Partial Differential Equations
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Module No # 07
Lecture No # 35
Tutorials of IBVPs for wave equation

Welcome to tutorial on initial boundary value problems for wave equation in lecture 4.9 we have solved an initial boundary value with Dilichlet boundary conditions. In this tutorial we consider some more problems where we will change the boundary conditions to mix boundary conditions we change the domain or from a finite interval to semi finite interval and so on. So we are going to solve 3 to 4 problems in this tutorial.

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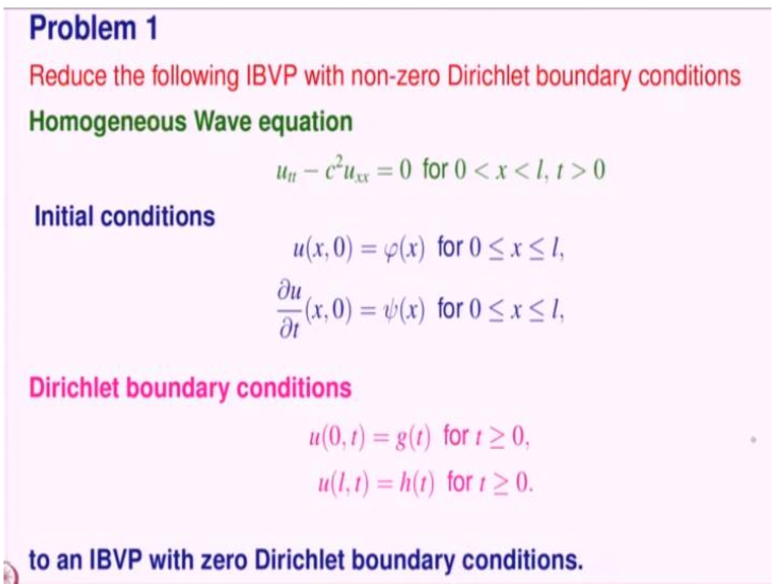

The first problem is reduce the following initial boundary value problem with non-zero Dirichlet boundary conditions here you have 0, t is g, t of l, t is h, t. This is the there is a problem in the last class we considered with g = 0 h = 0. At that time we mentioned that this problem can be reduced to IBVP with 0 Dirichlet boundary conditions so the problem is how to do it that is what we are going to discuss.

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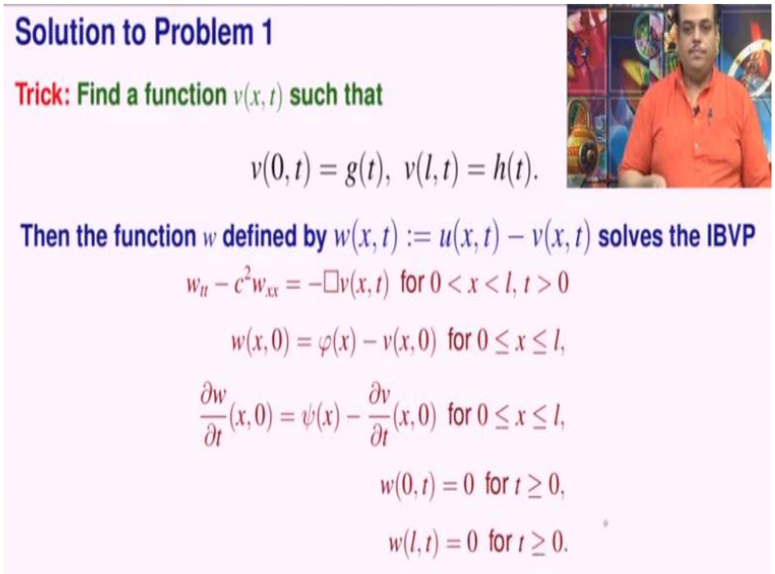

So trick is to find a function v such that it satisfies the boundary conditions v of 0, t is g t and v of l, t is h t. Suppose you find such a function then if you define a function w which is u – v where u is a solution that we want to find. And we have found a v satisfying these conditions. And if you look at u – v what problem is w solves w t, t – c square w x, x is equal to d'Alembert acting on v. Because d'Alembertion acting on u is 0 because you want to solve homornous wave equation.

Therefore minus d'Alembertian version v d'Alembertian is a linear operator therefore it distributes over u – v. Of course we have a new term but this is known term because if you know the function v you know the d'Alembertian of v. So it is a known function so we have got a wave equation which we started with had no source terms but now we have a source terms. And the initial displacement w of x, 0 is u x, o – v x, 0.

But u x, 0 should be phi x so phi x – v x, 0 similarly the initial velocity dou w by dou t of x, 0 is psi x dou v by dou t at x, 0 v is a known function. So this is a known function this also known function and the source term is also known function and we have 0 boundary conditions the Dirichlet boundary condition are become 0. Because u should satisfy G u of 0, t is G but v of 0, t is already G. Therefore G –G will be 0 so w of 0, t is 0 similarly w of l, t if you plug in here u l, t –v l, t u l , t supposed to be H and v l, t is h by constructing. Therefore the difference is 0 so w l, t is 0.

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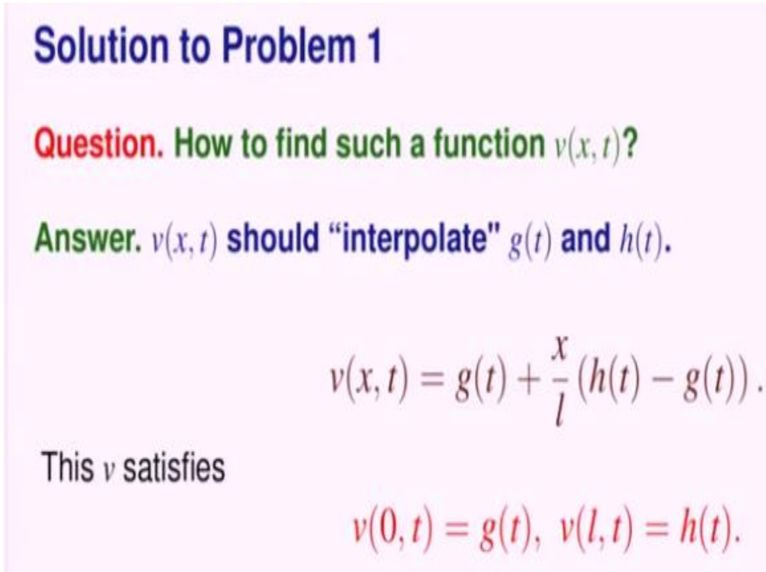

Now the question is how do we find such a function v? So v must interpolate the function v that should interpolate these 2 functions g, t and h, t. At x = 0 it should be g, t and x = l it should be h, t so the simplest function we can think of is this is also called linear interpolation. If you are familiar with terminology in numerical analysis you will immediately follow this why it is called interpolation.

So now look at v x, t when I put x = 0 this term is not there because x = 0 what I get is g, t when I put x = l, l by l is 1 so it is h – g under + g. Therefore we get h so this we satisfies what is required term and thus we have converted our problem of initial value problem to another initial value boundary problem with source term and the initial displacement and velocity have change because of this v and the boundary conditions became 0 boundary conditions that is advantage.

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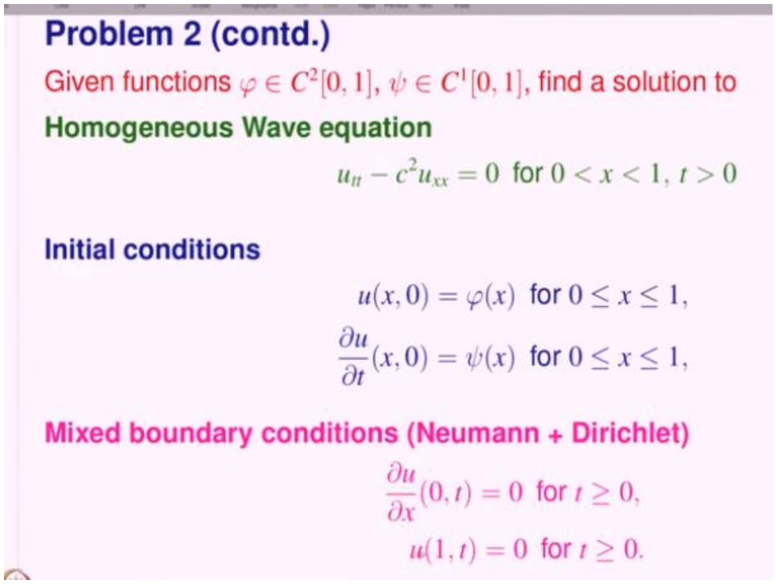

So let us look at a problem 2 in lecture 4.9 using first principles we solved in initial boundary value problem with Dirichlet boundary conditions. Now the problem 2 that we are going to discuss now is about solving an IBVO with mixed boundary conditions. We are still considering we are going to still consider finite interval 0, l in the lecture 4.9 we consider u 0, t and u l, t being prescribed. Here we consider derivative of u is prescribed at one of the boundaries and the function itself is prescribed at the other boundary.

Once again owe want to solve this using first principles so given phi which is c 2 0, 1 psi which is c1 0, 1 find a solution to the homogenous wave equation and initial conditions are as usual initial displacement is phi initial velocity is psi. Now comes to the boundary condition mixed boundary conditions because we are using derivative dou u by dou x at x = 0 that is given to be 0 and u is given at 1 x = 1 u of 1 t is given.

Of course it is given as 0 so 0 boundary conditions but the nature of the boundary conditions have changed derivative in one on the one boundary and the function on the other boundary is prescribed.

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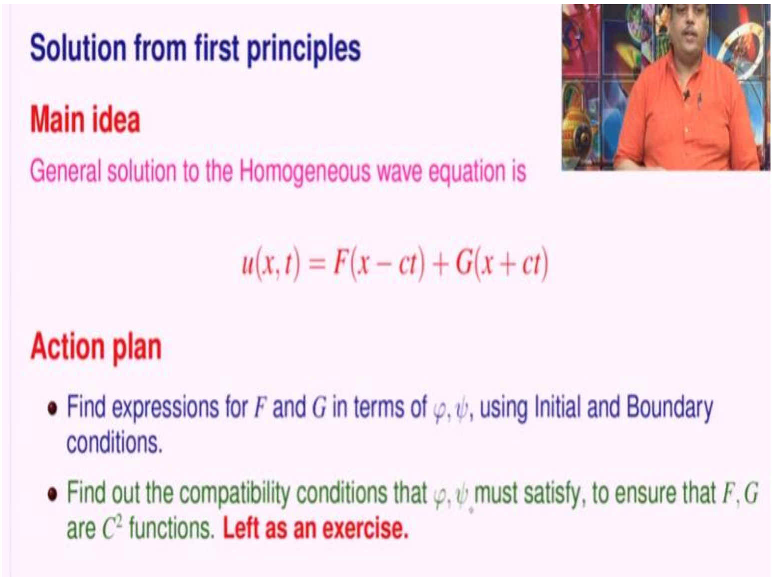

Now what is solution from first principles the idea is that u of x, t is given by F of x – ct + G of x + ct. So the plan is to find what these functions are F and G of course you want to solve the initial boundary value problem which is given by phi and psi therefore F and G we want get an expression in terms of phi and psi. Of course we have to use initial and boundary conditions for that and find out the compatibility condition that phi and psi must satisfy.

So that F that we construct and the G that we construct are actually c2 functions\ thereby u of x, t is actually a classical solution to the given IBVP. Of course that is left as an exercise to you we have exactly a same exercise with different boundary condition that is all. We have analyzed what should be the compatibility condition in that problem. Now it is similar you will see even the wave that we are going to solve the problem is similar. So you should be able to do this exercise please do, that.

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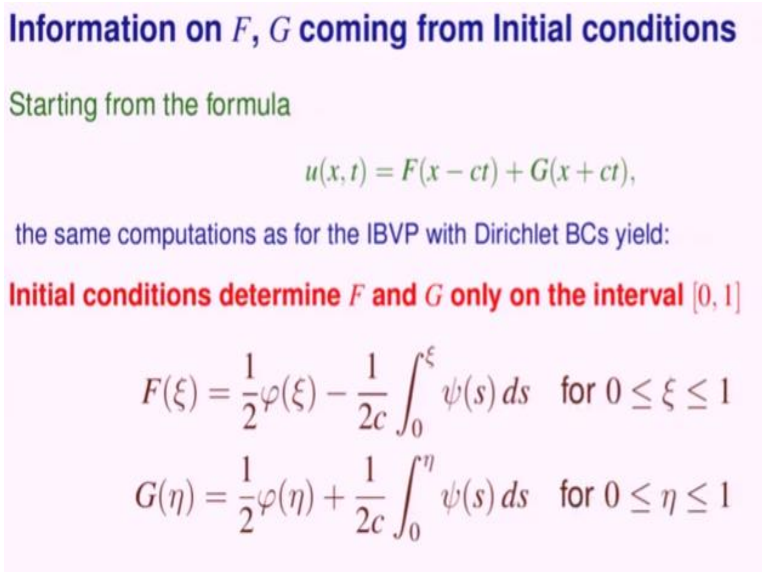

Let us start from this formula u x, t = F of x – ct + G of x + ct the same computation that we did for Dirichlet boundary conditions exactly same condition will come. If you looking at what happens to F and G are from the initial conditions? How much of F and G are determined by initial conditions? We get the same conditions I am not doing this computation because we have done this already twice.

First of all we have done in deriving d'Alembert formula for the Cauchy problem in R secondly we have done in lecture 4.9 where we solved IBVP with Dirichlet boundary conditions. So exactly same thing you get the same expressions for F and G remember the validity it is valid for psi between 0 and 1 and here eta between 0 and 1.

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Information on F, G coming from Initial conditions

Initial conditions determine F and G only on the interval $[0,1]$

$$F(\xi) = \frac{1}{2}\varphi(\xi) - \frac{1}{2c}\int_0^{\xi} \psi(s)\,ds \quad \text{for } 0 \le \xi \le 1$$

$$G(\eta) = \frac{1}{2}\varphi(\eta) + \frac{1}{2c}\int_0^{\eta} \psi(s)\,ds \quad \text{for } 0 \le \eta \le 1$$

Substituting in the formula $u(x,t) = F(x-ct) + G(x+ct)$, we get

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

for (x,t) such that $0 \le x - ct \le 1$ and $0 \le x + ct \le 1$.

So, when we substitute in this formula we get the solution in terms of x and t which is this valid for psi between 0 and 1 psi is x – ct and eta between 0 and 1 which is x + ct. So in terms of x t the validity is this.

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Information on F, G coming from Initial conditions

Initial conditions determine solution as

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

for (x,t) such that $0 \le x - ct \le 1$ and $0 \le x + ct \le 1$.

In other words, solution is determined in the region (**the region 0,0** in the figure)

$$\{(x,t) \in (0,1) \times (0,\infty) : 0 \le x - ct \le 1, 0 \le x + ct \le 1\}$$

The analysis so far is same as the one for IBVP with Dirichlet boundary conditions.

So initial conditions determine the solution in some region and that region is 0, 0 region.

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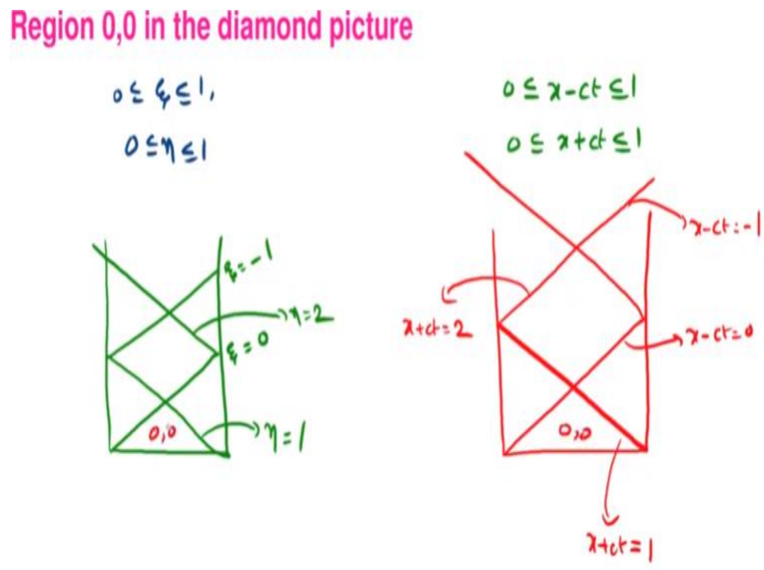

The 0, 0 region in the diamond picture in fact the diamond picture is exactly same as what we saw in lecture 4.9 it is described like this, 0 less or equal to psi less than or equal to 1 0 less than or equal to eta less than equal 1 in terms of psi eta. In terms of x and t it is described as s –ct less than or equal to 1 and x + ct less than equal to 1. Now let us draw the picture here this is the diamond picture we are talking about so on.

So this is psi = 0 this is psi =-1 these are the lines this line equal to eta equal to 1 this line is eta equal to 2. And let us I mean exactly same picture we are going to write x and t this is how it goes the picture this is the line x – ct= 0 this line is x –c t = -1 this line x+ ct = 1 this is x + ct = 2 and so on. And this is the region 0, 0.

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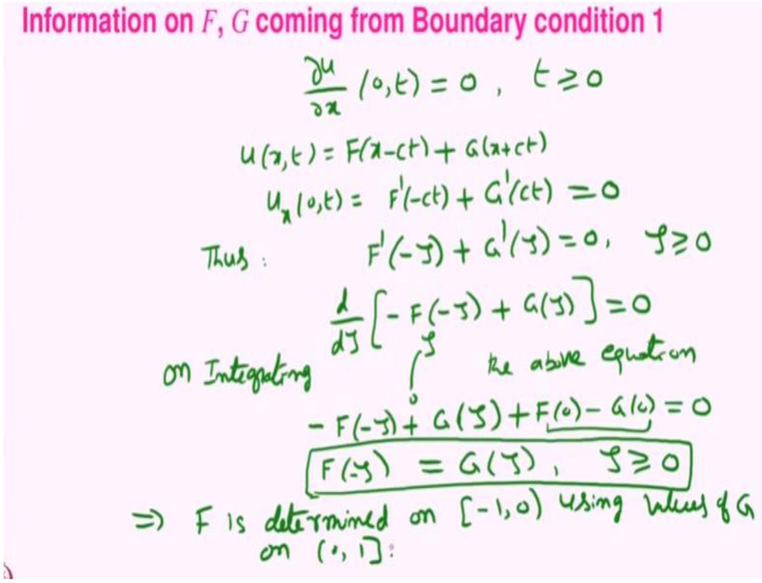

Now what do we get if we use boundary condition 1 how we have analyzed the information that we get from initial conditions. So let us see what we will get if we use boundary condition 1. What is the boundary condition 1? That is dou u by d x at 0, t = 0 this is a condition given to us. Now if we use this in this formula what is u x of 0, t it is F dash at –ct I am differentiating and then substituting x = 0.

So chain rule this equal to 0 so thus we have what do we have is? F prime of – zeta + G prime of zeta = 0 for what zeta? Greater than or equal to 0 we have to always, remember for what range of zeta this equation is valid. Now this I will write as d by d zeta –f of –zeta + G of zeta = 0 if you differentiate with respect to zeta we exactly get this. F prime of –zeta and 1 –n will come that will make it plus.

So this is same as that now on integrating from 0 to zeta the above equation derivative equal to 0 so fundamental theorem calculus that will give you –F of –zeta plus G of zeta + F of 0 –G of 0 = 0. And we have given some reason why we drop some constants like this both in the derivative of d'Alembert formula and also in the IBVP with Dirichlet boundary conditions in lecture 4.9. For the same reasons which will not repeat we can drop that and what remains is F of –zeta = G of zeta valid for zeta greater than or equal to 0.

So this is what we get from using the boundary condition 1.what does this mean? F is determined see G is known in interval 0, 1 therefore this formula give us there is a minus sign here this formula will give us the values of f n 1, 0. Using values of G on which interval 0, 1 so this is information we get from boundary condition 1.

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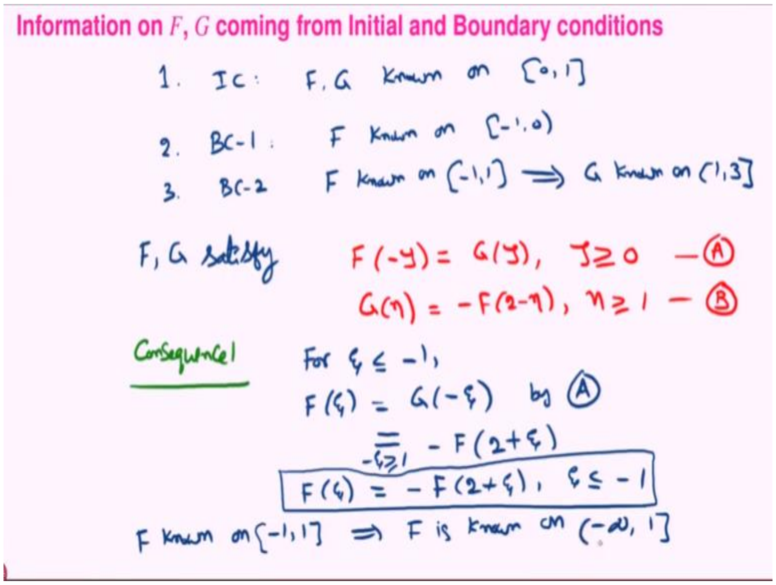

Let us see what we get from boundary condition 2 what is the second boundary condition? It is u of 1, t = 0 right so substituting the formula for u we get f of 1 – ct + G of 1 + ct = 0 valid for all t greater than or equal to 0. Here also t greater than equal to 0 so this means that F of 1 – zeta + G of 1 + zeta = 0 zeta greater than equal to 0. So that implies G of eta it means I am setting eta = 1 + zeta just changing the names of the variables.

Because now I am going to define G using the values of F on some interval so when eta, equal to 1 + zeta what is minus zeta? 1 – eta, s therefore G of 1- zeta is –F of 1 – zeta but what is minus zeta 1 – eta so that is equal to –F of 2- eta of course valid for what? Eta greater than or equal to 1 if you notice here 1 + zeta is always greater than equal to 0. Therefore 1 + zeta is always greater than or equal to 1 and I am replacing 1 + zeta with theta.

Therefore eta is always greater than or equal to 1 so what do I have? G of eta = -F of 2 – eta when eta is greater than or equal to 1. This is the information coming from second boundary condition. So let us write the information that we got from both initial and boundary conditions. So what did it says first one is initial condition it gave us F are known on this interval 0, 1. Second thing is b, c 1 boundary condition 1 that gave us F values.

So F known on -1, 0 because it is expressed in terms of values of G on 0,1 that is why we got this now the last one which we got b c 2 F known on -1, 1 implied that G is known on 1, 3. And importantly F G, satisfy some relations what is that F G satisfy. F of –zeta equal to G of zeta is greater than or equal to 0 and F of eta is equal to –F of 2- eta greater than or equal to 1. So let us call this as A call this as B.

Now let us write one consequence of this let us call it consequence 1 so for psi less than or equal to -1 F of psi = G of minus psi this is by A. If psi less than or -1 psi is negative and for negative things we know by A you can get this relation. But now psi is less than or equal to -1 is minus psi is greater than or equal to 1. When somebody is greater than or equal to 1 B is applicable so G of anybody is –F of 2 minus that.

So 2- eta but eta is minus psi here so it is 2 + psi so what we have let us, write down. F psi = -F of 2 + psi valid for psi less than or equal to -1. So this is the confusion we get now F is known on -1, 1 already this above relation means that F is known on minus infinity, 1.

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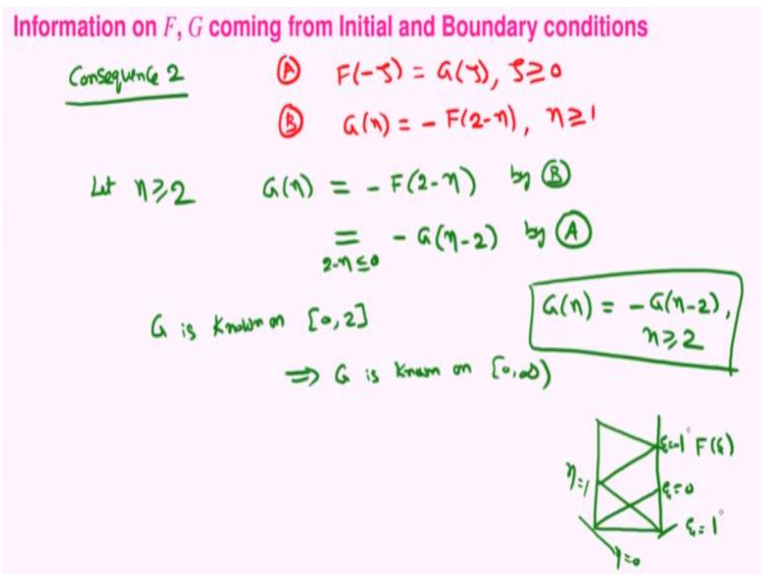

So let us look at the second consequence so let us briefly recall again what was A F of minus zeta = G of zeta if zeta is greater than or equal to 0 and B G of eta = -F of 2 – eta of course eta is greater than or equal to 1. Now let eta is to be greater than or equal to 2 then what happens G of eta = -F of 2- eta by B is applicable now 2- eta is less than or equal to 0 because eta is bigger than equal to 2. Therefore by A I get this equal to –G of eta -2 by A so therefore what we get?

G is known let us summarize what we get here G of eta = -G of eta -2 valid for eta greater than or equal to 2 from here it follows that G is known on 0, 2 implies G is known on 0, infinity. As you remember from last lecture and as well as from this picture this where it came from this is how things were right. Now F of psi is what is required and psi starts from 0 -1 and so on and of course psi is equal to 1.

So we need F values for psi which is less than or equal to 1 and G's which are this is eta = 0 eta = 1 and so on. So we actually need values of G from 0 to infinity and value of F from minus infinity 1 and that we have achieved these are the 2 consequences that because of that we know the values of F and G they are determined.

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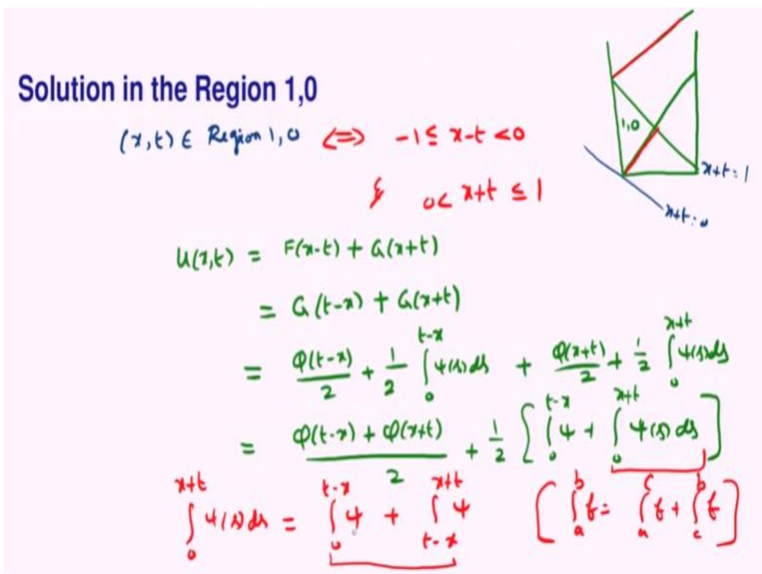

Now let us look at the solution in the region 1, 0 let us briefly let us draw the picture so this is the region 1, 0 region is this. We want to solve inside this so x, t belongs to region 1, 0 what does it mean? It says something about x –t and x + t this is x –t = 0 this is x – t = 1 it lies between that. So -1 less than or equal to x-t less than 0 this x – t = 0 x –t = -1 and what about x + t? This is x + t equal to let us use blue colour this is x + t = 0 this is x + t = 1.

So this is the meaning of x, t belongs to region 1, 0 now what is u of x, t it is F of x – t by definition that we started with. Now x – t is between -1 and 0 therefore F of x – t is negative and we have determined the values F of minus zeta is G of zeta. So this is nothing but G of t – x this stays as it is because x + t is between 0 and 1. So G is known in 0, 1 now let us substitute the expressions for G that we know F and G.

In this case only G is relevant and 0, 1 we know that expression for G using that what we get is phi of t – x by 2 + 1 by 2 into 0 to t – x psi S ds this is G of t – x. G of x + t is phi of X + t by 2 + 1 by 2 0, 2 x + t psi S ds. Now let us club the like terms so this is equal to phi of t – x + phi of x + t by 2 + 1 by 2 0 to t – x psi + 0 to x + t psi of S ds. So let us analyze what is there in these brackets the integral in the brackets and we will then come back to this.

So we have determined the solution now we would like to express it as d'Alembert form that is why we would like to do little more work. So let us look at these integrals that we have 0 to x + t psi S ds o to t –x psi + t – x 2 x + t psi. We can always write this so what I am using here is something from the calculus we know a to b have always equal to a to c F + c to b F. Whether or not c belongs to interval a, b does not matter only thing F should be defined in such interval then we can always do this.

So with that I get this expression therefore this term now becomes this I will substitute with this. And then see what we get? So what we get is 0 to t – x i is also coming one more time here. So 2 times so let me write that.

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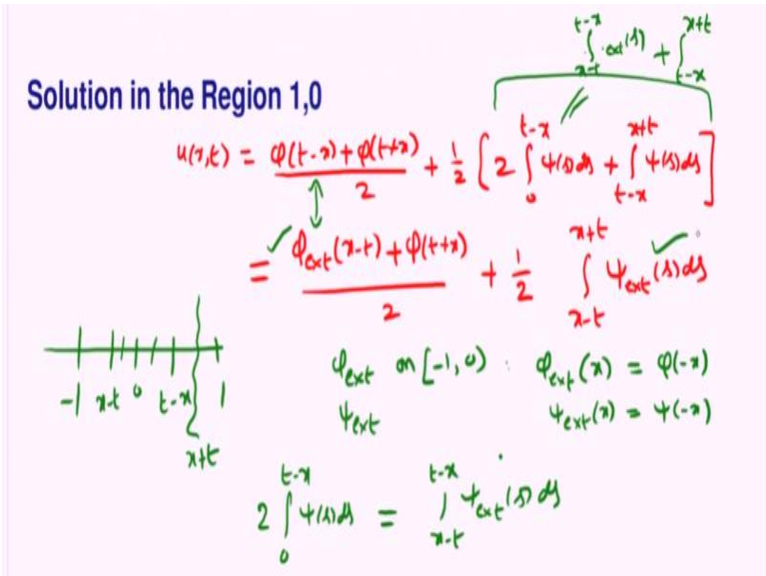

What we get is phi of u of x, t = phi of t – x + phi of t + x by 2 + 1 by 2 into 2 times integral 0 to t – x psi S ds + t – x to x + t psi S ds. This is what we get now here I would like to extend phi so let me write phi x and then I would like to have x – t here t + x is in 0, 1 there is no need to extend. So I just simply use phi here by 2 + 1 by 2 and here what would I like to have is x – 2 to x + t psi extend then this is in the d'Alembert form.

Now if you look at this first term this suggest that we must define phi as an odd function in some even function. Because phi of t – x I want phi of x – t no –n here therefore even I want to define as even. So what we do is the small picture we will draw here 0 so we have x – t is here in this region 1, 0 so -1 is here x – t is here. And then t – x is here and then 1 is here and we have x + t this distance is same as this distance because x – t and t – x are equal distance from 0.

So what we do now is that phi ext we want to define on -1, 0 whatever is needed only we will do. So phi ext of any x = phi of –x because –x will be in 0, 1 interval and there we know phi already it is given there. So phi – x similarly psi ext on this psi ext of x equal to psi of –x if we do this term is taken care we got this equality. Now we have to worry about this why these 2 integrals put together is equal to this integral.

So what is this 2 times 0 to t – x psi S ds this is nothing but x – t 2 x + t psi extended because the even function and this interval x – t, x + t is symmetric about 0 it becomes 2 times that. Therefore the one which is here is precisely integral x – t 2 x + t psi extension now this is not x – t x + t it is actually t - x so this is t – x that is equal to 2 times this. And what is next is t – x to x + t so if you combine you get this so therefore we have got the d'Alembert form also.

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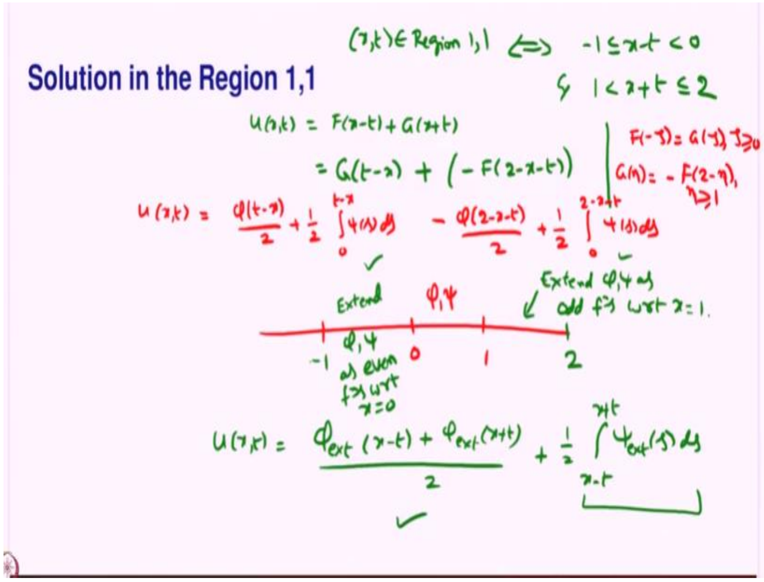

Let us solve the problem in the region 1, 1 so x, t belongs to region 1, 1 that means that -1 is less than or equal to x – t is less than 0 and 1 less than x + t less than or equal to 2. So therefore u of x, t = F of x – t + G of x + t because x – t it between -1 and 0 F of zeta will be G of minus zeta. Therefore it is G of t – x now x + t is between 1 and 2 and when you are more than there is a formula for G in terms of F which is –F of 2 – x – t this is what we have.

So what did we use? We use 2 formulas what are they? F of minus zeta = G of zeta if zeta is greater than or equal to 0 and G of eta equal to –F of 2 – eta if eta is greater than or equal to 1 we use this. Now using this expression for G we get u of x, t equal to phi of t – x by 2 + 1 by 2 integral 0 to t – x psi S ds this is the first term. Second one F of 2 – x – t that will give us phi of 2 – x – t by 2 + 1 by 2 integral 0 to 2 – x – t psi S ds this is what we have.

So now we need to define some extensions let us do once for all 0 is here so between 0 and 1 phi and psi are known. So between -1 and 0 so extend functions as even functions extend phi psi as even functions with respect to x = 0. In 1, 2 here u extend as odd functions extend phi psi as odd functions with respect to x = 1. If you do that you get this formula u, x t = phi extended of x – t + phi extended of x + t by 2 + 1 by 2 x – t to x + t psi extended of S ds this is what we have.

In this part is clear so we just discuss why this equal to sum of these 2 integrals we are going to discuss that.

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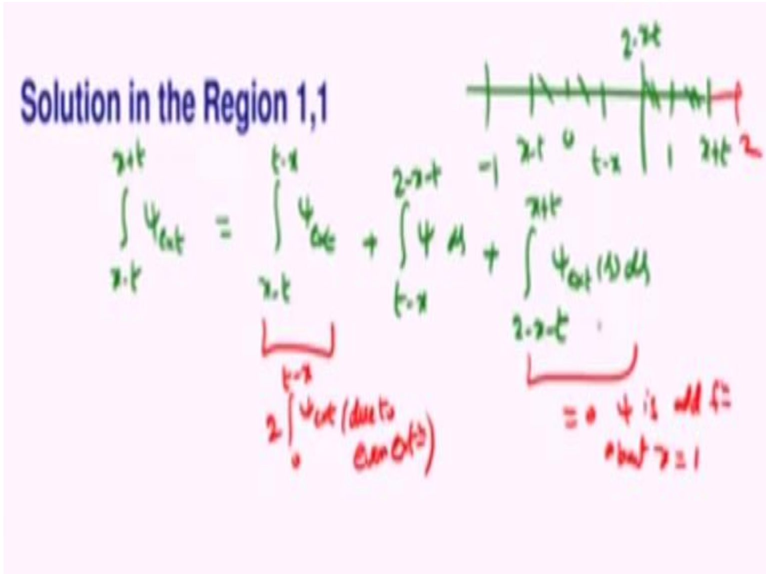

So x – t to x + t of psi extended is x – t before that maybe worth drawing a picture 0 here x – t is here of course -1 is here this symmetrically placed here. This length is same at this length and then we have a 1 here and we have x + t here and here we have 2 – x – t this distance is same as this distance. So x – t to t – x psi extended + t – x to 2 – x – t of psi no extended because we are in the interval 0, 1 therefore it is psi extended coincides with psi ds + 2 – x – t into x + t psi extended.

Now here the integral is 0 sorry the first integral is actually twice the integral from 0 to t – x because extended function psi was extend as the even function due to even extension that is why. Now this term is 0 because psi is about odd function about x = 1 that is how we have extended the functions to the right side of 1 as odd functions up to this interval 2 therefore that is 0 and this is an interval which is symmetric about 1 that is what we saw the distances are same from one and therefore integral is 0. So this is the reason why we have the d'Alembert formula.

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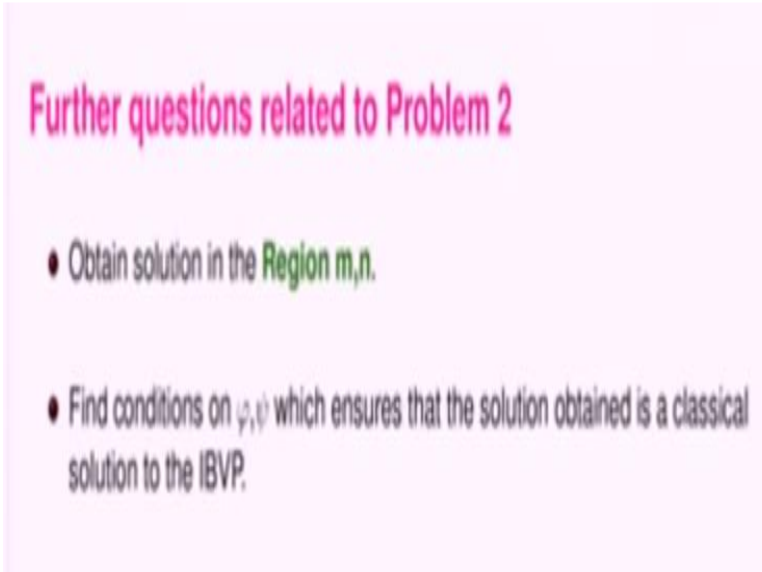

Further questions related to problem 2 obtain solution in the region m, n we have obtained only in the region 1, 0 and 1, 1. So get a formula for m, n also and find conditions on phi and psi which ensures that the solution obtained is it classical to you IBVP and express the solution in the d'Alembert form. We have done this in the 2 regions that we considered do it for general region m, n as well.

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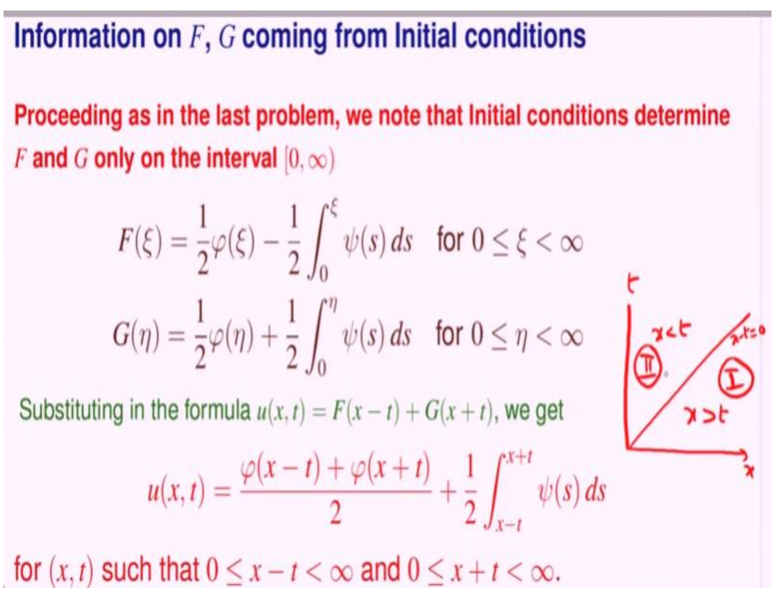

Now let us look at the third problem 3A because the same problem has the 2, 3 parts. So as a first part what we are going to do is we are going to consider an IBVP on semi-infinite intervals. In lecture 4.9 we said this is much more, simpler than a bounded interval we will see why it is much more, simpler. So given functions phi in C2 of 0 infinity and psi which is C1 of 0 infinity find a solution to the homogenous wave equation and initial conditions as before the displacement is phi and initial velocity is psi.

Now there is only one boundary because the domain is 0, infinity there is a boundary only at x = 0. So the boundary condition is again once again we consider Dirichlet boundary condition U of 0, t = 0. Now proceeding as in the last problem we note that the initial conditions determine F and G only on this interval 0, infinity no surprise again. It so these are the expressions for F and G exactly the same as before.

So substituting in this formula we have this expression which is a d'Alembert formula of solution and it is valid in this region x – t is greater than or equal to 0 x + t is greater than or equal to 0 in this region. So this region let us see how it looks so now what we have is picture is like this x here t, here x – t is playing a role this is x – t = 0 here x is always greater than t, here x is always less than t.

So we have determined the region x t greater than equal to 0 and x + t greater than equal to 0 actually is this region let us call it as 1. So in this region 1 initial condition determine the solution so what remains to do is to find the solution in the region too. In the region 1 looks like the 1, 0 we had earlier so essentially there are only 2 regions now. Whereas if you; are in a finite interval case it had many infinitely many regions.

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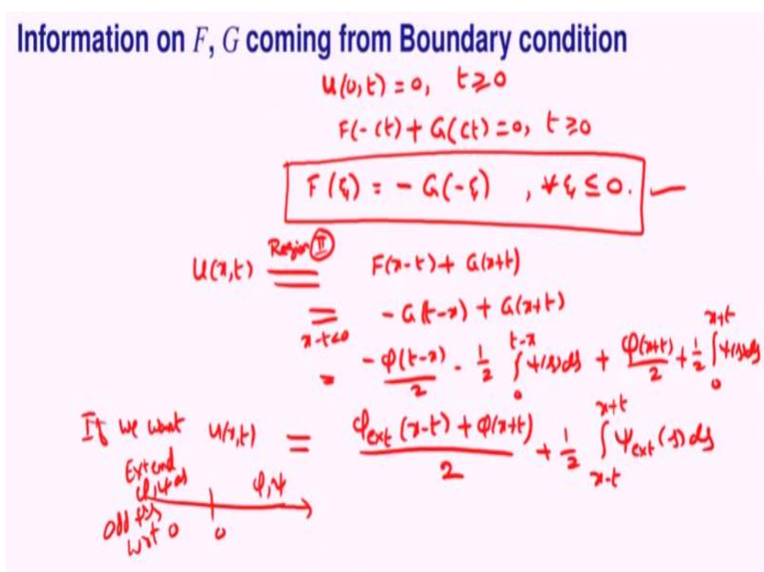

Let us see what is the information that we get from the boundary condition? So we have only one boundary condition which is u of 0, t = 0 t greater than or equal to 0 so what we get is? F of –ct + G of ct = 0 t greater than or equal to 0 so that means F of psi = -G of minus psi for all psi less than or equal to 0 so this is information that we can. So therefore u of xt in region 2 this is the original formula F of x – t + G of x + t.

But now that becomes x – t is negative using this relation this is –G of t – x + G of x + t we can substitute the expression for v for phi for G in terms of phi and psi that will give us –phi of t – x by 2 – half 0 to t – x psi S ds + G of x + t is phi of x + t by 2 + 1 by 2 integral x + t psi S ds. So this is nothing but phi extended of x – t + phi of x + t by 2 + 1 by 2 integral x – t to x + t psi extended of S ds.

If we want this if we want u x, t equal to this we have to tell how we have to extend there is no need to extend the other side. Because this side phi and psi are given so we have to get only this side and this suggest of phi of extension of x- should be equal to –p of t – x. That means extend as odd functions extend phi psi as odd functions with respect to 0 so we have this. So please check the validity of this equation.

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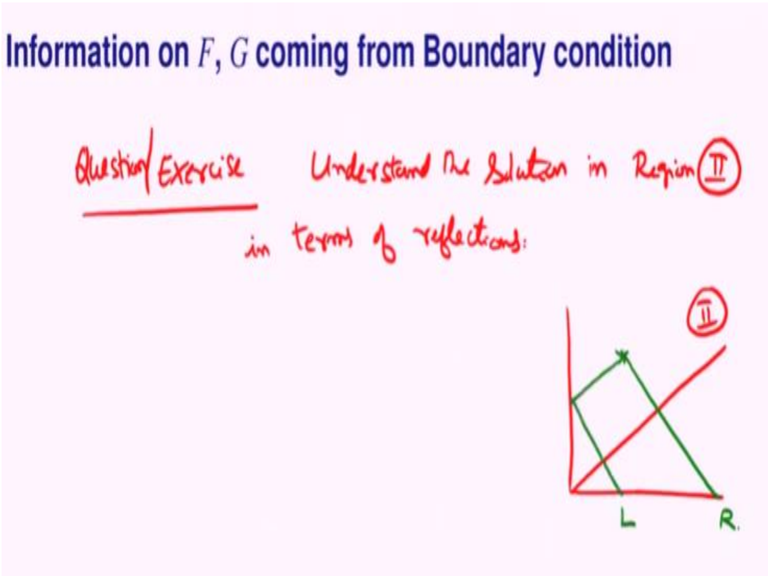

Now a small question which is an exercise understand the solution in region 2 we have obtained in terms of reflections something like this I will just indicate but I will not do it. So this is the region 2 so take a point here go like that there is R right I mean then if you come here shift the other one that will be L in terms of this.

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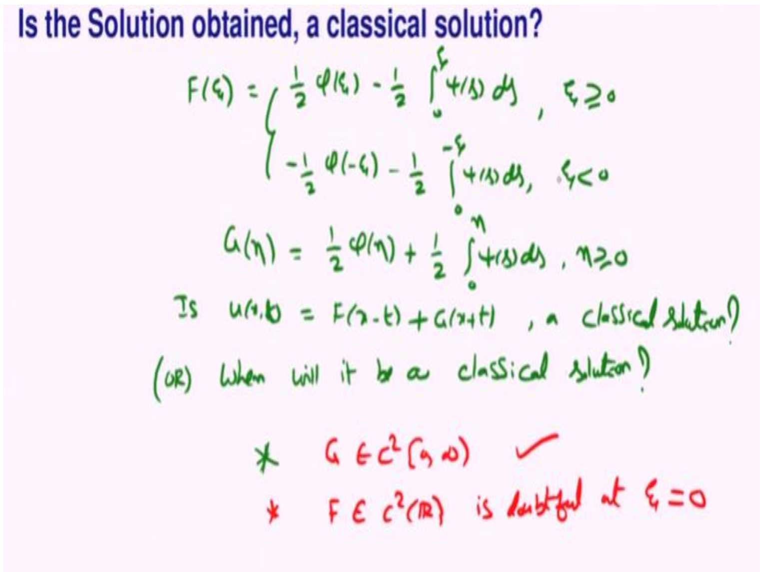

Now the question is the solution that we obtained is it a classical solution? So let us write down the solution that we obtained once more what we got is F of psi = half phi of psi minus half 0 to (()) (45:14) psi S ds if psi is greater than or equal to 0 and minus half of phi of minus psi minus half of 0 to minus psi S ds if psi is less than 0. And for G of eta equal to half phi, eta + 1 by 2 integral 0 to eta psi S ds.

So is u of x, t = F of x – t + G of x + t where F and G are given above a classical solution that is a question. Or when will it be a classical solution? That means we are indirectly asking for conditions on phi and psi whether they need to satisfy any conditions or extra conditions on phi and psi which we called sometimes compatibility conditions. We are looking for such things such condition are necessary or is it automatic?

When will be a classical solution that is the question now let us observes that as far as G is concerned there is no doubt G is c2 of 0 infinity we; do not require values of G for negative values. So that is fine this is easy second one is about F is c2 of R we would like to have that because x – ct takes all the values in R. Now F is in ct 2 of R that is doubtful at some points at only one point psi = 0 in terms of x, t it is on the line x = t there is some doubt.

Otherwise there is no problem for this function is nicely defined in only at the interface psi =0 there could be some issues. So we will analyze that on the later.

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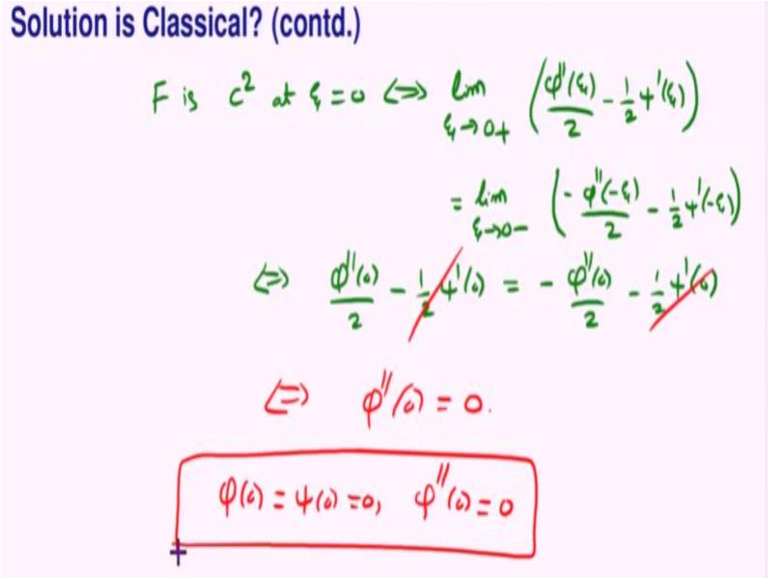

Let us analyze that so F is continuous at psi = 0 if and only if phi of 0 = 0 and F is c1 at psi = 0. So basically to conclude this what you have to do is we have split formula for F so pass 2 limit on both sides as psi = 0. You get something like phi of 0 by 2 = -phi 0 by 2 some such thing therefore you will get phi 0 = 0. So please do the computation in the next one I am going to do. F is c1 if and only if limit of the derivative I am going to take directly the derivative and derivative is phi dash of psi by 2- half psi-psi.

This limit is equal to limit on the other side psi going to 0- phi dash of –psi by 2 + half psi of –psi. So that is of and only if phi dash of 0 by 2 minus half psi is 0 equal to phi dash of 0 by 2 + half of psi of 0. So you see that this cancels and what we get is psi of 0 must be 0 so F is c1 psi – 0 if and only if psi of 0 is 0. So we got phi of 0 = 0 psi of 0 = 0 now we have to still ask is it c2 that we will put one more condition.

So F is c2 at psi = 0 if and only if I am taking the second derivative in the formula for F on both sides of psi less than 0 and psi greater than 0. And that gives me one side this is the limit of the second derivative this limit should be same as the R limit from the other side. So that is if and only if phi double at 0 by 2- half psi dash of 0 = -phi double dash 0 by 2- half psi double dash half 0.. So now this is goes off therefore if and only; if phi, double dash of 0 equal to 0.

So we have got 3 conditions phi of 0 psi of 0 must be 0 no conditions on first derivatives second derivatives should be 0. So these are the compatibility conditions if they are satisfied then what are got is indeed a classical solution and we have expression in terms of phi and psi as well as in terms of F. F itself expressed in terms of phi and psi.

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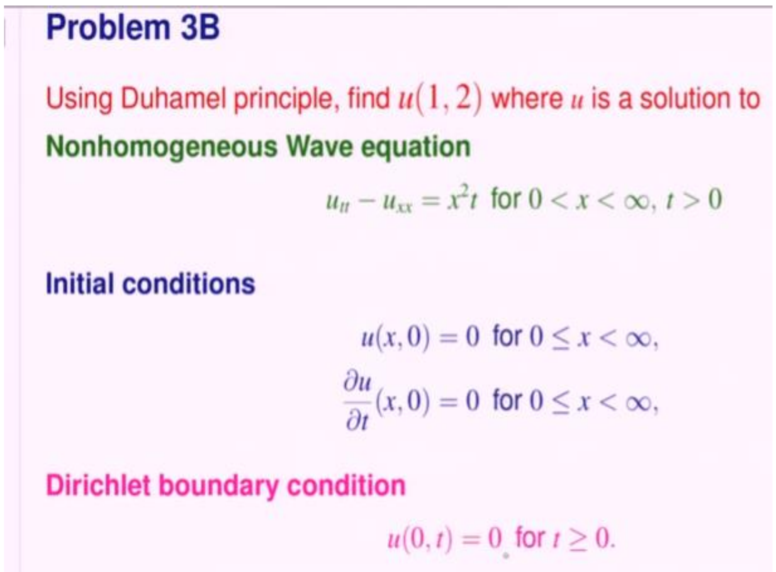

So let us move on to problem 3B using Duhamel principle find u of 1, 2 where u is the solution to u t, t – u x, x = x square t so we have a source term now. Initial conditions we are taking as 0 phi is 0 psi is 0 boundary conditions we take Dirichlet boundary condition as before. So we want to solve this problem non-homogenous wave equation with a source term given by x square t and 0 Cauchy data 0 boundary data. We want to solve this we want to use Duhamel principle.

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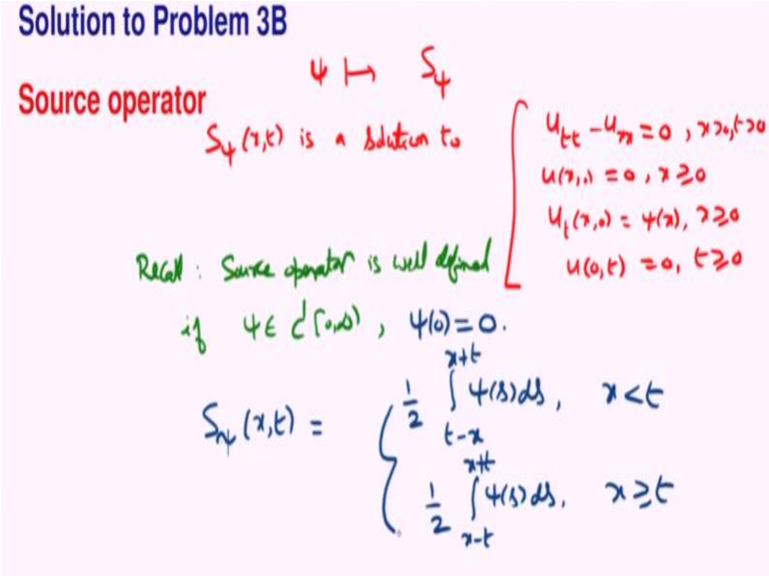

Duhamel principle what we need is a source operator so what is the source operator definition? It is the one which maps psi maps to S psi so S psi of x, t is solution to psi should come as initial velocity everything else should be 0. So u t, t –u x, x homogenous wave equation equal to 0 x positive t positive fine and u of x, 0 = 0 x greater than or equal to 0 u, t of x, 0 = psi of x = 0 and we have boundary condition that should be satisfied.

So given psi find the solution that is called S psi so recall that the source operator is well defined if psi should be c1 function and psi of 0 must be 0. This is part of the compatibility condition there is no compatibility condition on phi because phi is already 0. So it satisfies all the compatibility conditions. So now what is S, x S psi of x t what is an expression for this? It is 1 by 2 t – x to x + t psi S ds if x is less than t that means is in the region 2. If x, t is in the region 1 it is this this is in the region this is the expression for S psi.

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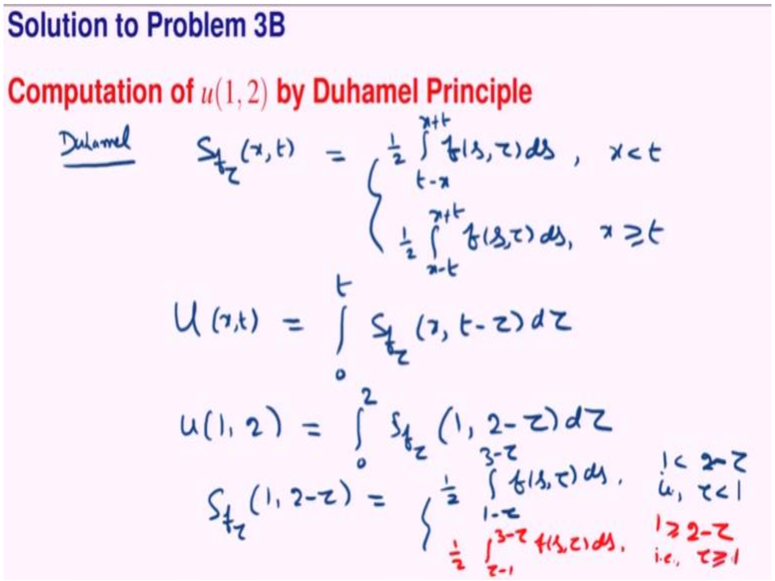

Now what do we need in the Duhamel principle we need to find what is the source operator corresponding to F tau. So what we need for that Duhamel S f tou of x, t and that is nothing but half integral t – x to x + t F of s, tau ds x + t and half x – t to x + t F of x tou ds x greater than or equal to t this is s of tau. Then u the solution at x, t the non-homogenous equation solution is given as a super position of this s of tau x t – tau d tau.

And what we want to compute is? U of 1, 2 that is what we ask to find so therefore t = 2 0, 2 s f tau x = 1 t = 2. So that is what it is what, is s F tau 1, 2 – tau you can substitute and get that formula. S of tau 1, 2 – tau – half t – x that is 2 – tau -1 there is 1 – 2 – 1 is 1 – tau this is 3 – tau F of s tau ds this happens if 1 is less than 2 – tau. That is tau is less than 1 and other one is half integral tau -1 to 3 – tau of F of s tau ds this is for 1 greater than or equal to 2 – tau which is tau greater than or equal to 1.

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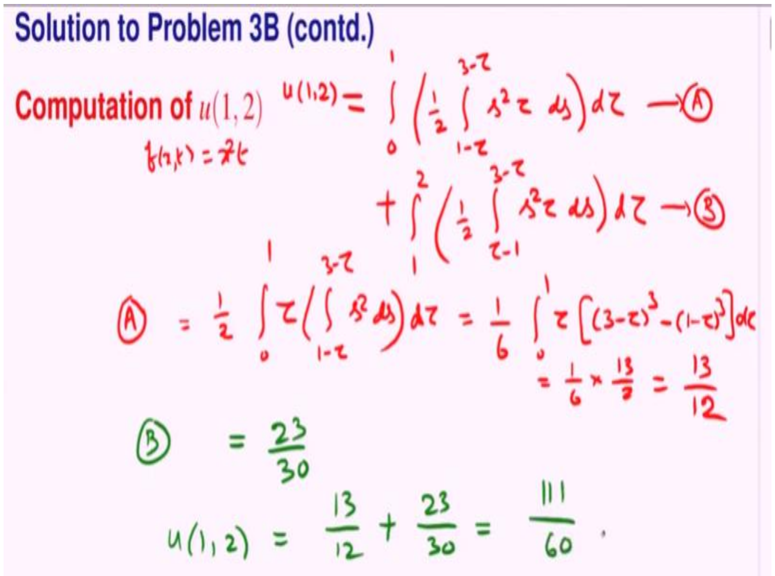

So therefore u of 1, 2 = it is an integral from 0 to 2 but we are not split that into 0 to 1 + 1 to 20 to 1 half 1 – tau to 3 – tau our F of x, t is x square t. So therefore integrand is s square tau ds and then d tau + integral from 1 to 2 of 1 by 2 tau – 1 to 2 3 – tau s square tau ds d tau. Let us call this term as A the first term and this as B. And we will compute them separately what is A and B. So A is half I have brought out half to the front integral 0 to 1 then tau is here then 1 – tau to 3 – tau s square ds then d tau.

That is nothing but s square integral will be s cube by 3 that 3 comes out and I become 1 by 6 into 0 to 1 tau into 3 – tau cube – 1 – tau cube d tau. This after computation becomes 13 by 2 therefore 13 by 12 that is what it becomes. Now the B we can compute B and that value comes out to be 23 by 30 which is matter of integration so please do it. Therefore U of 1, 2 is 13 by 12 + 23 by 30 on simplification this become 1, 1, 1 by 60. So this is the answer we have done this.

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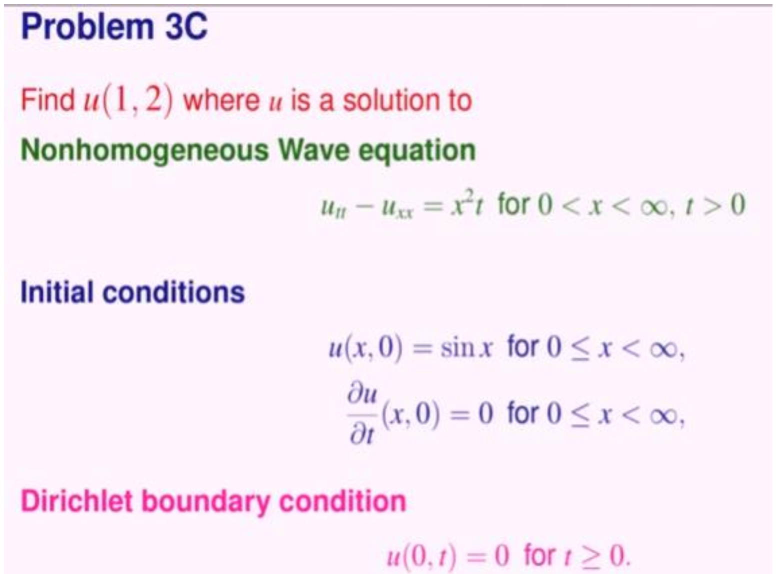

Let us look at the problem 3C here we need to find u of 1, 2 for the same non-homogenous equation but there is a Cauchy data here sin x other one is 0 the initial velocity Dirichlet boundary condition.

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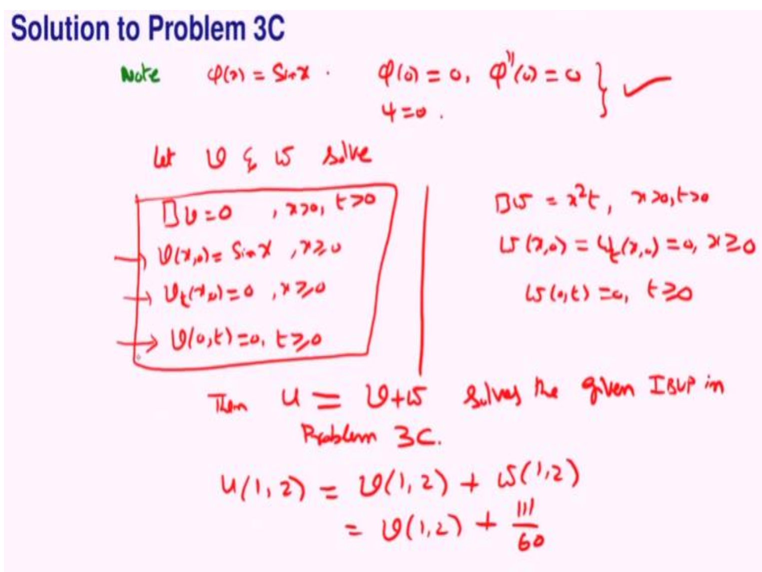

So this we solve using super position principle again now before we do anything let us note that phi x = sin x this is the initial displacement. This satisfies the required compatibility conditions that we found so that we have a classical solution. And psi is 0 therefore it satisfies F 0 = 0 so compatibility conditions are satisfied we are in classical solution so solution that we obtained is a classical solution.

Let v and w solve this is the d'Alembertian of v = 0 v –f x , 0 is sin x v t of x, 0 is 0 v of 0, t is 0 and w satisfies the non-homogenous equation x square t x positive t positive. And rest of the conditions are 0 conditions w of x, 0 w t of x, 0 are 0 and w of 0, t is also 0. Then u = v + w solves the problem that we posed in 3c solves the given IBVP in problem 3c. So therefore if you u of 1, 2 that is nothing but v of 1, 2 + w of 1, 2 but w of 1, 2 we already computed.

So that is this +1, 1, 1 by 60 so we have to simply compute v of 1, 2 in other words we have to compute this solution of the homogenous wave equation with initial displacement as sin x initial velocity 0 and boundary conditions 0. That we already obtained a formula for the solution we simply use this formula.

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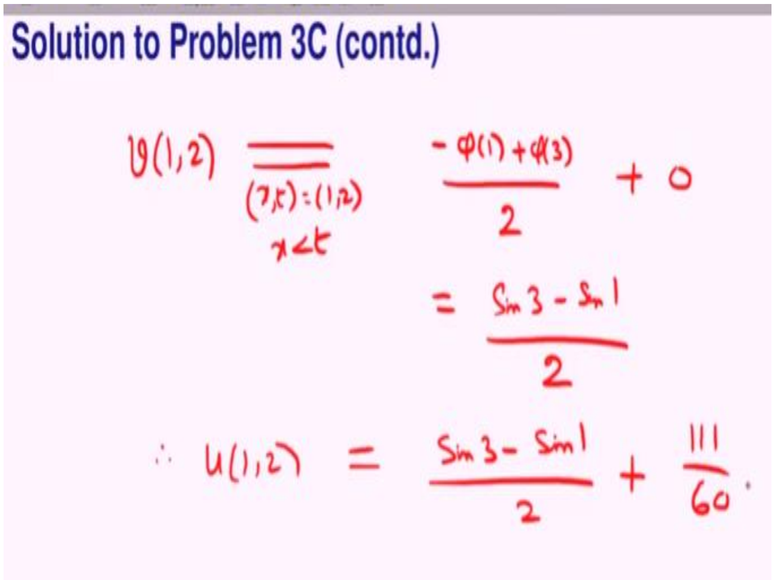

So v of 1, 2 x = 1 t = 2 so clearly x is less than 1 we are in the region 2 so we have to apply that formula. So that is given by –phi of 1 + phi of 3 by 2 psi is 0 so there is no other term and this is nothing but sin 3 –sin 1 by 2. Therefore u of 1, 2 is v of 1, 2 +w of 1, 2

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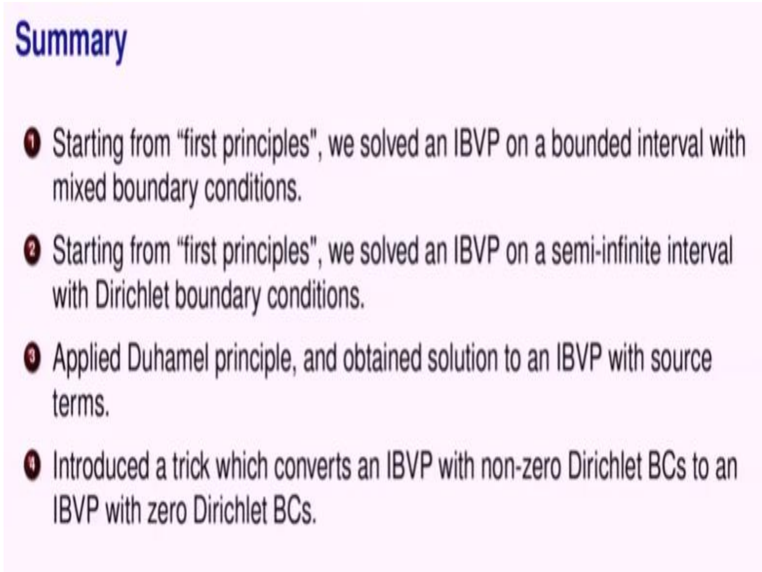

Let us summarize what we did in this tutorial we starting from first principles we solved an IBVP on a bounded interval with mixed boundary conditions starting from first principles again we solved in IBVP on a semi-infinity interval with Dirichlet boundary conditions. Applied Duhamel principle and obtain solution to IBVP with source terms introduced a trick that is problem 1 which converts with IBVP with non-zero Dirichlet boundary conditions to 0 Dirichlet boundary conditions IBVP thank you.