Partial Differential Equations
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Module No # 07
Lecture No # 34
Wave Equation for an interval in R – Solution to an IBVP from first principles

For wave equation we have so far discussed Cauchy problem is posed on R d cross 0; infinity that is x belongs to R d and t is positive. We have considered wave equations in 3 space dimensions name d = 1, 2 and 3. In this lecture we are going to discuss wave equation when we disposed on not necessarily on R d but on a subset of R d. In this lecture we are going to restrict ourselves d = 1. So we consider wave equation which is posed on sub intervals of R.

Because the equation is posed on sub intervals of R they have boundaries and the problem that we are going to considered is what is known as initial boundary value problem. And we are going to solve with using first principles in this lecture. So the out claim of the lecture is as follows.

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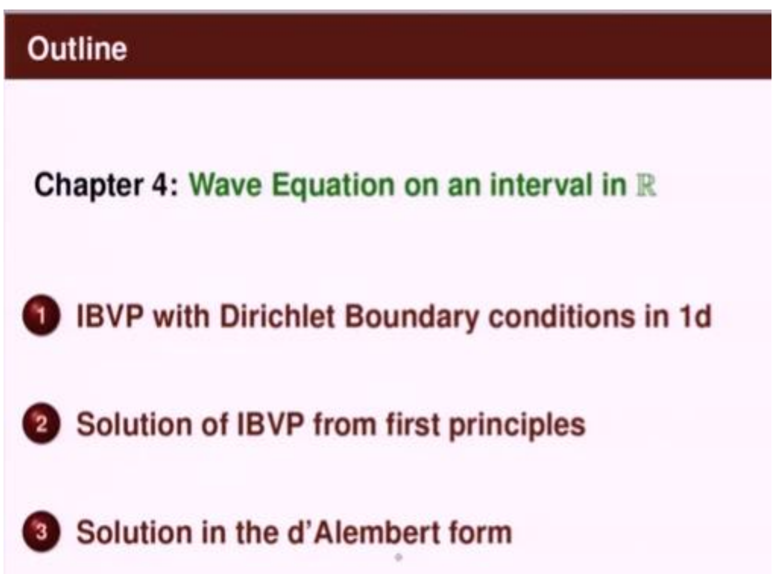

We state initial boundary value problem with what are called Dirichlet Boundary conditions in 1 space dimension that is in R. And then we solve IBVP using first principles I will explain what are the first principles mean here. In the end we would like to express this solution obtained in the form of this d'Alembert solution. Recall d'Alembert formula gave us solution to the Cauchy problem in R. Now we would like to have a similar formula even for a IBVP that is the goal here.

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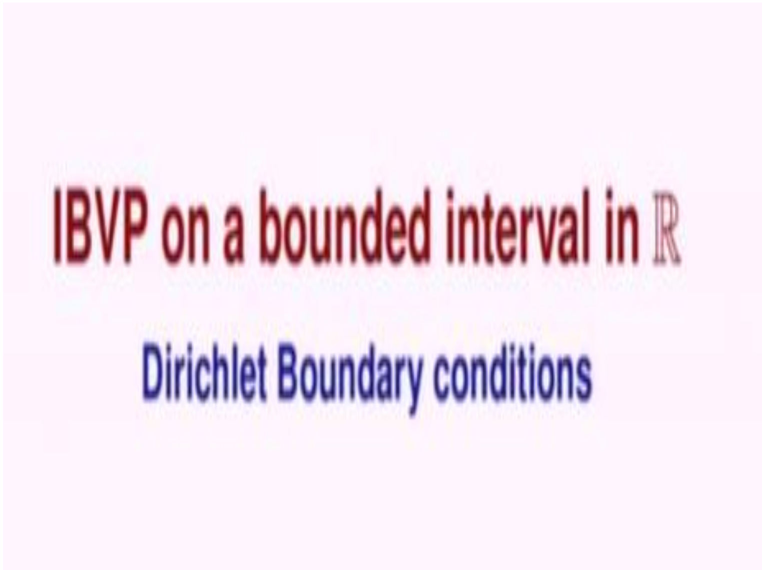

So IBVP and a bounded interval in R with Dirichlet boundary conditions what is it?

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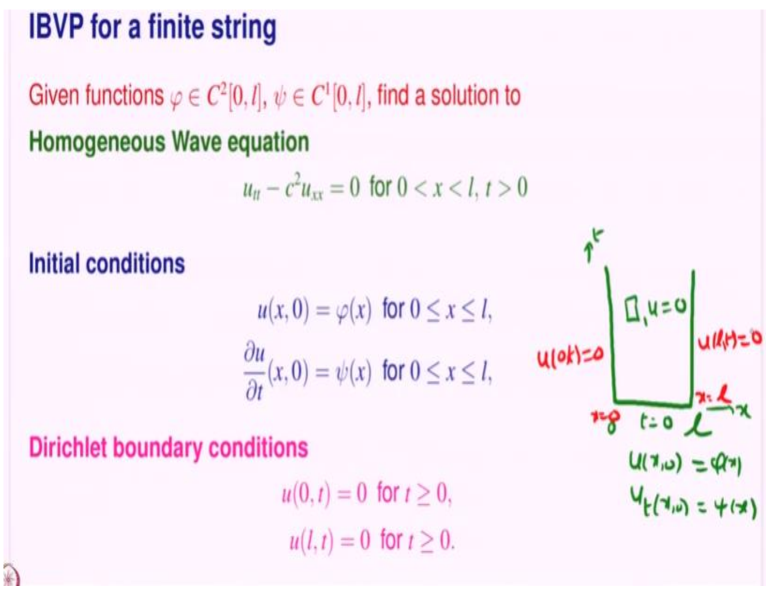

It is also called something IBVP for a finite string remembers the 1 dimensional wave; equation. We have modeled as a model for the transverse vibration of string so since the string is finite it is called finite string here 0, l that is the string position 0 to l. So given functions phi which is in C 2 on the interval 0, l and psi which is C 1 on the interval 0, l. Please note the closed intervals here. Find a solution to the homogenous wave equation it means the IBVP we are going to consider the homogenous wave equation.

Posed on the domain x in 0, l t positive with initial conditions the initial displacement is phi x and initial velocity is psi x. And Dirichlet boundary conditions these are new compared to the Cauchy problem in R that is because we are considering x in the domain 0, l. So it as 2 boundary points namely x = 0 and x = l and that there we described u of 0, t = 0 and u of l, t = 0. In other words this is domain we are considering 0, l carry this x this is t.

Here we want that the wave equation to be satisfied homogenous wave equation and this is t = 0 here we prescribe u x, 0 and u t x, 0 this is phi of x this is psi of x. And here these are the boundary points these is x = 0 and x = 1. Here we prescribed that u of 0, t is 0 u of l, t = 0 we have a introduced this initial boundary value problem in a tutorial in the lecture 4.3 earlier.

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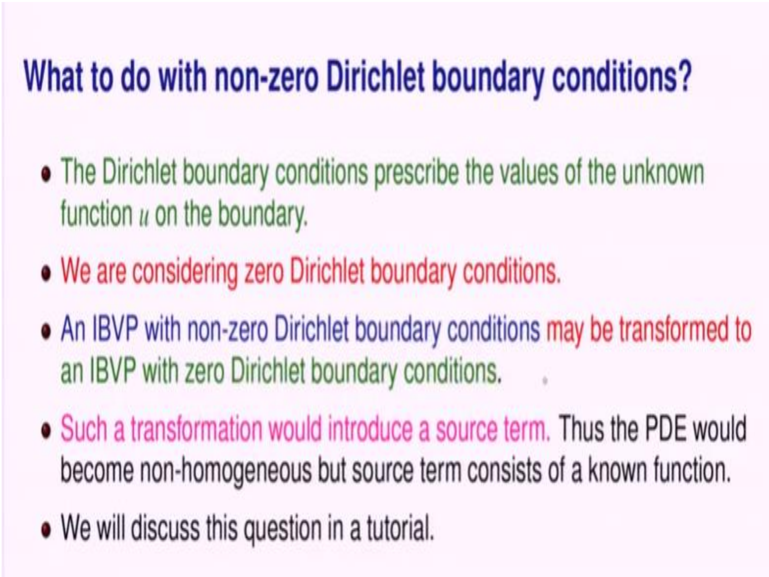

So what to do with non-zero Ditichlet boundary conditions? The Dirichlet body conditions prescribed the values of the unknown function on the boundary. We are considering 0 Dirichlet boundary conditions and IBVP with non-zero Dirichlet boundary conditions may be transformed to an IBVP with 0 Dirichlet boundary conditions but we may have to pay a cost that will be introduce a source term and also modify the initial speed and velocity initial displacement and velocity.

So the PDE would become homogenous but source of course consist of known function so that is something good. And we will discuss this question in a tutorial what is a question? How to transform a problem with, non-zero Dirichlet boundary conditions to a problem with 0 Dirichlet boundary conditions?

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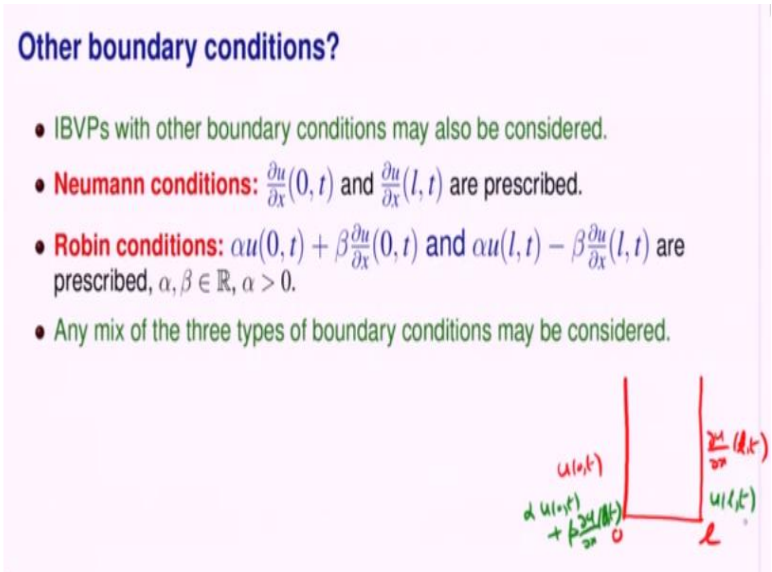

So what about other boundary conditions so IBVP is with other boundary conditions may also be considered? Newmann conditions that is dou u by dou x 0, t dou u by dou x at l, t are prescribed. In Dirichlet we are prescribing u, 0 t and u l, t instead of u prescribed dou u by dou x at the boundary points x = 0 and x = l. At the boundary x = 0 is actually a line and x = l for all times they are prescribed positive (()) (05:44).

Or Robin conditions which is a combination of a Dilichlet and Neumann so alpha times u 0, t + beta times dou u by dou x at 0, t. This is prescribed and alpha u l, t – beta dou u by dou l, t they are prescribed what are alpha beta they are real numbers alpha to be posed. So any mix of these times of boundary conditions may be considered n other words what are saying is this is a domain that we are considering this is 0 this l.

Here we may have u of 0, t and here we may have dou u by dou x of 0, t of l, t we may prescribe this. We may also prescribed this like this alpha u 0, t + beta dou u by dou x l 0, t. Here and here you prescribed for example u of l so any mix of conditions is this is also possible one can study.

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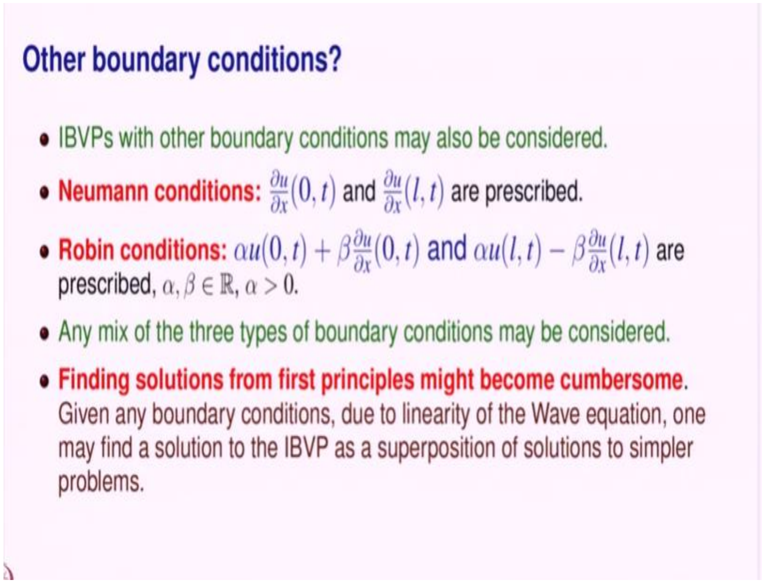

Finding solutions from first principle might become some cumbersome for this other boundary conditions and not so much for Neumann conditions but for problem conditions it is going to be a cumbersome. Given any boundary conditions due to linearity of the wave equation one may find a solution to IBVP as a super position of solutions to simpler problem. So when are solve an IBVP we should identify simpler problem out of them and then solve them.

And then super pose this solutions and that will be a solution thanks to the linearity of wave equation.

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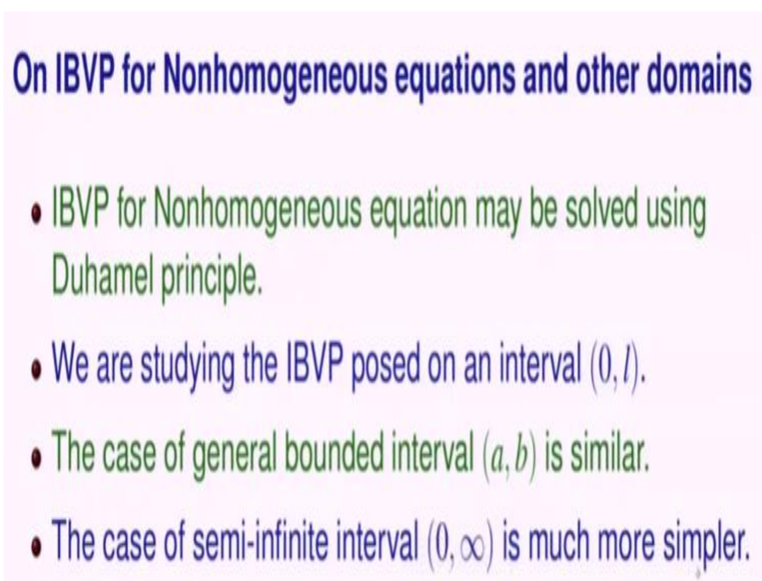

Or we have operator on IBVP for non-homogenous equations on other domains. So IBVP for non-homogenous equations will be solved using Dubamel principle exactly as we did for the Cauchy problem. We have studying the IBVP posed on interval 0, l there is only for convenience one could also consider in interval a, b is similar. One may ask what about an infinite intervals semi-infinite interval that means 0 intervals of 0, infinite that is much more, simpler.

So that is why we are not studying that instead we are studying a more complicated which is a finite interval 0, l case. This is much simpler the methods that we are following in this lecture. We will apply and solution can be obtained in much simpler manner.

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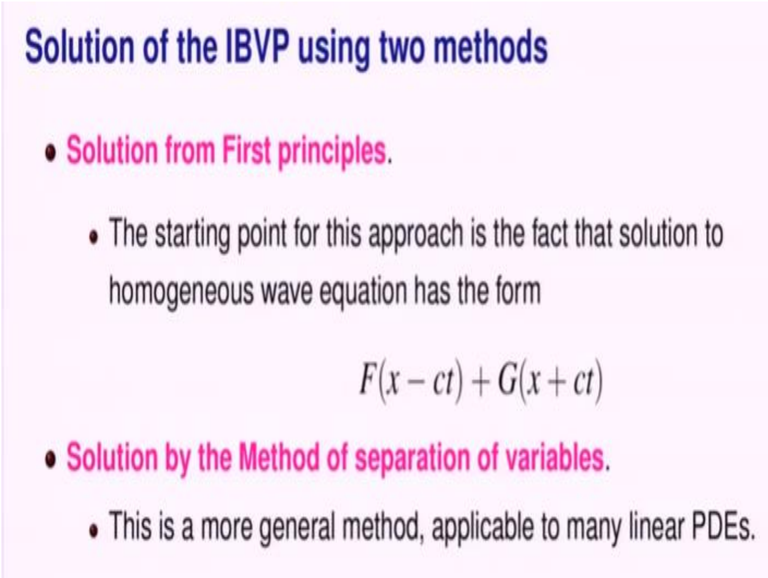

So there are 2 methods that we are going to discuss to solve the initial boundary value problem. One is solution from first principle that is what we are discussed in this lecture the starting point for this approach is the fact that solution to the homogenous wave equation is of the type F of –ct + G of x + ct. That means it is super position of a left moving wave and a right moving wave. Solution by the method of separation of variable as a method this we are going to discuss in next lecture.

And that is a very more general method it does not assume a specific knowledge of the general solution of the problem of the wave equation this is a very general method applicable for may linear equations. We are going to see later on in the course and that you will apply for this for the Laplacian equation and also heat equation.

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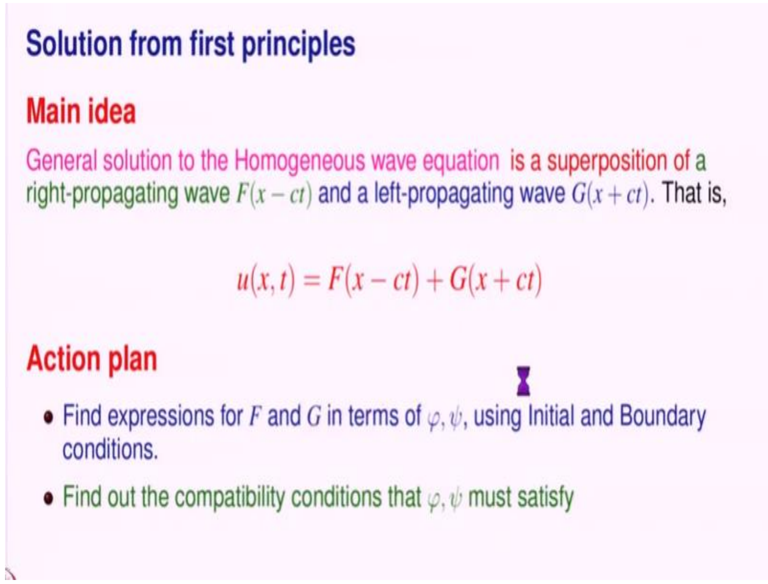

So let us start discussion of the solution of the IBVP from first principles. The main idea as explained before is at general solution homogenous wave equation is a super position of a right propagating wave and a left propagating wave. That is u x, t = F of x – ct + G of x + ct a general classical solution would be exactly this formula were F and G are ct functions. So what is the action plan for us? Find expression for F and G of course we are solving IBVP.

So therefore naturally the expressions for them involve the data in the problem namely phi and psi. How we get that? We have to use initial boundary conditions and find out the compatibility conditions that phi and psi must satisfy. It is actually not enough that phi is actually a c2 of 0, l and psi is c1 of 0, l as mentioned earlier. That is not enough they must satisfy some compatibility condition between them.so that the F and G are C2 functions.

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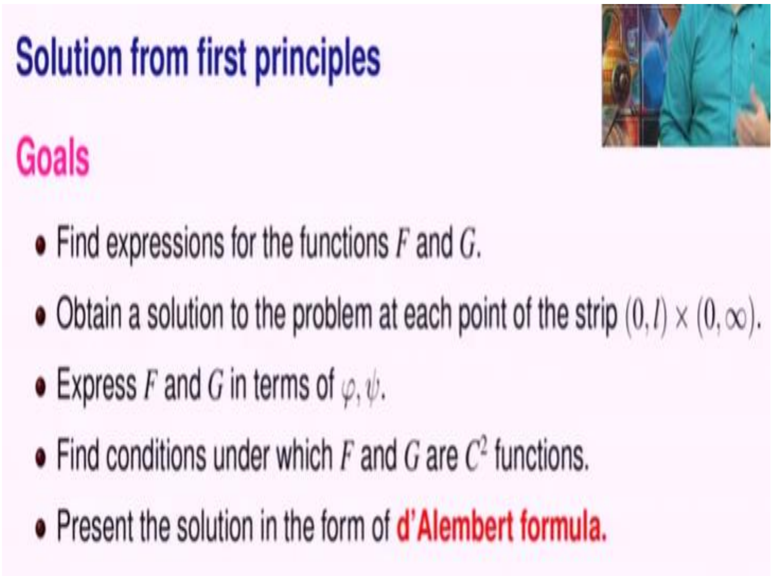

So what are the goals in this find expression for the functions F and G. Obtain a solution to the problem to the each point of the strip 0, l cross 0, infinity one side history. Usually strip is like this infinite thing but now we are starting stopping here. So obtain solution here 0, l cross 0, infinity express F and G in terms of phi, psi. Find conditions under which F and G are C2 functions and finally present the solution in the form of d'Alembert formula.

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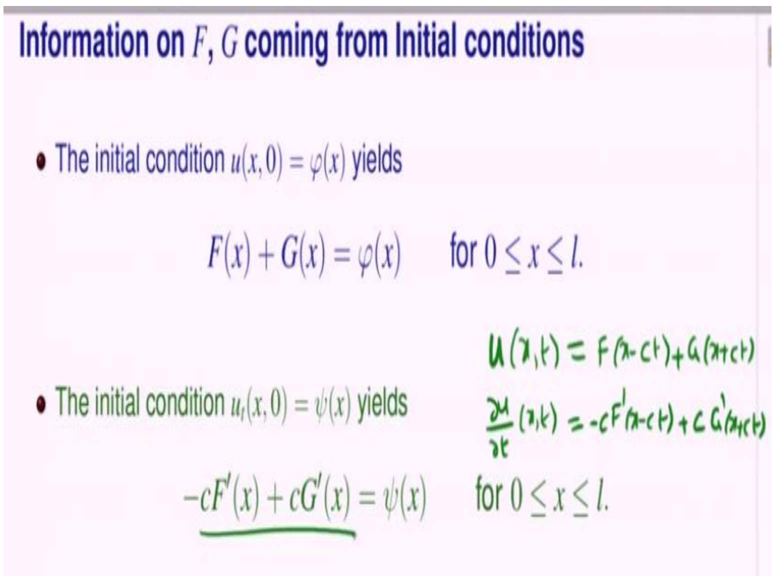

Now let us ask is information that we can get on F and G from initial conditions where initial condition u x, 0 = phi x gives us F x + G x = phi x. Always keep writing this domain for which this equality is valid this is very important if we ignore this we can easily make mistakes. We have another initial condition which u t of x 0 = psi x so on differentiating u of x t we did this earlier exactly same computation.

We did this while deriving d'Alembert formula so u t dou u by dou t of x t is by chain rule F prime at the point x – ct into derivative of x – ct with respect to t that will give us –c . Similarly G prime of x +ct and derivative of x + ct with respect to t c so, that is what exactly we have here. When you put t = 0 it reduces to this.

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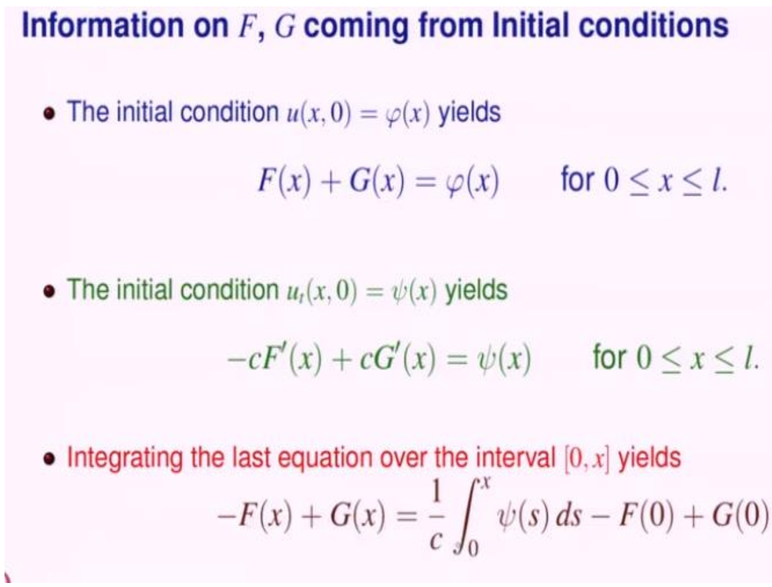

So integrating this last equation over the interval 0, x will give us this expression –F x + G x = 1 by c integral 0 to x psi s ds – F 0 + G 0. So now 2 equations in fact linear equations for F x and G x one equation is here one equation is here therefore we can solve for F and G.

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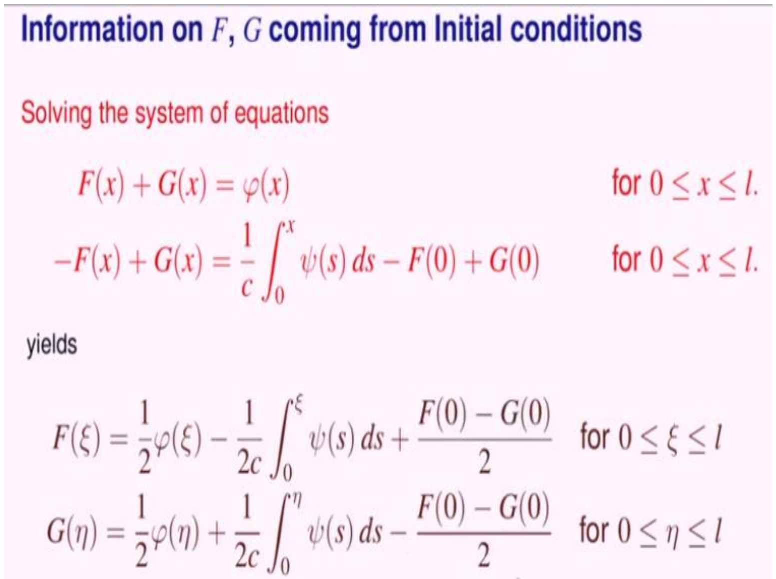

So solving this system of equation gives us F psi equal to this and G eta equal to this. I am using different notations here psi and eta because we know normally we use psi for x – ct that is what we are going to substitute later we will get a formula u x, t. I am here we are going to put x + ct that is why we are using eta. So when you u of x, t = F of x –ct and +G of x + ct something happen from here. But this will get cancelled this is F 0 – G 0 by 2 this is exactly minus of that so when you add this will be 0 it is a constant.

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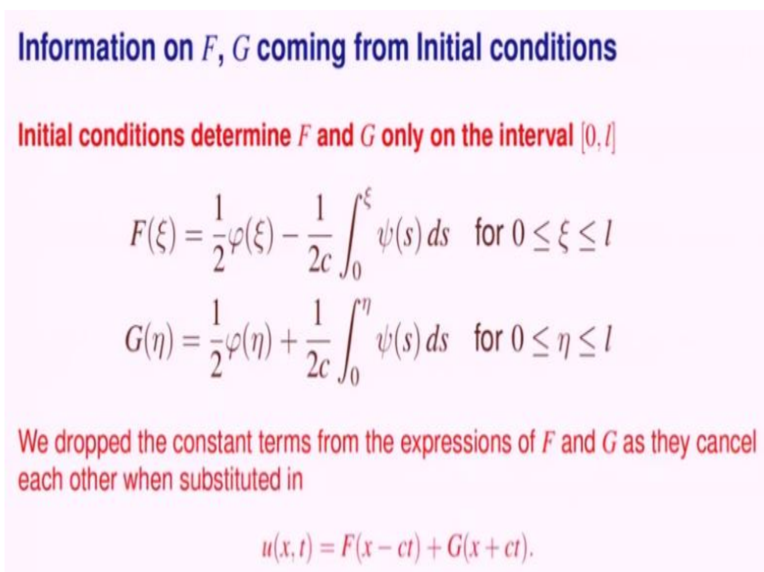

(refre time: 14:03)So finally what do we have initial conditions determine F and G only on the interval 0, l F and G are determined only on the intervals for psi and 0, l and eta in 0, l. And we dropped here the constant term for reason that I have explained earlier.

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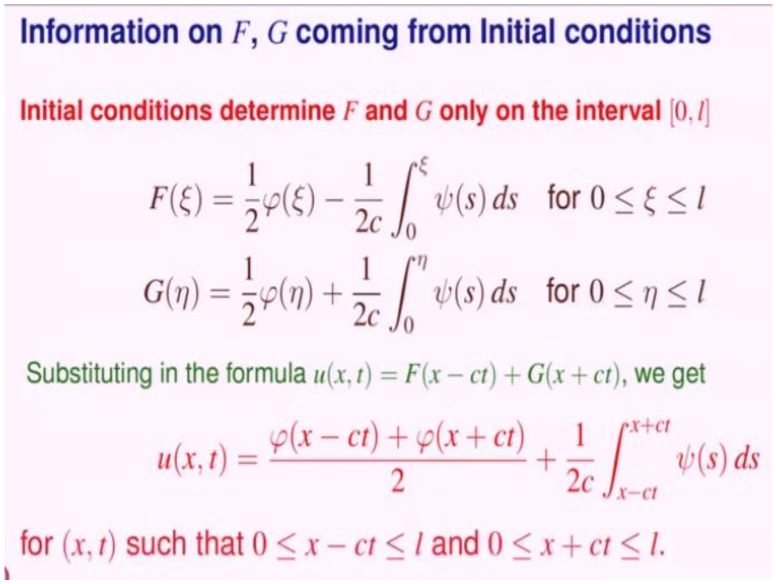

When we substitute in the formula u x, t = F of x – ct + G of x + ct what we get is the d'Alembert solution exactly d'Alembert solution. What is this domain where psi is between 0 and eta is between 0 and l that is exactly same as all those x, t for x – ct is between 0 and l and x + ct is between 0 and l.

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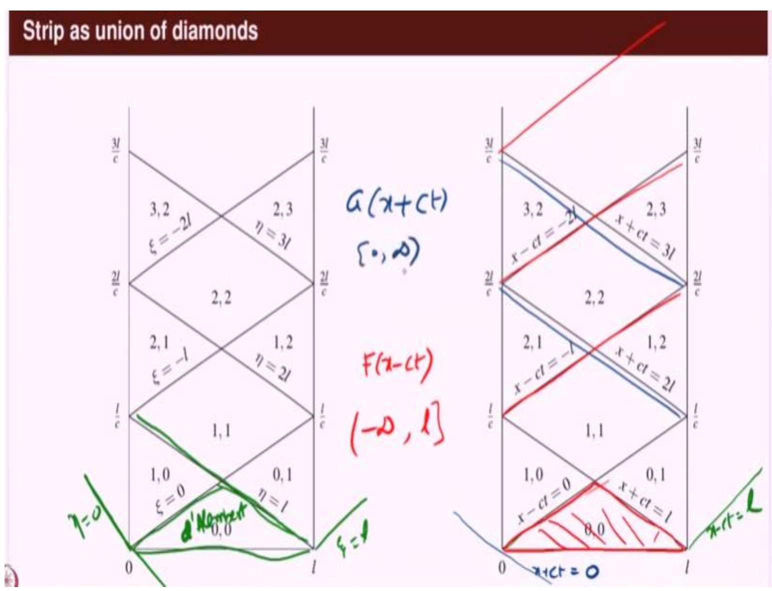

So initially conditions determine solution by the d'Alembert formula but the solution we obtain only for few xt but not for all xt in the strip that we wanted to solve for. It is only here in this region.

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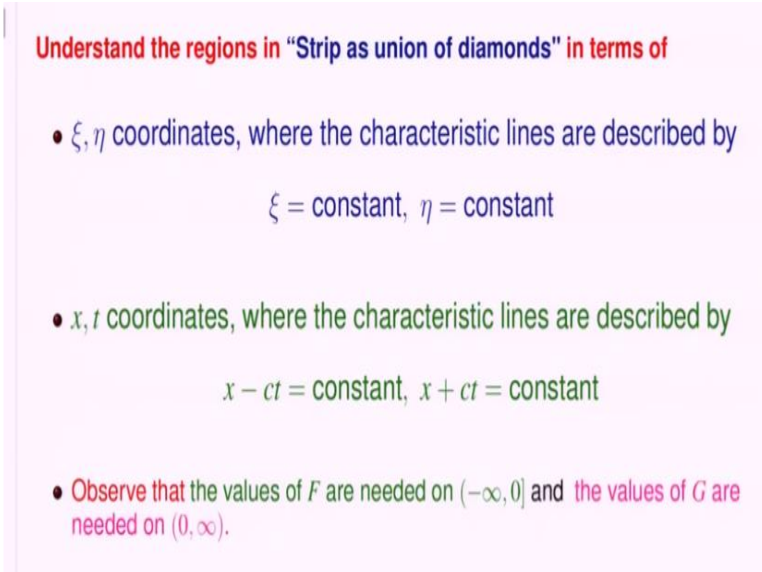

That means solution is determined in this region which is written here xt in 0,l cross 0, infinity such that x – ct lies between 0 and l and x + ct lies between 0 and l. We have figure on next slide let us look at that and also the interpretation in terms of psi and eta. Psi between 0, l eta between 0, l look at this picture this is exactly this is the region where the d'Alembert formula gives us solution.

If you look at this is x – ct = 0 and if you just write line here this is x – ct = l so this is psi = 0 so this psi =l. So psi lies between 0 and l similarly eta lies between eta = l is here this line eta = l and parallel line here would be eta = 0. Common part of that is precisely this triangular region that is where we are d'Alembert solution. So the initial conditions determine the solution in this triangular region.

So this picture is sometimes called diamond picture strip as a union of diamonds in terms of psi eta coordinates where the characteristic lines are psi is equal to constant and eta equal to constant. In terms of xt coordinate there are x – ct equal to constant x= ct equal to constant observe that the values of F, are needed on minus infinity is 0 and G are needed on 0 infinity why is that? Let us go back to the picture.

So this picture we obtain as you see x – ct = 0 x = ct = -l -2l and so on -3l and so on. Therefore the values of F, are need because we have an expression f of x – ct values are needed only from this l to minus infinity here. Similarly if you look at G what do we need? Is a here this is x + ct =

0 x + ct = l here x + ct = 2l, 3l and so on therefore since we have G of x + ct we need that G is defined on 0, infinity that is all we need. So values of F, are not positive real numbers after l and the values of G are needed for negative real numbers. And the various regions in this picture the boundaries are the lines the characteristic lines.

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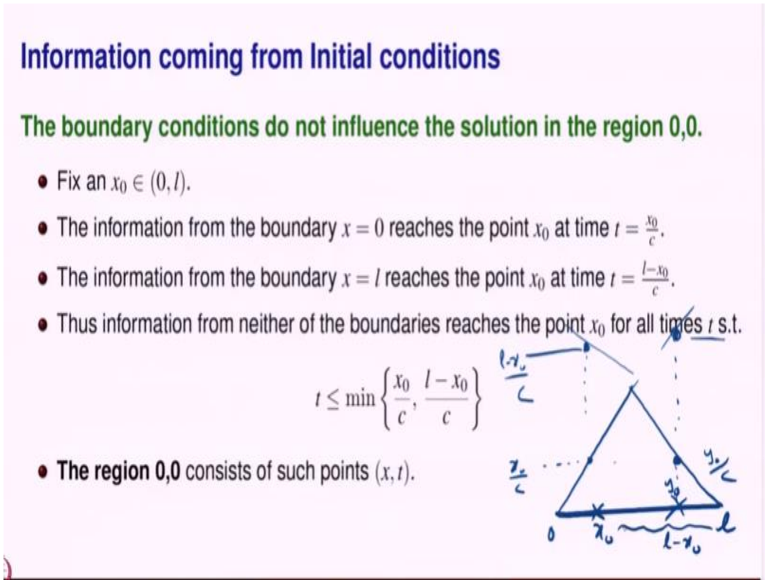

Now if you see the 0, 0 we have not used boundary conditions that mean boundary conditions do not influence solution in the region 0, 0. So fix an x naught in 0, l the information from boundary 0 = l reaches the point x naught at time x naught by c example at we will draw the picture at m. And the information from the boundary x = l reaches the point x 0 at time. Because the distance to the boundary is l = x naught speed is c therefore l – x naught by c is a time taken.

Thus information from neither of the boundaries reaches the point x naught for all times t which is less than or equal to min both of them minimum of this 2 times. Information does not reach x naught and the region x, 0 consist of such points x, t this is the point 0 this is the point x, 0 this is 0 this is l. So the distance is x, 0 speed is c. Therefore the time is x naught by c this distance is l – x naught therefore the time taken is here l – x naught.

If course in this as you see x naught is very close to 0 than l that is why this time is less if you have taken y is 0 here. This will be the corresponding time y, 0 by c and this information from here will be here this time. In any case if you take minimum of the times up to here the information does not reach. And this is true of every point in x, t set of all, those x, t is the times and spatial points they are the ones which are in the region 0, 0.

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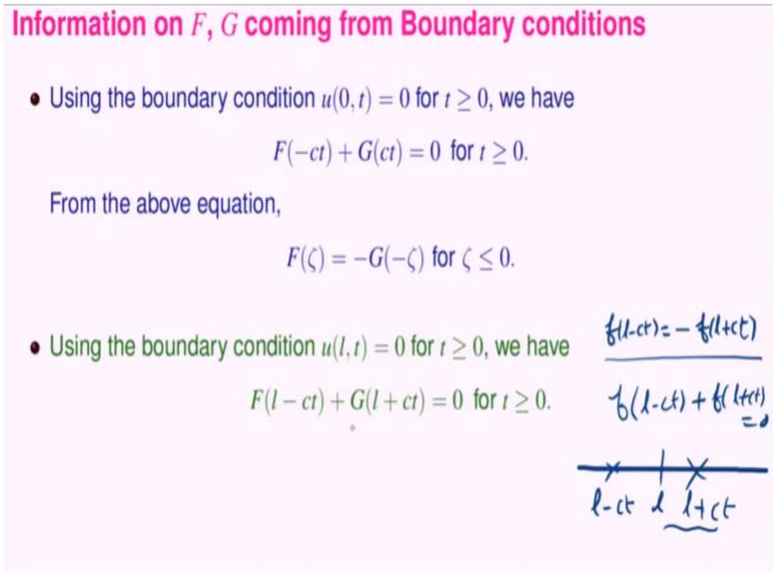

Now let us see what information we can get using the boundary conditions what are the boundary conditions is? u 0, t = 0 so what we get is F of –ct + G of ct = 0 because the f = 0 we get this equation. Now I observe here that F ct c is always positive so ct is positive –ct is negative so F can be enhance for negative real number provided you know the G for the corresponding positive real number ct. What do I now about G?

G we have already determined in the interval 0,l so therefore now F can be given meaning or defined or F is determined in the interval –l to 0. Because G is known on 0, G l that is what is the idea we are going to use. So we write F of zeta = -G of –zeta for zeta less than equal to 0 this if of course every zeta but only thing is that I know the formula for G only when –zeta is in 0, l. There are 2 aspects here one this is a relation satisfied between F and G this is true for every negative zeta.

There is no doubt here this is true but do you know the expression for F that can be done only when zeta is in –l, 0 because G is known only in 0 to l. Now let see the other boundary conditions u of l, t = 0 substitute in the formula for u we get F of l – ct + G of l + ct is 0 for t greater than or equal to 0. So this condition how do we see you have l here this is a point l + ct that means u travel ct to the right of l and you travel to the left side of l this is l – ct l + ct.

So F at l – ct + G at l – ct is 0 that means F at l – ct is – of G at l + ct see added be in same function imagine it was some function F of l – ct + F of l + ct = 0. Imagine this is something different from what we are considering here then this actually means what F of l – ct = F of l + ct. What does this mean? It means that the values at l + you travel ct distance to the right side and to the left side the values are tide like this it just means.

Imagine l = 0 what is it F of –ct = -F of ct it means F is hard here about 0 at these condition what is called F is odd about the point l. But here it is not the same here it is F and G.

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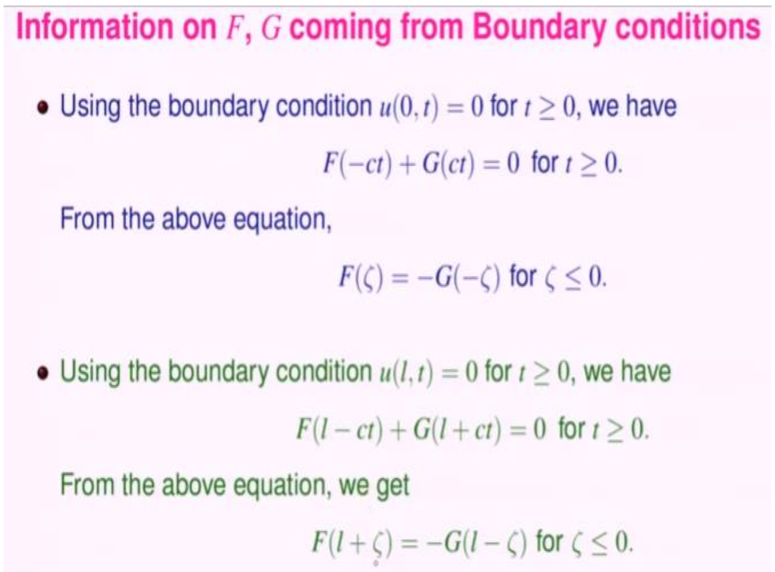

So this will give us F of l + zeta = -G of l – zeta for zeta less than equal to 0 because ct can be any positive real number. Therefore this can be any negative real number.

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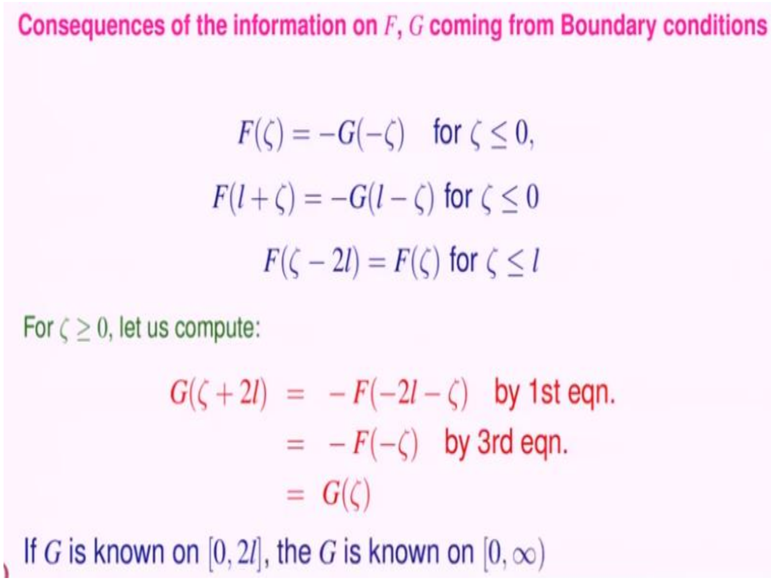

Our goal is to find F and G right so these are the information that we got. This we got from one boundary condition this we got from the, another boundary condition. Now let us compute for zeta less than equal to l F of zeta -2l that is equal to –G of 2l – zeta. This is actually of the minus of the inside thing so I am using the first equation now just this is rewriting of the same thing l - zeta – l. Now I am going to use the second equation I need the argument to be negative right yes because zeta is less than equal to l zeta – 1 is negative.

So –G of l minus any negative quantity is given by F of l+ that negative quantity done by second equation. But what is this F of zeta so F of zeta -2l = F of zeta for every zeta less than or equal to l. What does this mean we have l here take any zeta and take zeta – 2l the values of F are the same it means it is a periodic function on the left side of l. So if you know the value of F on this interval –l, l you know the values everywhere every negative number thanks to this relation.

This follows just for boundary conditions so if is known on –l, l then f is known on minus infinity to l. For zeta (()) (26:14) equal to 0. Let us know show a similar thing about G of zeta + 2l is –F of -2l – zeta by first equation and that is equal to –F of minus zeta because we already showed the periodicity which is G of zeta? Therefore if G is shown on interval 0, 2l then it is known everywhere 0 to infinity.

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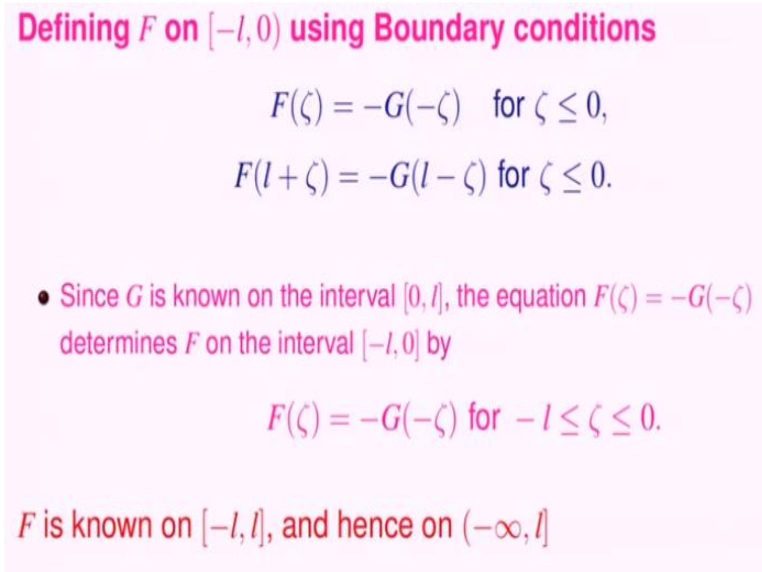

Now let us define here on –l, 0 is in a boundary conditions we have already noted how to define G is known on 0, l therefore F is known on –l to 0 by this formula. Therefore now F is known is determined on –l, l therefore on minus infinity, l.

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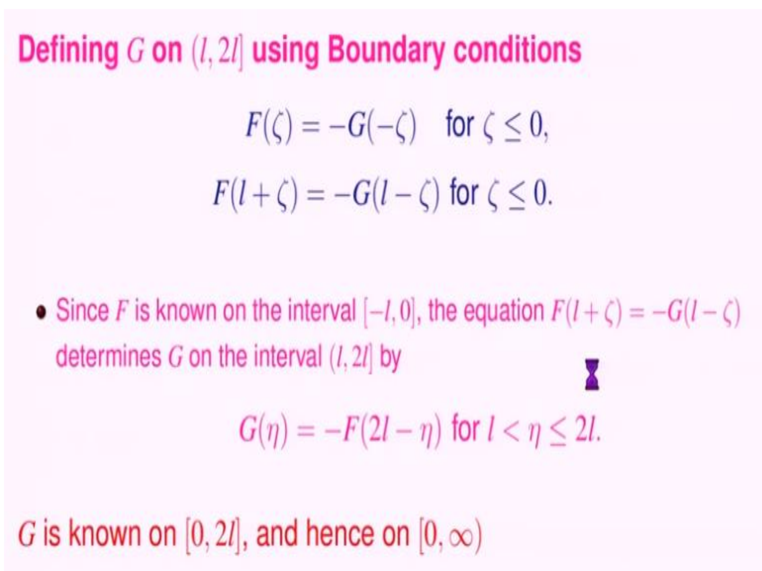

Now let us do the other one for G is already known on 0, l let us define it on l, 2l so that G will automatically determine due to periodicity to 0 to infinity. Now F is known on –I, 0 so we are going to use this second condition because this condition is saying I have l here 0 here –l here. I want to define in this region up to 2l in this region I want to define. Therefore from the second equation we get this G eta = -F of 2l- eta for l less than eta less than or equal to 2l. G is known on

2l therefore it is known on 0 infinity so, please stop here and convince yourself about this relations where this relation we have derived.

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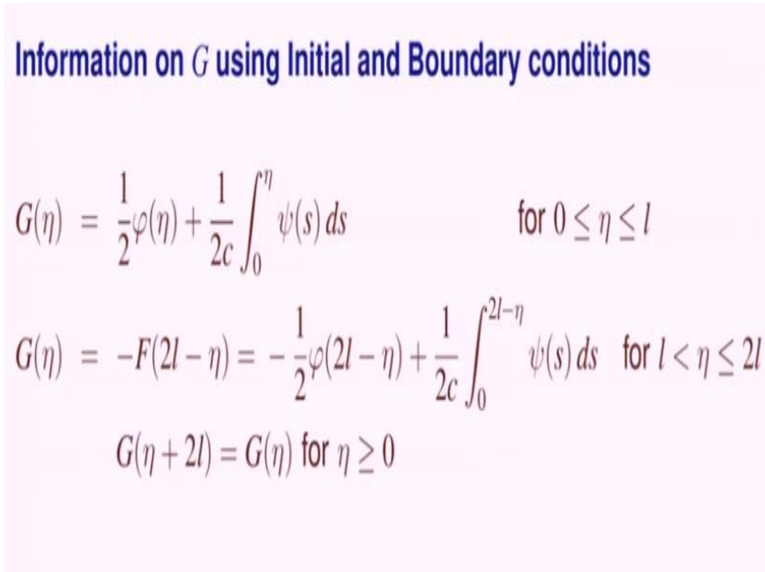

So the information that we get and using both initial and boundary condition is as follows. Using initial condition we follows this formula for F on the interval 0, l using the boundary conditions we have extended not extended we have determined F on the interval –l, 0 this is just a formula for GF –psi into –GF-psi and the periodicity. On G similarly initially we have determined on interval 0,l then using the second boundary condition we have got this expression on l, 2l and we have periodicity.

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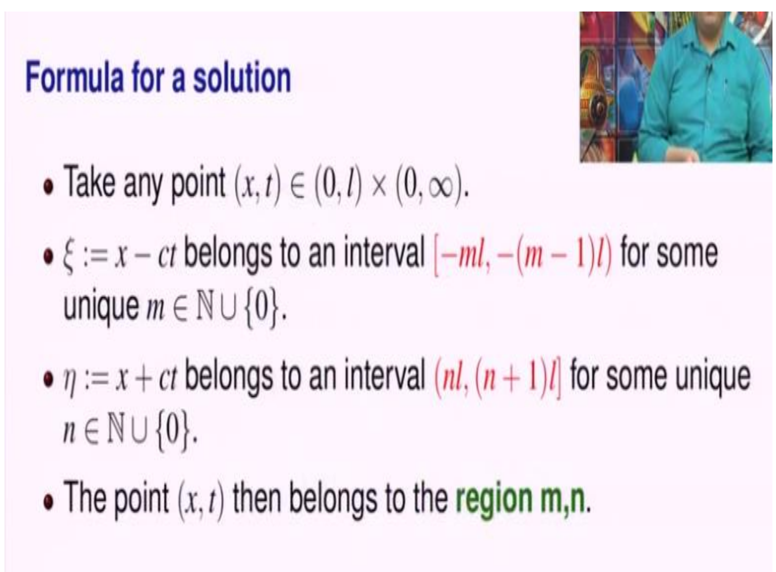

So now we need to write the formula we need to determine F and G everywhere so now we are going to write the formula for the solution. So take any point x, t in 0, l cross 0, infinity. Now if you take somebody a point in that diamond picture it is going to line some region means what? It is going to live in between psi equal to some number and psi equal to number +1. Similarly eta equal to a number and equal to that number +1 so that is what we are setting up the notation.

Now x – ct belongs to some interval of this (()) (29:29) in fact unique m is unique in n union 0 – ml, m – 1 into l. Similarly eta x + ct that belongs to interval of this time for some unique m. So this is actually determining the region m, n we have marked the region 1, 2 in that picture. So general region m, n is characterized by this.

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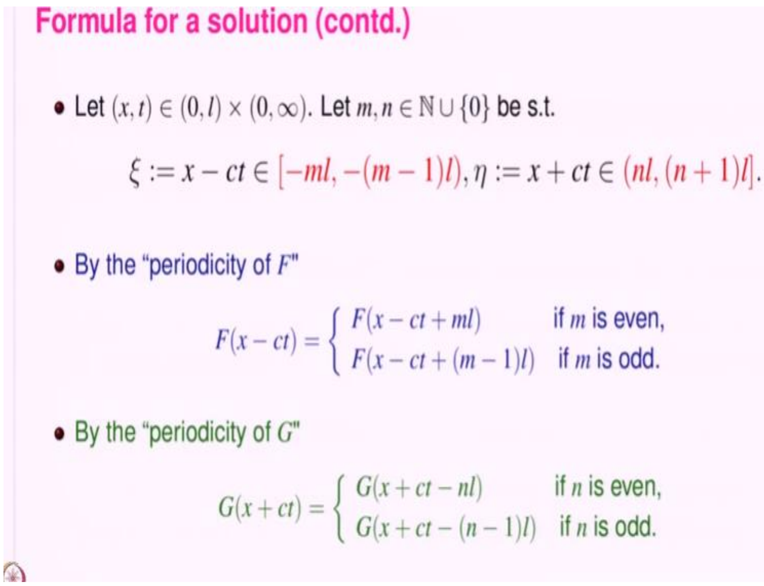

For example I take 2, 3 this is m, n m is 2 right so it lies between what? Psi equal to –l to psi equal to -2, l it is here. Psi = -1 to psi equal to -2l what about eta? This point I have taken so it is going to lie between here which is that eta equal to 4l and eta equal to 3l and this is the region that we get finally. So let psi belongs to let x, t belongs to let xt belongs to 0, l cross 0 infinity let m, n be such that psi is in interval x – ct is in this interval x + ct in this interval.

That is the region m, n now we need to write F of x - ct but as such we have formula for only on l rest is periodicity. So we are going to use periodicity and bring this value x – ct into the interval – l, l. Similarly x + ct we are going to bring it to 0, 2l by translation. So by the periodicity of F of x – ct has this formula if m is even it is F of x –ct + ml. If m is odd it is m – 1 interval I request you to do this computations by yourselves. Similarly for G we have an expression.

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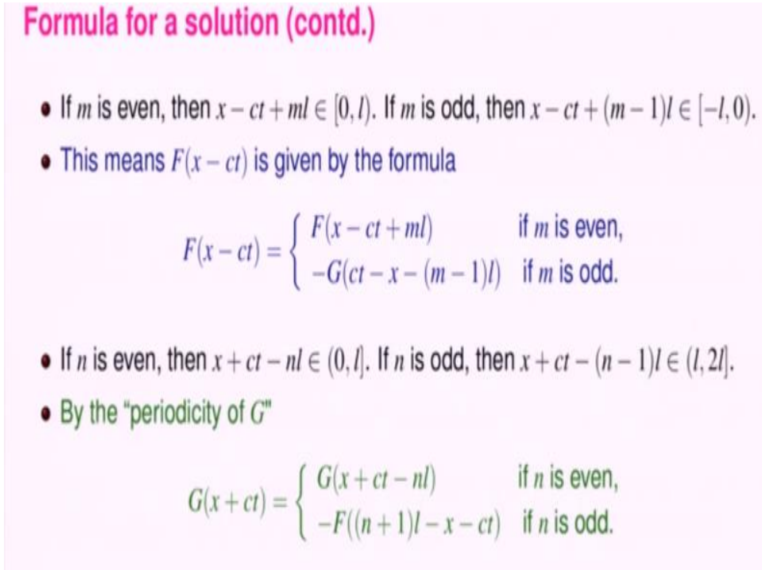

So if m is even x – ct + ml is in G value now we have to be careful because F and G are F is known on –l, l formulas are different on –sl to 0 and 0 to l. Similarly for G formulas is different in 0, l and l to 2l that is why we are doing this more final classification if m is even then x - ct + ml is in 0, l. If m is odd this is 1 which F uses so this will be in –l, 0 and I know the formula for F on that. So therefore F of x – ct is given by F of this if m is given and F of this if m is odd. Similarly for G we can write down we had this formula.

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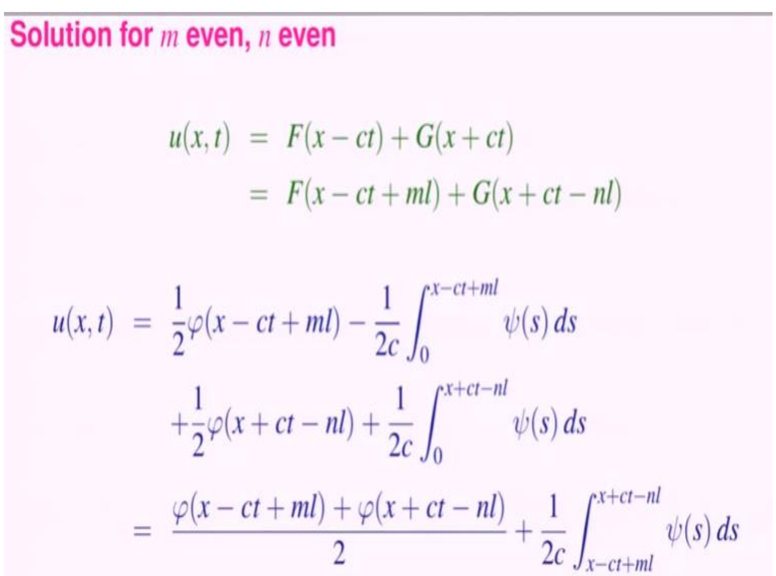

Now this is something that you must verify by yourself of m is even and n is even because I have to write now u of xt = F of x – ct + G of x + ct. But the values of F and G depend on whether m and n are depend on odd so I have 2 cases for m and 2 cases for n so in overall I will have 4 cases. So this is expression for u this is expression in terms of phi and psi. m even n odd we get this and we this formula this is for m odd and n even.

Note the final formula not in the d'Alembert formula because d'Alembert wants only phi here but there is a minus here and x – ct here but something else easy here.

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Solution for m odd, n odd

$$\begin{aligned} u(x,t) &= F(x-ct)+G(x+ct) \\ &= -G(ct-x-(m-1)l)-F((n+1)l-x-ct) \end{aligned}$$

$$\begin{aligned} u(x,t) &= -\frac{1}{2}\varphi(ct-x-(m-1)l)-\frac{1}{2c}\int_0^{ct-x-(m-1)l}\psi(s)\,ds \\ &\quad -\frac{1}{2}\varphi((n+1)l-x-ct)+\frac{1}{2c}\int_0^{(n+1)l-x-ct}\psi(s)\,ds \\ &= \frac{-\varphi(ct-x-(m-1)l)-\varphi((n+1)l-x-ct)}{2} \\ &\quad +\frac{1}{2c}\int_{ct-x-(m-1)l}^{(n+1)l-x-ct}\psi(s)\,ds \end{aligned}$$

So we will do that later so let us see m odd and n odd again mixing the formula for F is this G is this so this is the expression. Now I am going to substitute the values of G and F that I know and that becomes this is the formula.

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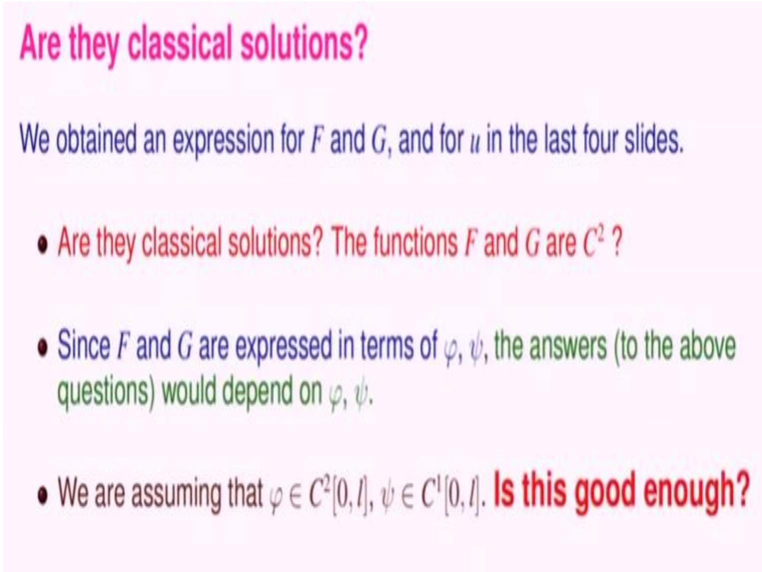

So now a question is are they classical solutions? We obtain an expression for F and G and for u in the last 4 slides. As a classical solution are the function F and G is C 2 functions. Since F and G are expressed in terms of phi and psi answers will depend on phi and psi. So we are assuming that phi is C2 close and role C l and psi is C 1 of 0, l is this good enough?

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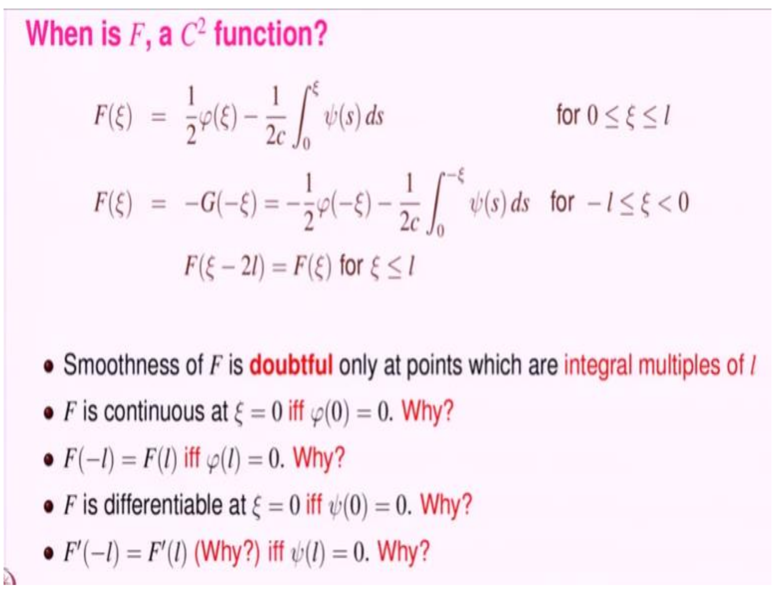

When is F of C 2 function this is what we know about F there is no doubt about smoothness of F when you are not at these points like 0, l – l and integral multiples of l because inside functions are nice this is nice so it will be smooth function. So the doubt is only at points which are like –l, 0, 2l, -2l etc., Because that is where we have brakes in the formula so F is continuous at psi = 0 that means let us see here psi is greater than equal to 0 right.

So you pass to limit as psi goes to 0 what you get this integral terms goes off what you get is half phi of 0. And here from here you can get half phi of 0 both have to same phi is if F is continuous which means phi of F is 0. Now these are all similar considerations F of -l = F of l if and only if phi l is 0. Please answer this questions for yourself F is differentiable at psi = 0 if and only if psi of 0 is 0 F dash of –l is F dash of l why do we have this?

Because it comes from here these holes for F of course it is hold for F dash double dash and so on. And that is true if and only psi l is 0.

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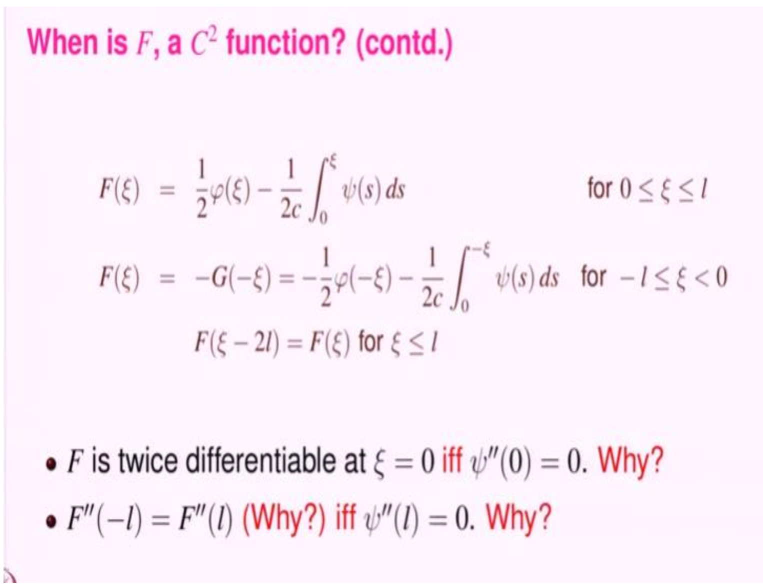

F is twice differentiable psi = 0 if and only if psi double dash of 0 is 0 similarly F double dash at –l is same as F double dash l if and only if psi double dash of l is 0.

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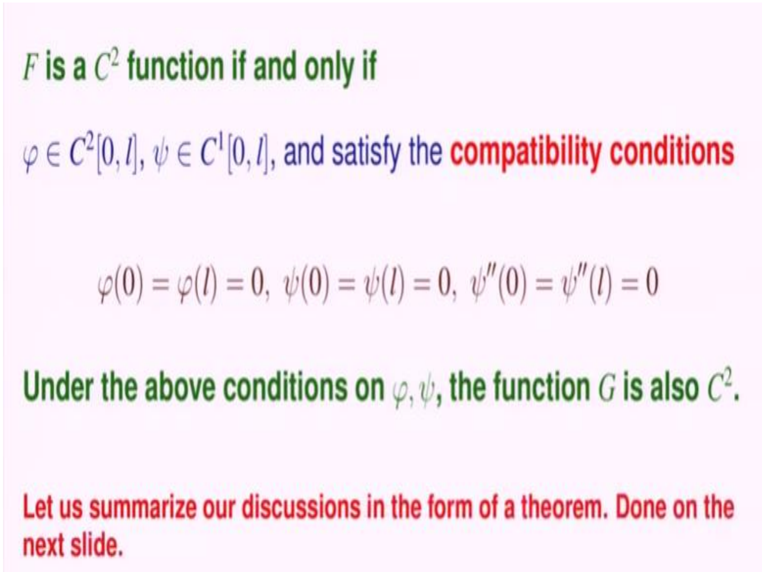

So F is C 2 function if and only if the following thing happens phi and psi satisfies the following compatibility conditions these are called compatibility conditions. Phi at the end point 0 and l are 0 psi is also 0 at the end point 0 and l so is psi double dash under the above conditions phi and psi the function G is also C 2. So we have a classical solution if phi satisfies these 2 conditions and these compatibility conditions. So let us summarize our discussion in the form of a theorem it is done on the next slide.

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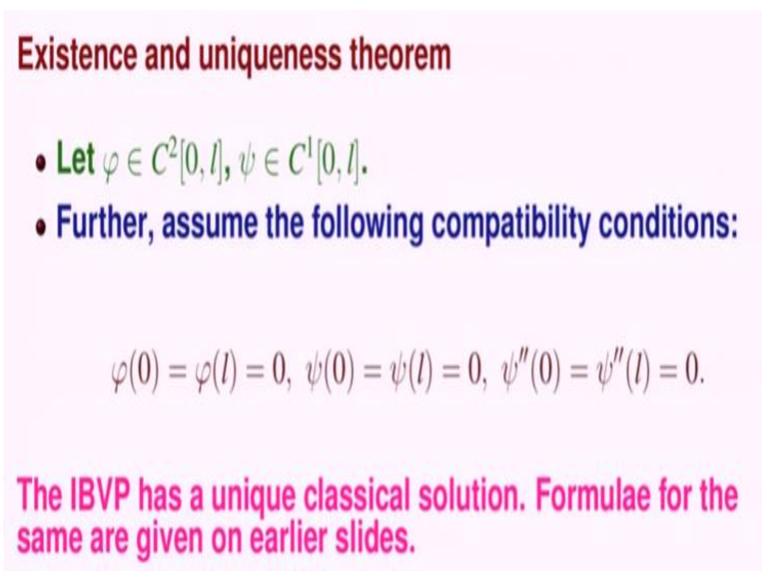

Existence uniqueness theorem let phi and psi have this smoothness further assume the further compatibility conditions. These 3 sets of compatibility conditions the IBVP as a unique classical solution this is IBVP remember this is a Dirichlet boundary conditions. If you are considering Neumann boundary conditions the compatibility conditions will change.

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So let us get the solution d'Alembert form that is another goal that we have.

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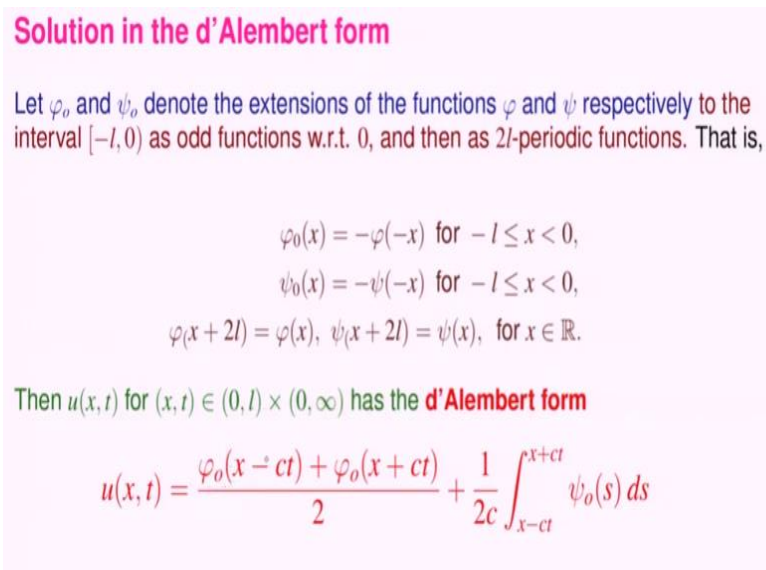

Or that what we have to do is phi and psi which are defined only on 0, l we need to extend to R. Because we need to take phi of x – ct psi of x + ct and so on they should make sense that means phi and psi should be defined for every real number. Let phi naught and psi naught denote the extensions of functions phi and psi respectively. To the interval l, 0 as the odd functions with respect to 0. And then as 2l periodic functions to r.

So phi 0 of x is –phi of –x psi 0 of x is –psi of – x this is what is defining are extending the function phi and psi as odd function to the interval –, 0. After that we can extend it to R then u of x, t as the d’Alembert form this is exactly what we like. Phi 0 of x – ct + phi 0 of x = ct by 2 + 1 by 2 C x – ct x +ct psi naught of S ds this is exactly how the d’Alembert formula looks for the Cauchy problem in 1 d. But now not phi and psi in terms of phi naught and psi naught that is the only change.

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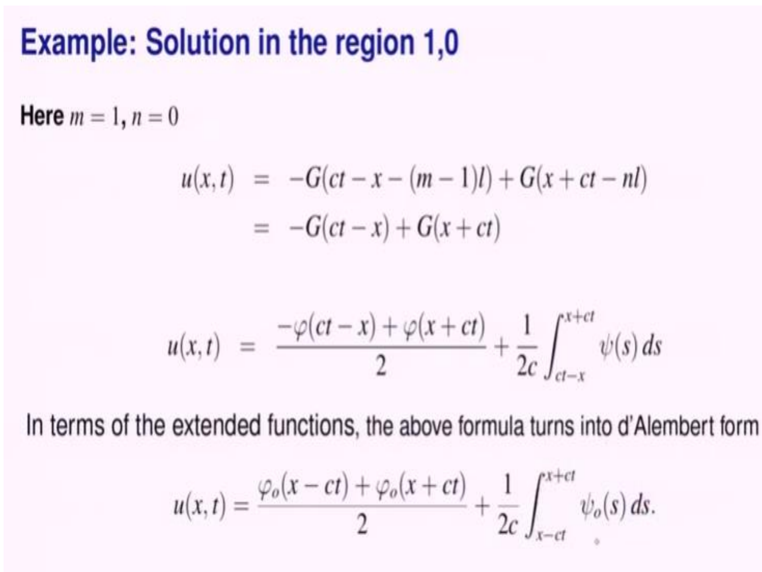

Let us look at an example of solution in the region 1, 0 means m means n is 0 so this formula which we have taken from the slide where the solution m is odd n is even. I get this expression and this is a formula we get. Now in terms of the extended functions it becomes this because ct – x will be in –l, 0 and we have extended phi 0 as an odd function. Therefore this quantity is precisely phi 0 of x – ct. This is between 0, l therefore there is no change it is same.

And this integral becomes this, integral that is left as an exercise for you though I will be doing on some other domain later on.

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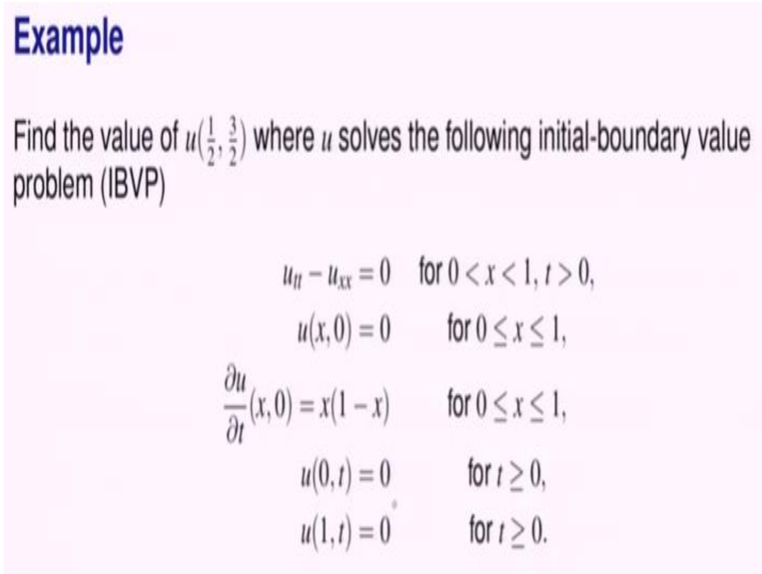

So find the value of u of 1 by 2 3 by 2 where u solves the IBVP given here where I have given psi equal to x into 1 – x phi is 0 as usual the Dirichlet boundary conditions which are 0 here.

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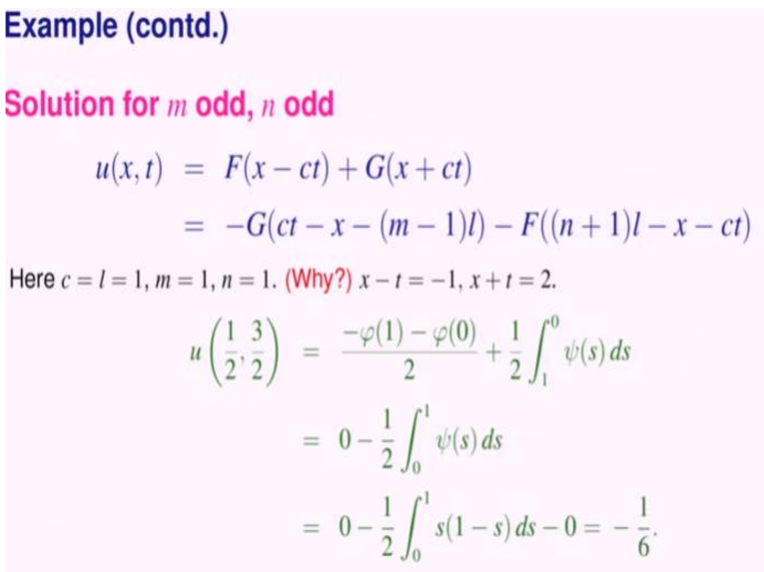

So solution for m odd n odd will apply here because we have to look c is 1 so x – t is half -3 by 2 half -1 x + t is 2. So m = 1 n = 1 that is what you have to explain reason I have written here. Formula is precisely this is here and needs a matter of computing this integral – 1 by 6.

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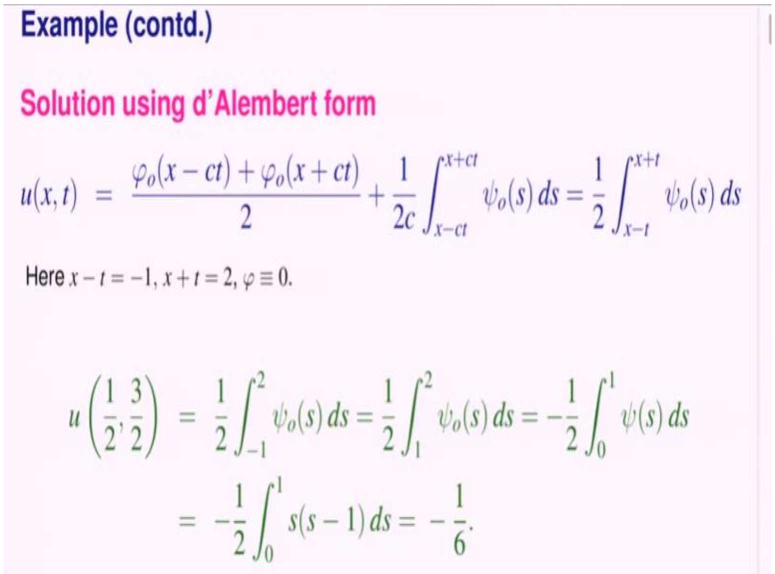

Now let us get the solution using d'Alembert form because on the last slide we have used the formula which is coming from this slide which gives a solution for m odd n odd. So now we need to extend our function phi naught and psi naught but phi naught is 0 no need to extend. So this is the formula so we need to know what is psi naught now this formula half -1 to 2 psi naught of x.

Now psi naught is an odd function therefore if you integrate it will -1 to 1 it will be 0. Therefore this integral is essentially from 1 to 2 and 1 to 2 how it will be defined psi by this manner. Then please do this computation by yourselves of course you get the same answer.

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Now let us illustrate the region 1, 2 where we see the connection between the d'Alambert form solutions and reflections.

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Solution in the region 1,2

The region 1, 2 is given by

$$\{(x,t) \in [0,l] \times [0,\infty) : -l \le x - ct \le 0,\ 2l \le x + ct \le 3l\},$$

and the solution in this region is given by

$$u(x,t) = F(x-ct) + G(x+ct) = F(x-ct) - F(2l - x - ct),$$

which in terms of the initial data take the following form

$$u(x,t) = \frac{-\varphi(ct-x) + \varphi(x+ct-2l)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct-2l} \psi(s)\,ds.$$

So region 1, 2 is algebraically even by this inequalities and solution is given by this you can write from the slide m is odd and n is even you get this.

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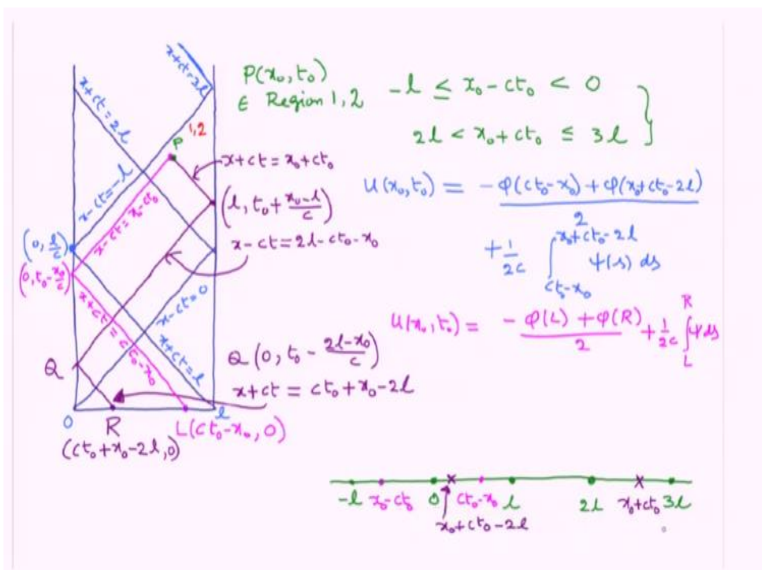

Now look at this picture here I am in this region 1, 2 by take a point P I want to find the u at x naught t naught P is x naught t naught. It is in the region 1, 2 of course region 1, 2 is characterized by these inequalities that you know. So what I am saying here how do I get this solution? So I have to do every time the m even n odd find which m which n etc., how it comes?

Look at this point now through this point there are 2 characteristics which pass through one is this pink colour which is moving towards right side.

And this is the violet colour which is moving towards the left side as te increases but what we have to do is? We have to do backward thing so take the point P we want to come towards x axis because that is where our initial data is here. So therefore follow these characteristics and come back you will hit the boundary. When you hit the boundary change over to the characteristic line from the other family and you hit this point.

This computation can be easily done after all this is straight line and you know this equation x = 0 is x = l you come here and just 1 reflection you do you come here. Now you come from this side you hit this and then switch over to other family characteristic passing through that point and then you hit again this boundary and again you switch over and call it R. So this is l and this is R and exactly the coordinates are identified what is R what is l?

Now this is a formula that we have written down on the previous slide no the formula are in the region 1, 1 is given like this. Now this is just for a placement where what is happens so –l, 0, l, 2l, 3l now x naught + ct naught lies between 2l and 3l so it is somewhere here. and x naught – ct naught between –l and 0 so it is here. But what we are is ct naught – x naught which his coming this side and this we are translating by 2l so that it fit into 0, 2l in this interval.

That is where we know the function F and G is right G is known 0, 2l so this is x + ct so G is responsible they come here. And what is l ct – x naught what is r? Is this point why are we doing this picture is because we want to get the d’Alembert formula. In d;’Alembert formula what is integral that is between x naught and -ct naught to x naught + ct naught.

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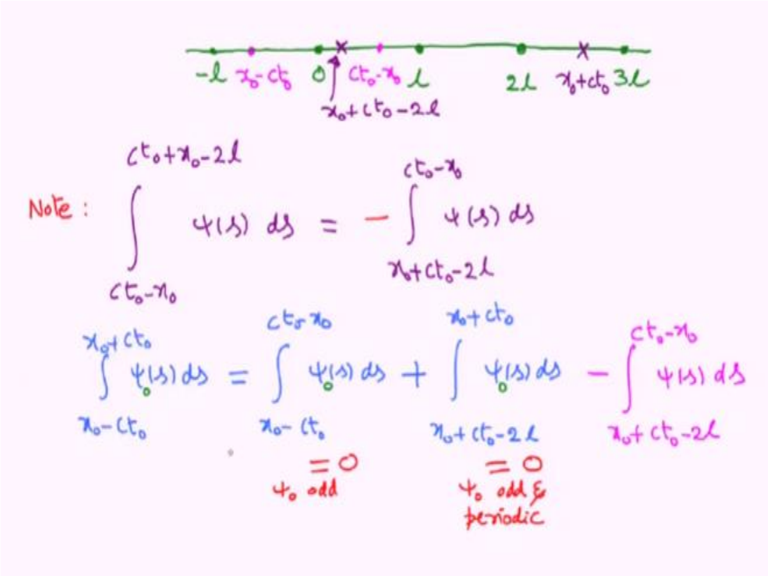

So let us look at this is integral we have this integral is nothing but minus of this just switching the limits even this. Now this is the integral that appears in the formula that i am writing as between x naught –ct naught – x naught that is come here to here and again from here to here and this is repeated so I subtract but this is 0 because the odd function and this is symmetry interval on this 0.

Here it is odd as well as periodic therefore we this is an integral on length 2 and interval it will be 0. So what remains is just this in other words? What we have is precisely the d'Alembert form?

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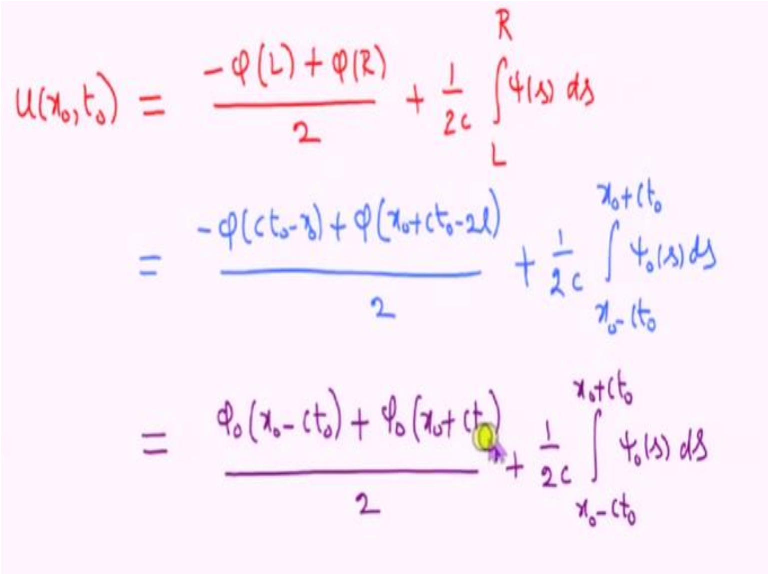

So this is integral we got instead of L to R x naught – ct naught x naught + ct now this is P of x naught – ct phi naught of x naught – ct naught. And this periodicity is phi naught of x naught – x + ct naught.

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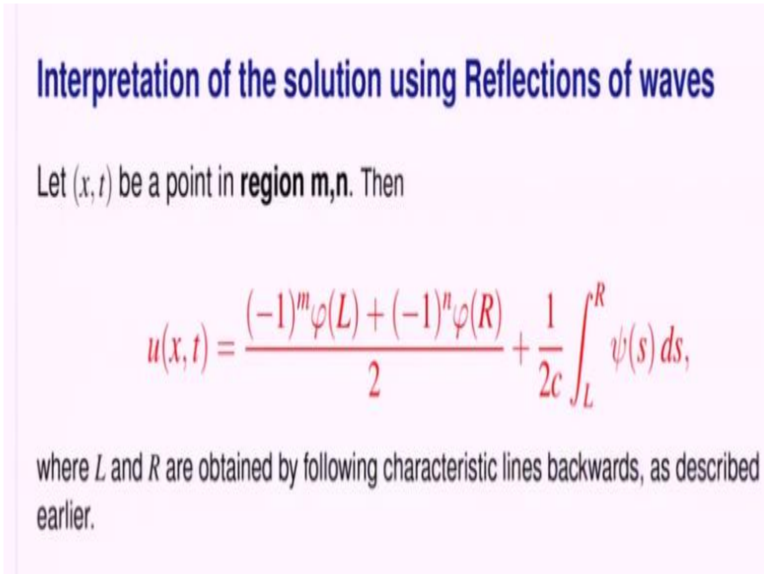

Interpretation of the solution using reflections of waves let x be a point in the region m, n then the formula is this. The L and R exactly as I suggested we have to come to L and R but that is multiplied with -1 power m -1 power n respectively. And it is 1 by 2c integral L to R L and R is obtained by following characteristic lines backwards as described earlier.

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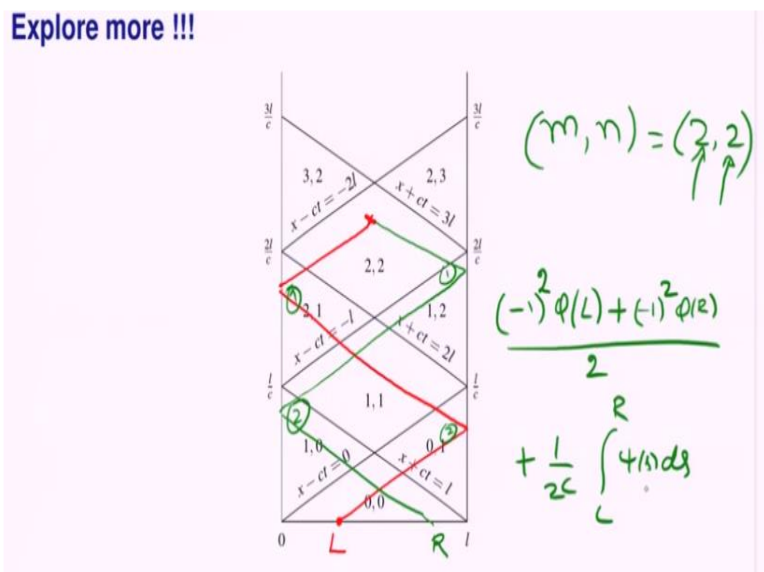

Let us also do one more point and try let me take a point here so first thing is go like this of course these are characteristic lines which is the family of parallel lines this one. So this point you can compute the coordinate this is L and less to the R. So this here m, n our m, n is 2, 2 how many reflection we made one reflection here and 2 reflections. Similarly here one reflection for this side and 2 reflection so, this is actually the number of reflection in that we are going to do.

You can identify this L and R and the formula as before is applicable what, is the formula? -1 power m in this case it is 2 into phi at this point l + 1, l power again n is 2 phi is R by 2 + 1 by 2 C integral L to R psi S ds. This formula holds is a very easy way to determine the solution.

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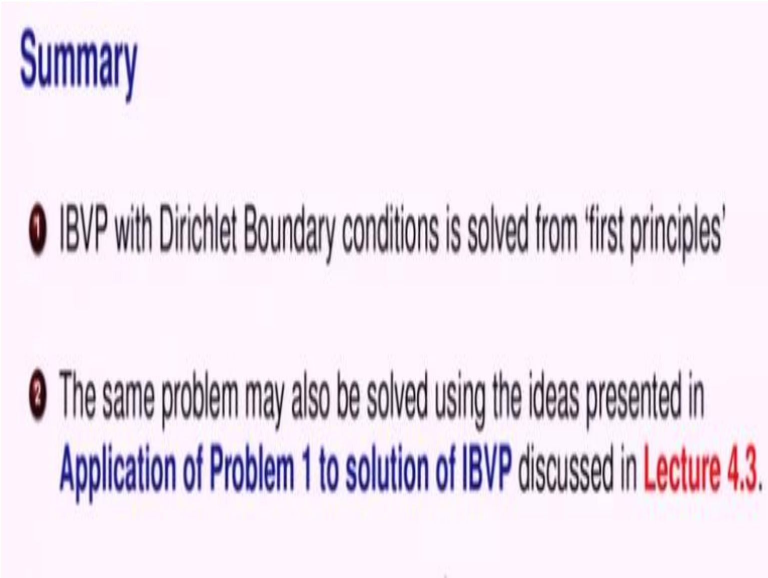

So let us summarize what we did you considered the IBVP with Dirichlet boundary conditions it is solved from so called first principles. We are calling it first principle because we have obtained to at the beginning we obtained the solution to the homogenous wave equation to transforming into this system of characteristic coordinates where the equation read W I eta = 0 and solution was a function of psi + a function of eta that is why we are calling this first principles.

And the same problem in also be solved using the idea presented in tutorial which is called lecture 4.3 there we did a problem 1 as an application of that problem to solution of IBVP this IBVP can also be solved. Thank you.,