definitely the domain of dependence names intersect a, b for this point R and if you take a point here. Let us call it S then also going to intersect the interval a b, the domain of dependence of that. So, therefore, the domain of influence of the interval is this particular set.

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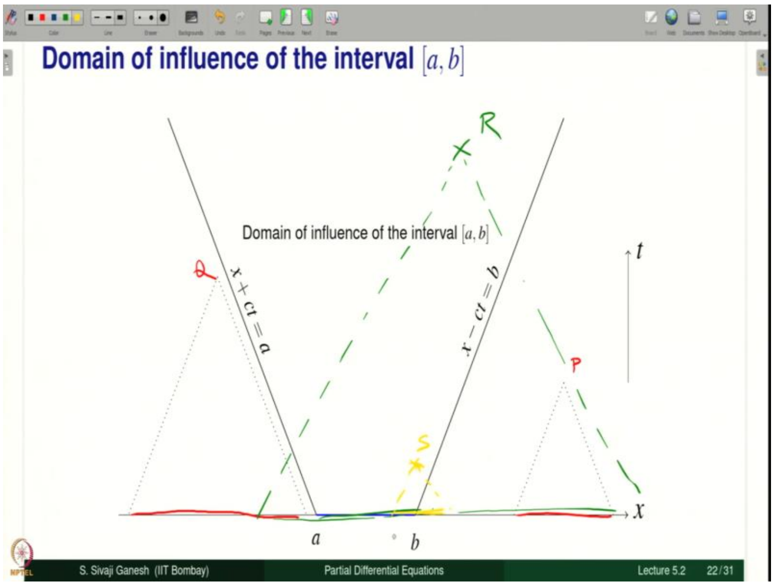

So, here once again we have this picture, if you take a point here, this is the domain of dependence for P is not intersecting a b it is the point Q, the domain of dependences here not intersecting. On the other hand, if you take a point here, R, then is going to intersect, it is much bigger than a b but definitely intersects a b and if I take a point here this is S then like that, still intersecting. So, this is precisely the domain of influence of the region a b.

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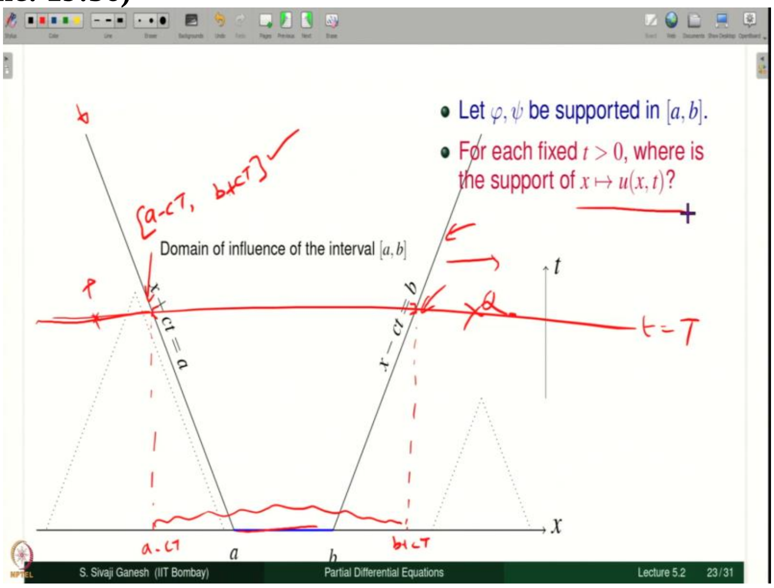

Let phi psi be supported in this interval a b can taking such a data that means phi and psi are 0 outside this interval a b. So this side phi and psi are 0 and this side phi and psi 0 for each fixed t positive where is the support of x going to u of xt so let us fix time. So here is the time. So, there is a t = some capital time T. Now, if you notice a point here, a b will not influence of this point P or any point which is to the left side of this particular line.

It will not influence therefore, solution is 0 and similarly to the right side of this line to this side, if you take any point Q u at Q is also 0. Therefore, only on this really it may be nonzero. So, therefore, the support is contained in this interval, what is this point and what is this point that you can check it is going to be an interval actually this is a, a has moved this side by time t we are going to see you can compute c a – ct.

Similarly v has moved this side to this point this is b + ct, this is a - ct. So support will be contained in this. So support might increase, of course, you see originally a b is a support for phi and psi and now the support is here. For the solution at the time t = t it might increase. But notice support is still compact. The support phi and psi inside a b it means it is a compact set support is a compact set. Now the support inside this interval it means this function has compact support.

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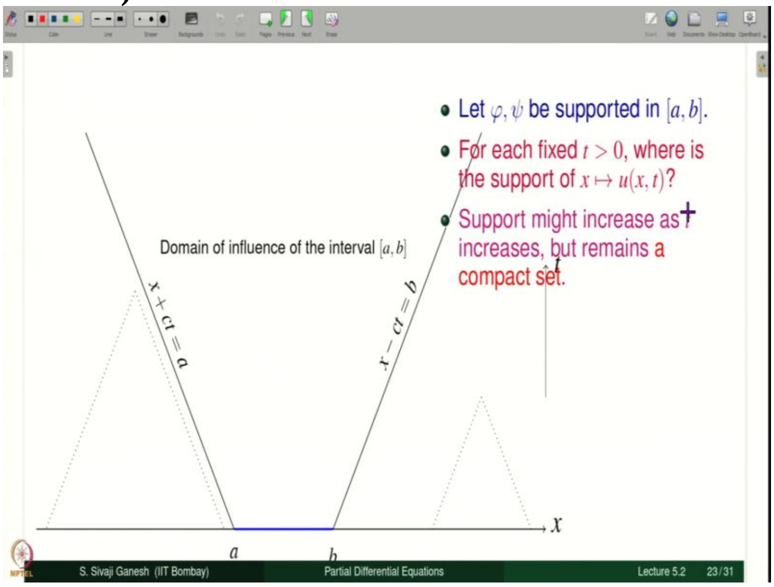

That is an interesting observation about the propagation of the initial disturbances. Let us look at the 2 dimensional wave equation and domains of dependence and influence for them.

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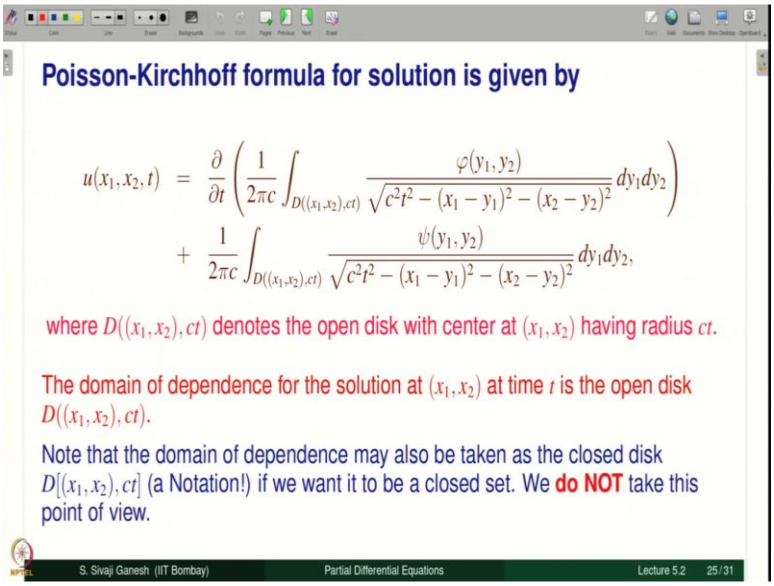

This is a formula for the Poisson Kirchhoff formula for the solution of the Cauchy problem for the wave equation in 2d. Now, if you notice the formula depends the values of phi and psi only on this disk therefore, the domain of dependence for the solution at the space time point x 1, x 2, t is this disc of radius ct with centre x 1, x 2. One may also consider close this that is not a problem.

Because it is an integration it is an integration so the boundary what is the difference between the closed disc and the open disc it is a boundary and that does not make any change to this integral it does not affect the integral. So there is no plus here there. So, if you want it to be a close set, but there is no need for asking that. So, we do not take this point of view.

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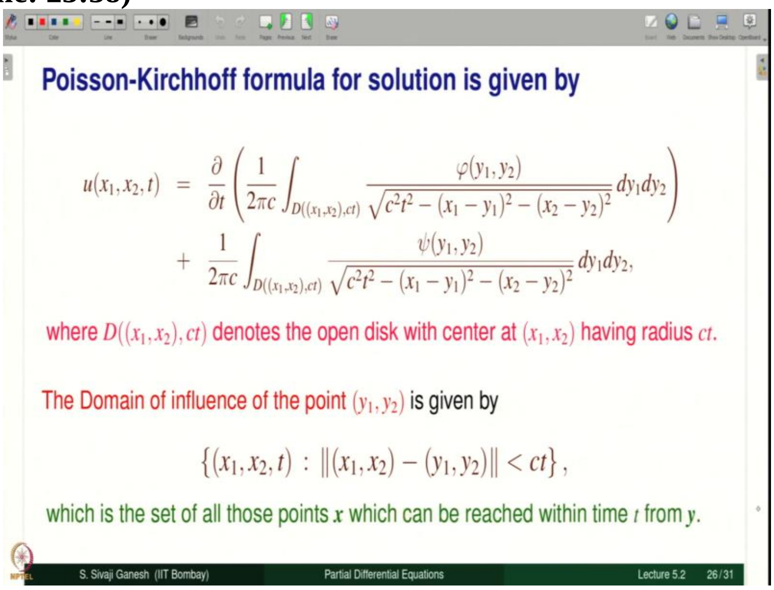

Now, what is the domain of influence that is given by a set of all space time points such that the distance between x 1 x 2 and y 1 y 2 is less than ct the y 1 y 2 x 1 x 2 distance is less than ct that is a domain of influence please convince yourself about this answer everything comes from this formula. So, this is the set of all those points which can be reached within time t from y what is this this is a distance between the bold face x and bold face y that less than ct. If you divide distance with c, so distance by speed is less than this t. So, that means you reach within, within the time t from y to x r x to y.

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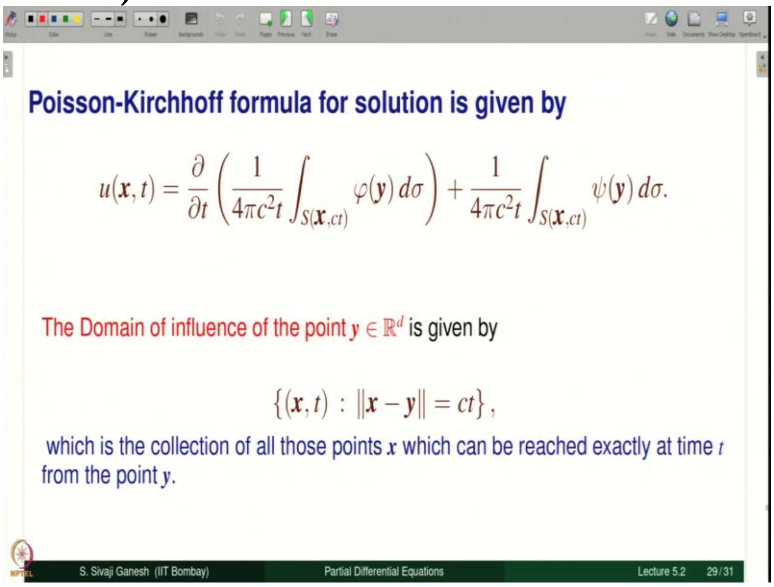

So, 3 dimensional wave equation what are the domains of dependence and influence. These formula Poisson Kirchhoff formula, s of x, ct, ct is this phi. So therefore, the domain of dependence is s of x ct because that is where the integrals are on and domain of influence of the point y in R d is now x - y = ct because of this fear earlier it was a disc that is why it was less than t. Now these phi therefore, distance is precisely ct. In other words, those points which can be reached those points x which can be reached exactly at time t from the point y.

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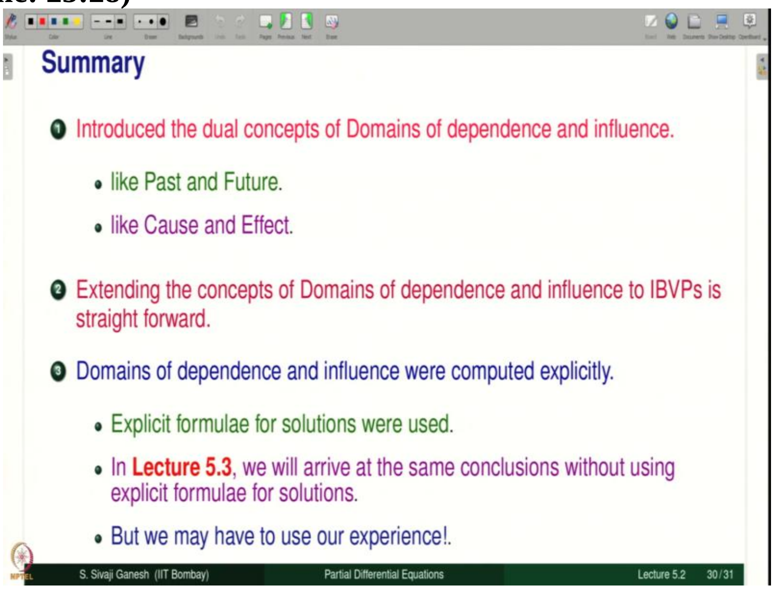

Let us summarise we introduce the dual concepts of domains of dependence and influence, like past and future like cause and effect, extending the concepts of domains of dependence and influence for IBVP straight forward. Domains of dependence and influence were computed explicitly, explicit formulae for the solutions were used. In lecture 5.3, we will arrive at the same conclusions without using the explicit formula solutions. But of course you have to use something. Thank you.