Partial Differential Equations
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Module No # 06
Lecture No # 32
Non-Homogenous Wave Equation – Duhamel Principle

Welcome to this lecture in this lecture we are going to discuss about the non-homogenous equation how to solve it using a very general principle called Duhamel principle.

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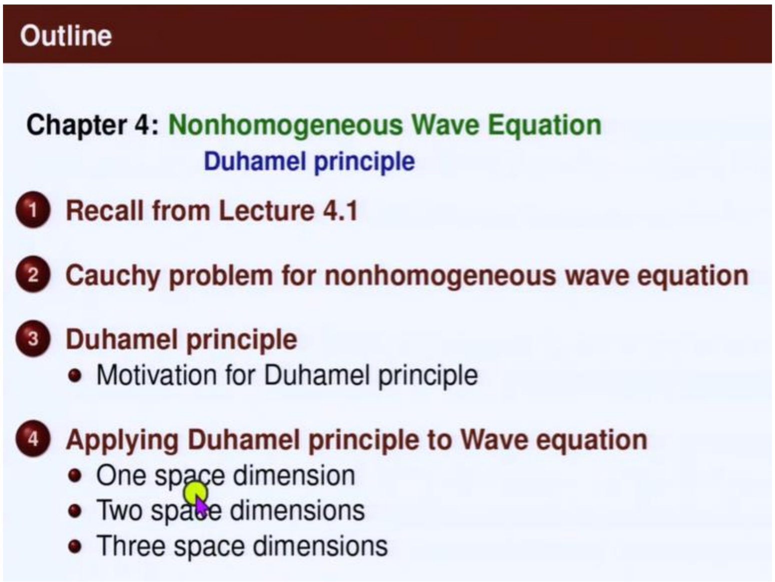

So out for the lecture is as follows first we recall certain things from the lecture 4.1 and then we state the Cauchy problem for non-homogenous equation. Then we introduce to Duhamel principle after giving a motivation for it then we apply to handle principle to wave equation will do for 1, 2 and 3 space dimensions.

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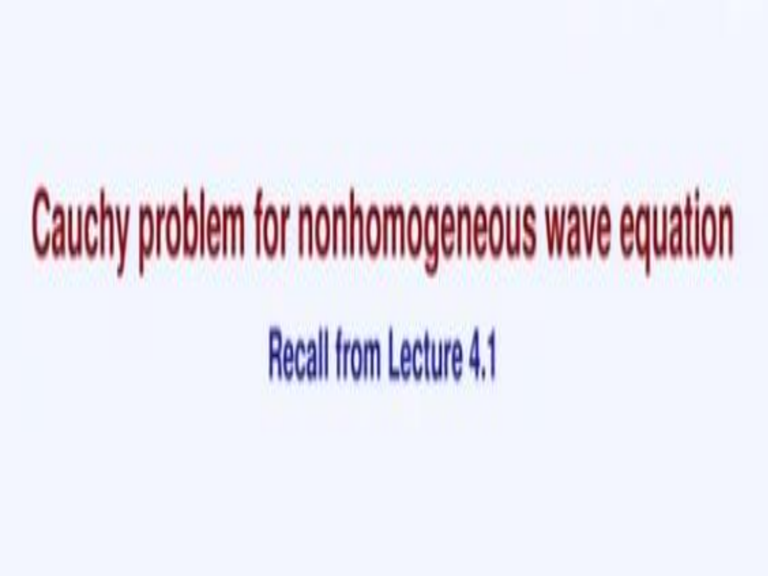

So let us recall from lecture 4.1 certain things about Cauchy problem for non-homogenous wave equation.

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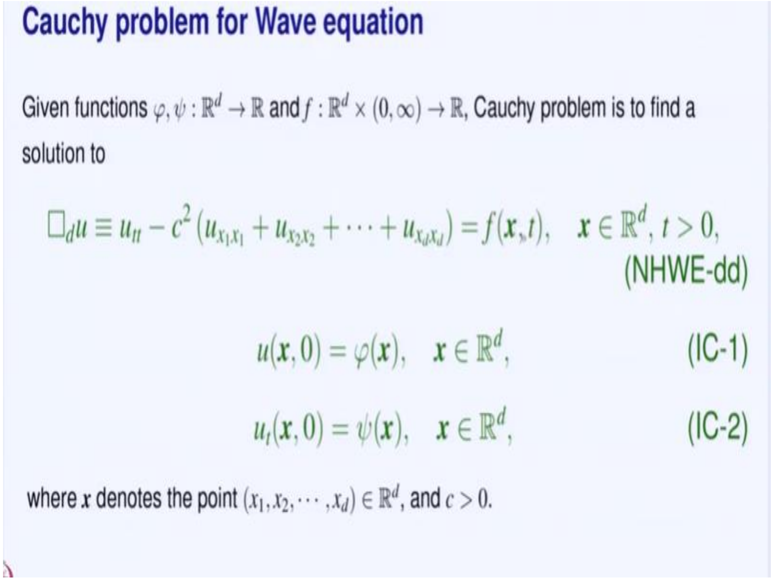

So given functions phi psi and F Cauchy problem is to find a solution to the wave equation this is a d'Alembertion operator equal to F on R d cross 0 infinity. Such that u of x is phi x u t of x 0 is psi x. Phi and psi are generally referred to as Cauchy data and F is referred to as source term.

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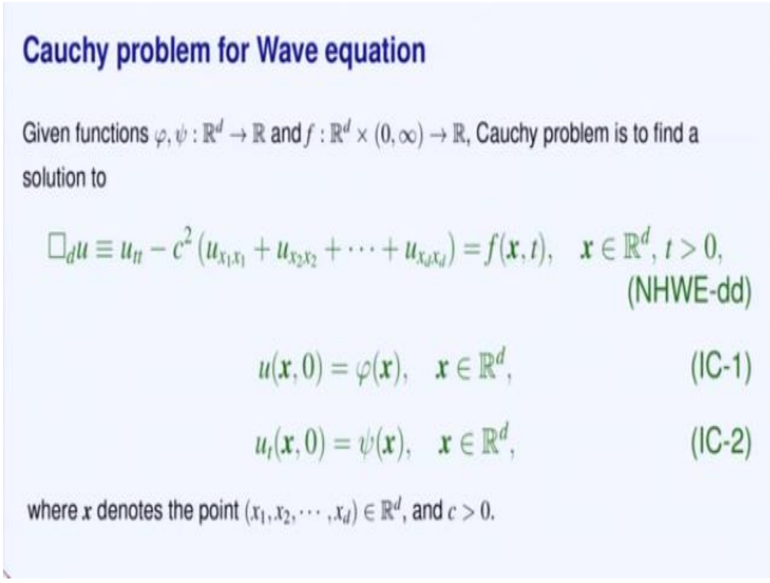

The d'Alembertian operator is a linear operator since d'Alembertian applied to u + v is d'Alembertian acting on u + d'Alemberitan acting on v therefore a solution to the Cauchy problem which was stated on the earlier slide that is the non-homogenous wave equation with the Cauchy data that may be obtained as some of 2 functions v and v tilde where v the solution to the Cauchy problem where the wave equation is without the source term it is called homogenous wave equation.

When there is no source term is referred as homogenous wave equation. And v tilde is the solution to the non-homogenous wave equation but with 0 initial conditions we will say it more clearly on the next slide what the problem emphasize is.

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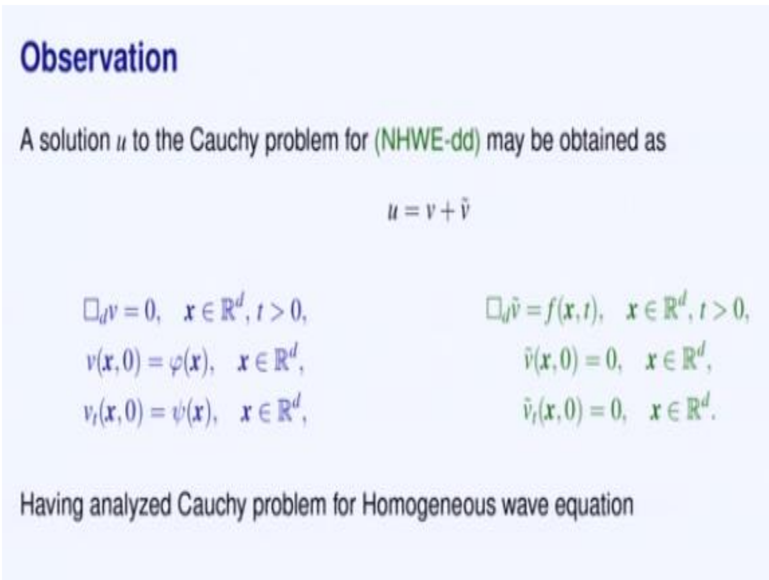

So u = v + v tilde and v satisfies this homogenous wave equation and the Cauchy data and v tilde satisfies homogenous wave equation and 0 Cauchy data 0 initial condition so when you add v + v tilde by the linearity of d'Alembertian operator d'Alemberitian of u will be d'Alembertian of v + d'Alembertian of v tilde that will be 0 + f therefore f. So you will get the non-homogenous wave equation satisfied by u and we look at the first initial condition v x, 0 is phi x v tilde if 0 is 0.

Therefore u of x, 0 is phi x similarly u, t of x, 0 is v t of x, 0 + v tilde t x, 0 one of them is psi and other one is 0. Therefore the addition will psi the sum will be psi so therefore u indeed satisfies non-homogenous wave equation. And recall that we have already analyzed these problems the homogenous wave equation and the Cauchy problem for we have analyzed already. Therefore it is enough that we analyzed this how to solve this problem then we know how to solve the non-homogenous Cauchy problem for the wave equation.

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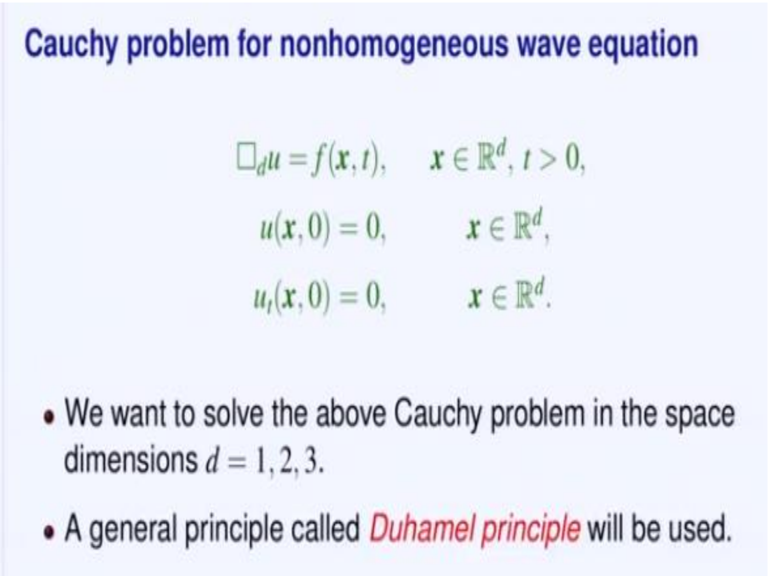

So let us look at the Cauchy problem for non-homogenous wave equation with the homogenous data or genius initial condition or 0 Cauchy data. And we want to solve this Cauchy problem in dimension 1, 2, 3 because we are discussing only wave equation in 3 dimensions 1, 2 and 3 these are the 3 dimensions in which we are discussing wave equations so far. So general principle called Duhamel principle will be used.

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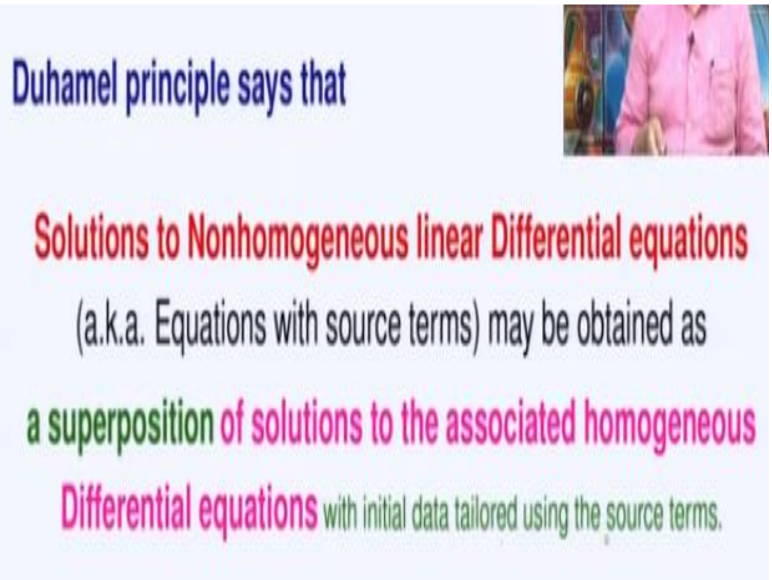

So what is Duhamel principle? It says solutions to non-homogenous wave differential equations that is also known as equation with source terms. Maybe obtained as the super position of solutions to the associated the homogenous differential equations. In other words what it says this is a rough statements precise statement we are going to see later. If you want to solve non-homogenous equations it is enough to solve homogenous equations that are what it is.

Of course the initial data will be tailored using the source terms after all we want to solve the homogenous equation that means source terms is given to us. We had to use that and make our manufactured this homogenous differential equation in corresponding Cauchy problem essentially the initial data will be tailored using the source terms.

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Motivation for Duhamel principle we will give using initial value problem for a second order linear ODE.

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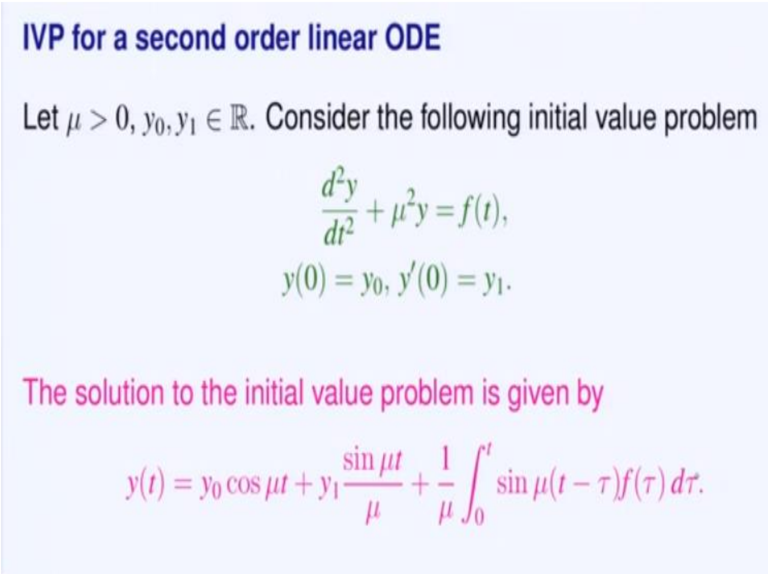

So initial value problem for a second order linear ODE is given by this d2y by dt square + mu square y by f t y of 0 is y 0 y dash 0 is y 1. So in this f y 0 y 1 are given and we are supposed to find a solution to this ordinary differential equation satisfying this 2 initial conditions. If you notice this looks like wave equation because in wave equation you have u 2 t term that is like d 2 u by dt square but instead of mu square y.

We have minus laplasian y equal to f of x t naturally wave equation being an equation being an partial differential equation you have both x and t variables. But one can view wave equation to also has an ordinary differential equation. Here ordinary differential equation y is a function of t given any t y of t will be real number. In the wave equation given any t u of t will be a function of x that is the difference we will come back to this later.

So first let us see how Duhamel principle works in this example so this solution to the initial condition value problem one can explicitly solve and write down as this y of t is y, 0 cost mu t + y1 sin mu t by mu + 1 by mu integral 0 to 2 t sin of mu into t – tau into f tau d tau. This is a formula one can verify this is already used to this course on ODE and got this solution. Now we want to see each and every term closely. And then we will see how Duhamel principle comes into here.

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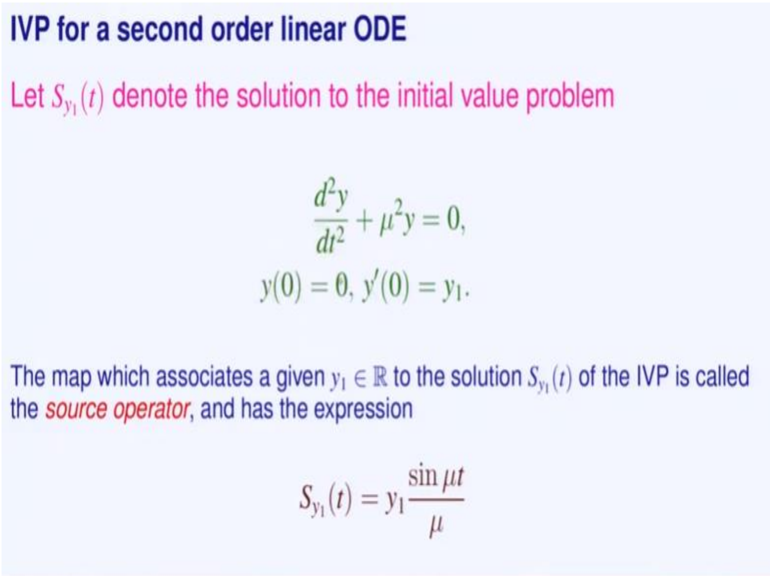

So before that introduce this homogenous equation remember Duhamel principle is something to do with the homogenous equations now where the. So we will make the equation homogenous and the initial data has to be tailor made using this source term. So with that we will mind let us discuss homogenous ordinary differential equation with initial conditions where y of 0 is 0 that is initial position is 0 and initial velocity is y1 y prime 0 is y 1.

A solution to this we will denote it as S y1 of t so S for solution y 1 remember the y prime 0 is y 1 and solution for this ODE. And ODE is background homogenous ODE and this map which maps given y 1 given number to this solution S y 1 of t is a problem is called the sourced operator and of course we know the formula and that is given by y 1 times sin nu t by mu. So S y 1 of t is solution of this homogenous differential equation and with y1 as the initial speed or initial velocity and initial displacement or position is 0.

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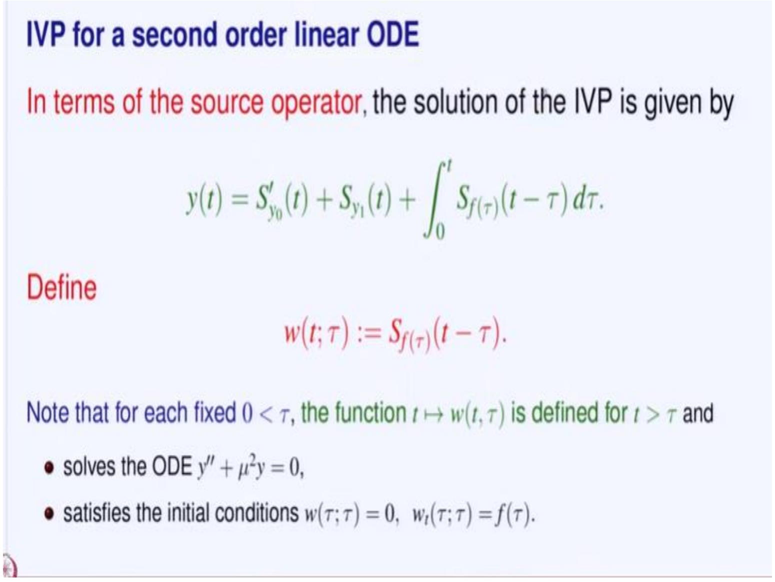

So in terms of the source operator the solution of the IVP is given by this y t = S y naught of t d by dt of that S + y 1 of t + integral 0 to t S f tau t – tau d tau. What is S f tau? It is the solution operator what does it do it maps f of tau is given as the initial velocity S of tau of t will be the solution of the given ODE. Let us define the integrand to be a function of t with tau hanging around.

So w of t: tau is S of tau f t – tau we will observe what happens to this w what is that w satisfies? So each tau positive the function w, as the function of t is defined for t bigger than tau and it is solves the ODE. So it is the solution to the homogenous equation and it satisfies initial conditions input t equal to tau w will be 0 and w t of tau will be f tau. Remember that is how will source operator is defined like that.

When t is tau it is defined as f tau at 0 so that is why that is 0 because s of tau solves with 0 initial displacement. And this is the velocity so that is d by d t of this quantity had 0 will be of, f. Now observe this first 2 terms some of the first 2 terms solution to the homogenous equation ordinary differential equation and the Cauchy data is satisfied that is why I have 0 is y 0 y dash of 0 is y 1 is satisfied.

Now this term essentially is solving the non-homogenous differential equation with a 0 initial condition. And how it is obtained as an integral? We can think integral with a certain some therefore generalization of the sum therefore this is a super position of the quantity is inside.

But what are the thinks inside and therefore this is a super position of the quantity is inside whatever the things inside they are solution is homogenous equations.

So this is the explanation of motivation Duhamel principle coming from initial value problem for second order linear ODE.

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Now let us apply Duhamel principle to wave equation.

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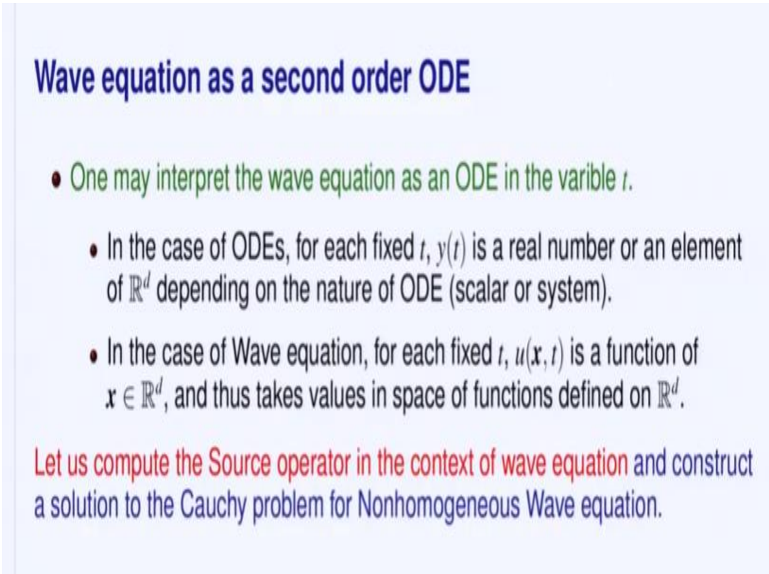

So with wave equation as a second order ODE one may interpret the wave equation as an ODE in the variable t in the case of ODE’s for each fixed t y of t z is real number r n element of R d depending on whether dealing with a scalar equation or a system of equations. In the case of wave equation for each equation of f u of x t is function of x in R d and this R d is in space of functions defined an R d.

So let us compute the source operator in context of wave equation and construct to the solution to the Cauchy problem for non-homogenous wave equation.

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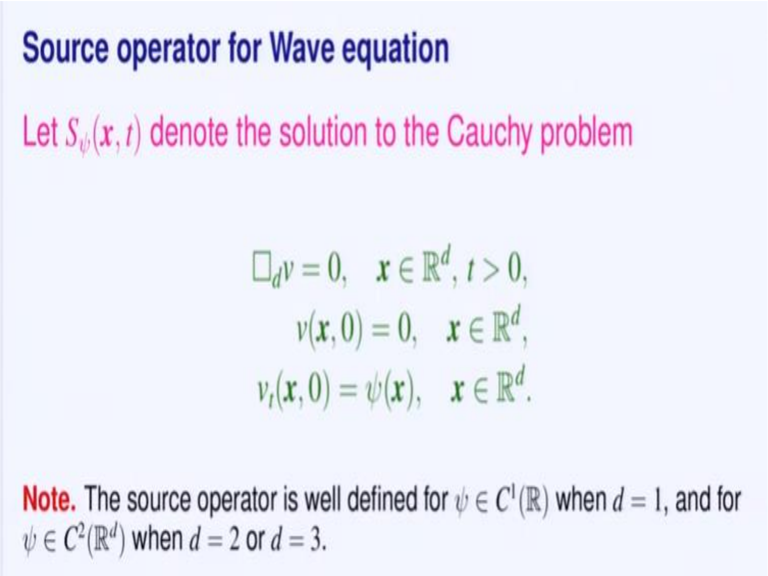

So let us see what is the source operator for the wave equation? What is that x psi of t for the ODE? For the wave equation s psi of s, t what is this psi represents? The initial velocity and initial displacement is 0 and you solve the homogenous equation exactly as it was the case in ODE. So psi of x, t let it denote the solution to this Cauchy problem for homogenous wave equation and 0 initial displacement and initial speed as psi.

The source operator is well defined we have to ask when it is well defined that means we have to ask when is that we have a classical solution to this. If d = 1 we need psi to be in c 1 because ewe have d'Alembert formula which is applicable. And then for d = 2 or 3 we need psi to b in c 2 that is one the source operator is well defined.

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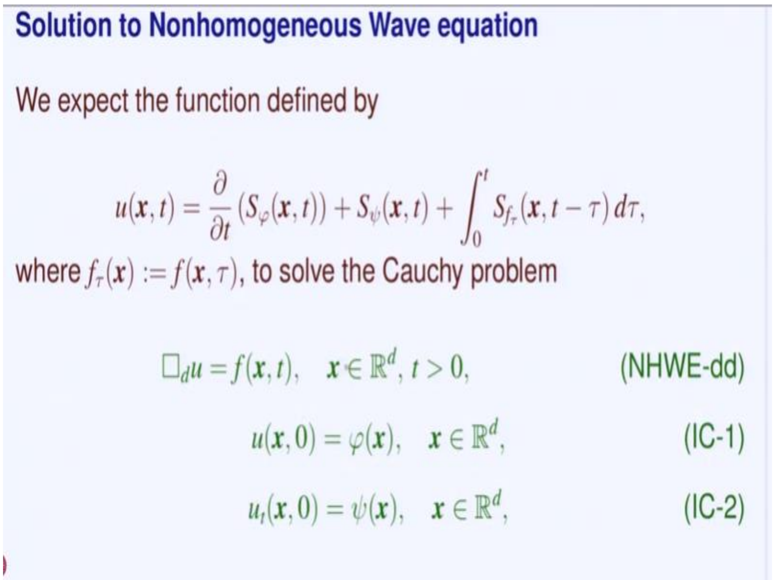

Now we expect the function this is precisely that S y is 0, dash now it is y 0 is phi and y1 is psi in the context of wave equation this is S phi 1 and this is integral. 0 to t s f tau which we now looks subscript tau half t –tau x dependence is S so we write that. So this we expect to solve the non-homogenous wave equation and the Cauchy data. So f tau of x stands for f of x, tau so we expect that to solve this Cauchy problem let us call this as an non-homogenous Cauchy problem.

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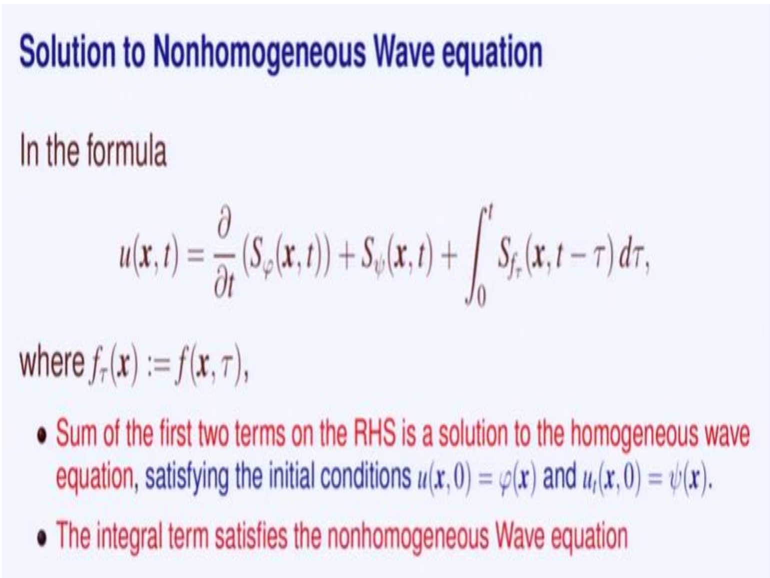

So we have to check that so in this formula some of the first 2 terms on the RHS namely these 2 terms is a solution to the homogenous wave equation. And satisfies the initial conditions what are those? Initial displacement is phi and initial velocity is psi so these condition are taken care by these 2 terms. So therefore what we expect is? These terms will be solution to the non-homogenous equation with a 0 initial data.

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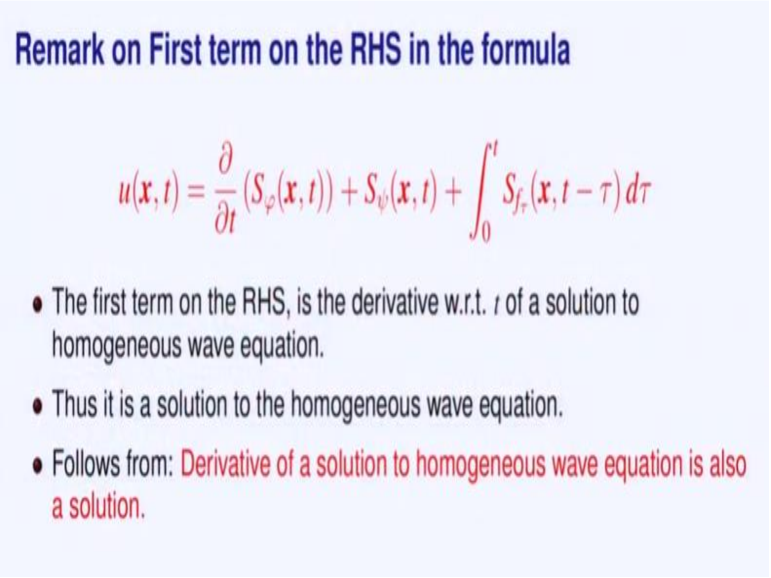

Now remark on the first term we have to check that these are indeed the solutions to homogenous wave equation we are going to do that. And they satisfies the Cauchy data we will check that so now let us concentrate on the first term it is dau by dau t of S phi of x, t. It is a derivative with respect to t of what f phi of S, t what is S, t? It is a solution to psi wave equation by definition.

Solution to the wave homogenous wave equation initial wave displacement 0 initial velocity phi x now what we have the first term is the time derivative of a solution to the homogenous wave equation. Therefore it is itself a solution one can check that if you have a homogenous wave equation and let us say u is the solution to that then dau u by dau t is also solution. Dau k u by dau t is also solution we have already mentioned this in one of the tutorials earlier otherwise you can check.

So it is a solution that is no doubt so it follows from derivative of the solution is a solution for the homogenous wave equation.

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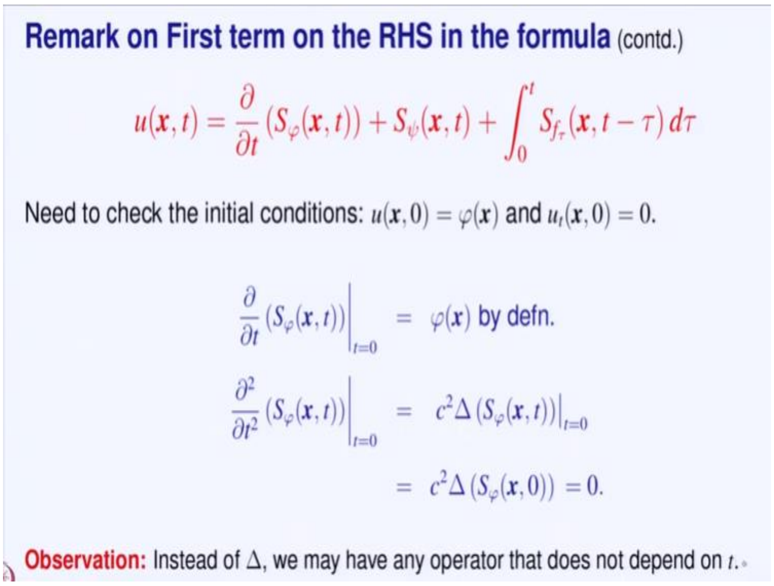

Now we have to check the initial data so this first term satisfies initial displacement phi and initial velocity to be 0. So let us check initial displacement that is dou by dau t x phi of x, t at t = 0. We want to check this is phi x this is by definition because (()) (15:55) S phi is solving homogenous wave equation with initial velocity as phi this is exactly that dou by dou t of that is initial velocity.

So that is phi this is by definition now we have to check these condition initial velocity is it 0 or not that means. We have to take this term differentiate with receipt to t that means essential dou 2 dou t square of S phi and check that is 0. So this is what we have to check dou 2 by dou t square of S phi of x, t at t = 0 compute t what it is. By the equation because it satisfies homogenous wave equation right so u t, t = c square laplacian u so that is what I have write.

But now I will evaluate at t = 0 that is why both side I out t = 0 now what is laplacian? It is differential operator with respect to x what am I doing here. I want to take the value of t = 0 therefore I can as well take the evaluated t = 0 and then take the derivative with respect to the other variable space variables. Or take the derivatives and then take t = 0 both are same because the evaluation I am going to take is for t and the derivative that I have front is laplacian that is with respect to x.

Therefore when I do that I get this but what is s phi of s 0 by definition of s phi the initial displacement it is right solution. So therefore that is 0 therefore it is 0 now if you notice instead of laplacian you could have heard any operated that does not depend on t this is an interesting important observation. So that Duhamel application principle is not limited to wave equation only you can apply u t, t equal to any operator l of u ss long as l does not involve the variable t.

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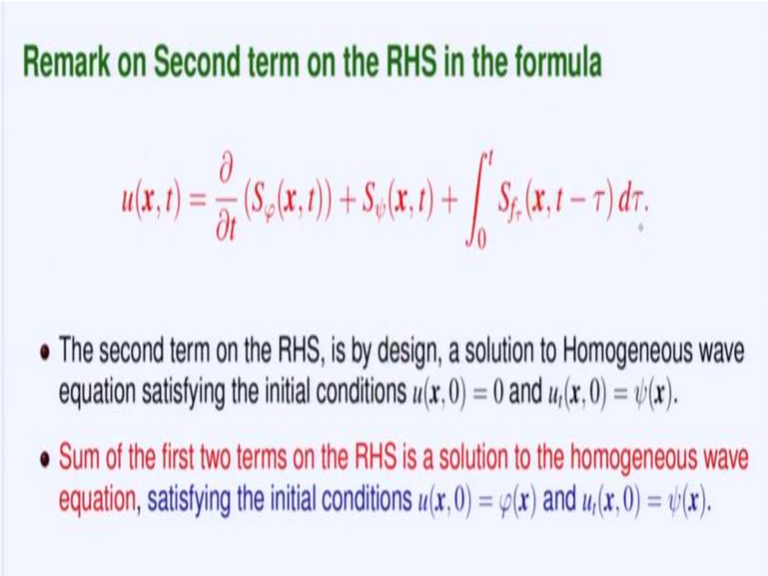

So therefore we check the first term satisfies homogenous wave equation and it satisfies the initial conditions u of x, 0 equal to phi u of x, 0 = 0. Now let us move to the second one which is more, straight forward because just a different condition the second term is by design or definition a solution with a homogenous wave equation with initial displacement 0 and initial velocity as psi that is the definition.

Therefore the sum of the Cos 2 terms is going to solve homogenous wave equation because both of them are solution to the homogenous wave equation and initial condition will be now u x, 0 equal to phi and u d x, 0 equal to psi done. Now we have to concentrate on the third term now what about the third term?

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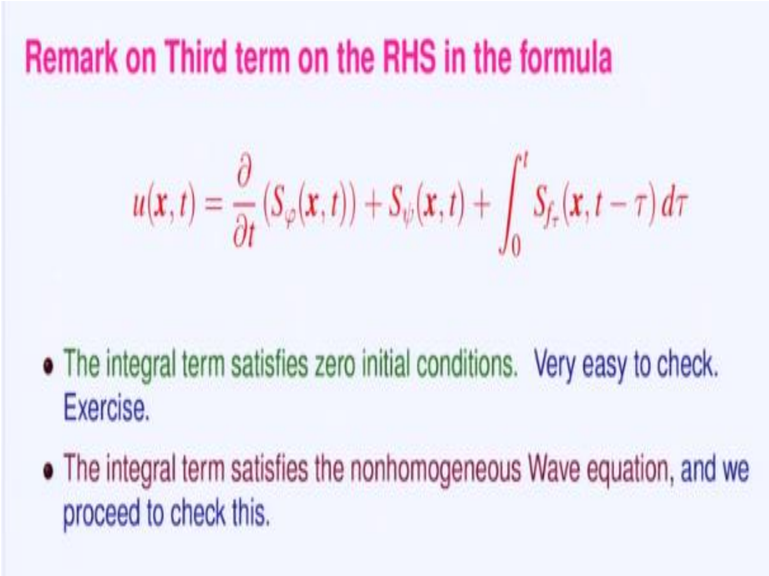

So the integral terms satisfies 0 initial conditions because if you check t = 0, integral collapses there is no integral will be 0 and check that the derivative is also 0 that I leave it as exercise very easy to check. So what remains to really check is that it satisfies the non-homogenous wave equation then we would have proved that this is solution to the non-homogenous Cauchy problem.

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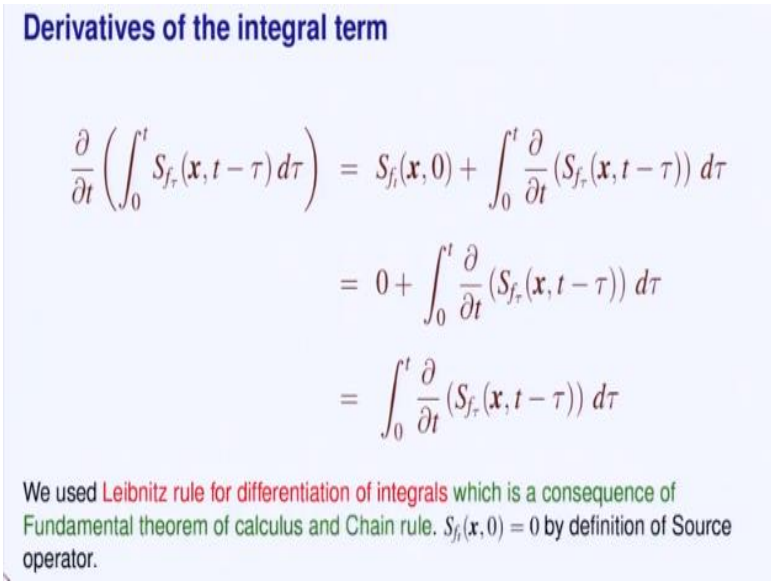

Let us do this checking so first we have to compute the derivatives right if you want to check this is the solution the wave equation we have to compute 2 time derivatives and 2 x derivatives. Let us compute the first derivatives dou by dou t of this now we are going to use Leibnitz rule. Whenever you have integrals which depend on variable with respect to differentiation the rule how to differentiate that particular integral I am going to follow that.

So therefore you get this derivative equal to the inside thing evaluate at tau equal to t that will give you this. Because tau becomes t S t x, when it was t it is 0 so you get this plus we are going to differentiate inside the integral. So 0 to t remains as it is and dou by dou t of the inside quantity so we have this. What is S f t of s, 0 it is 0 by definition that is how the source operator is defined. And not change the integrand as it is so one at a time now let us differentiates once more.

So we use Leibnitz rule for differentiation of integrals here and it is a consequence of fundamental theorem of calculus and chain rule. S of t of x, 0 is 0 by definition of the source operator. So, first derivative of integral term is this now you will compute the second derivative.

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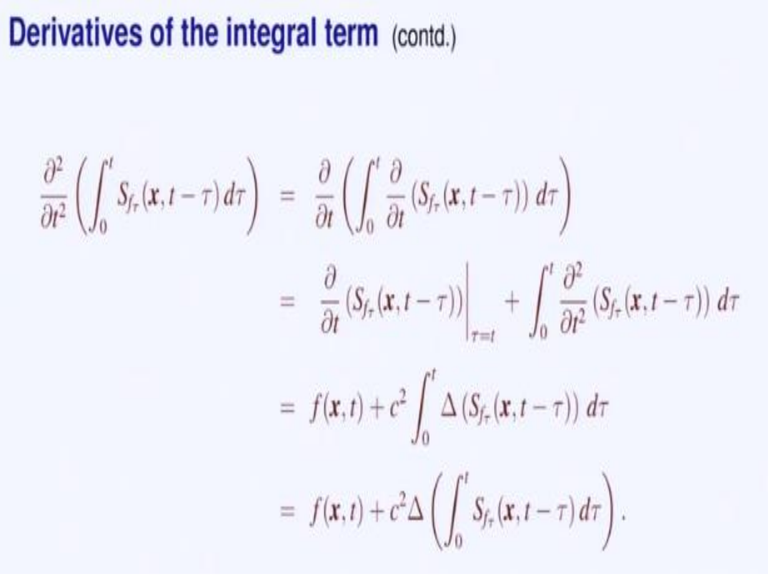

So this is being carried forward from the previous slide this is a first derivative of the first integral term therefore first derivative becomes second derivative of the first derivative. Now exactly again by the Leibnitz rule this will be the integrant evaluated at tau = t + differentiate inside with respect to t. In this quantity is precisely f of x, t once again from the differentiation of the source operator.

Now here we see that dau 2 by dau square of some solution right s of tau after all is a solution to the wave equation. Therefore dau 2 by dau t square is written in terms of the Laplacian so it is c square Laplacian of whatever is here. And Laplacian comes outside on behalf this.

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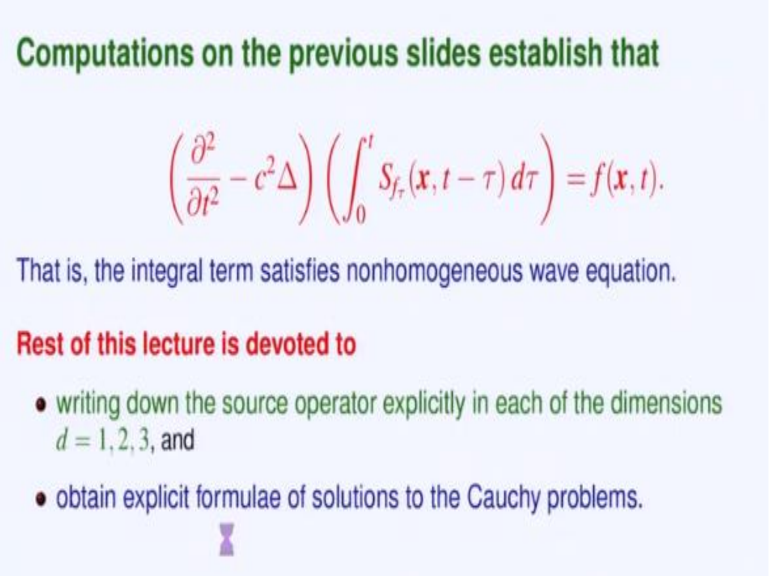

Therefore we approved that this integral term satisfies dau 2 by dau 2 square –c square Laplacian equal to f x, t. Take this term to the LHS then precisely we have this that is integral terms satisfies non-homogenous wave equation. And rest of this lecture is devoted to writing down; the source operator explicitly in each of the dimensions 1 to 3 and obtain explicit formula or solutions to the non-homogenous Cauchy problems.

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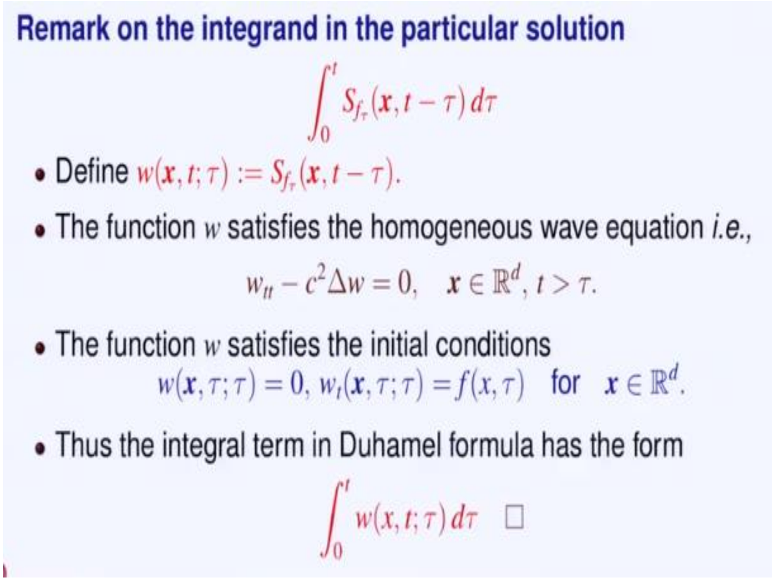

Before that let us give a remark on the integrand in the particular solution we may call this thing as the particular solution this is a nomenclature we use even in ODE’s it is a one particular solution. That means it is one solution of the non-homogenous equation. So I have before we already introduced this in the context of ODE’s so integrand you call it as w of x, t, tau then the function w satisfies the homogenous wave equation and it satisfies initial condition when t = tau w is 0 and derivative of w when t = tau is f of x tau.

See in other words you see this is a super position of solution to the homogenous equation with initial data which is tailor made using the source terms this explains Duhamel principle.

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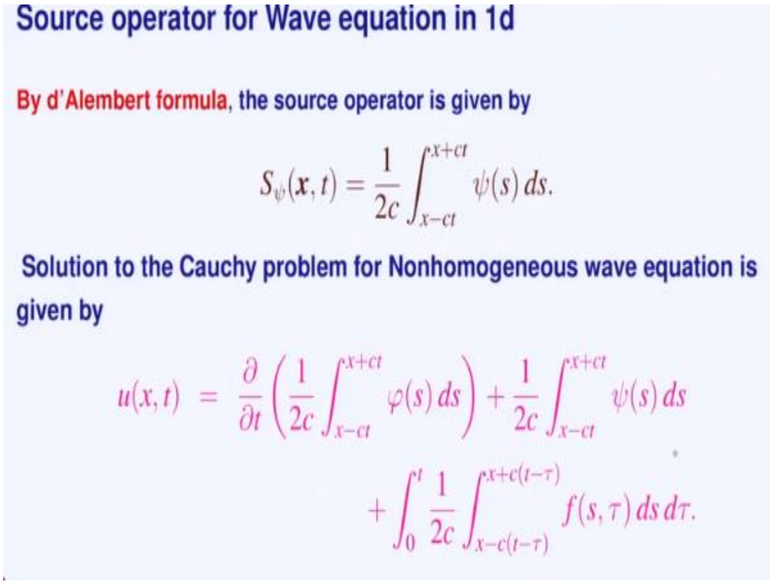

So let us obtain explicit expressions we have already solve homogenous wave equation on Cauchy problem so it is a just matter of writing down what is the source operator is? So d'Alembert formula the source operator S psi is this is 1 by 2 C integral s – ct x + ct psi S ds. So solution to the non-homogenous wave equation is this dau by dau t of S phi + s psi + the integral term 0 to t s psi of some t – tau x tau dt which is there on the previous slide.

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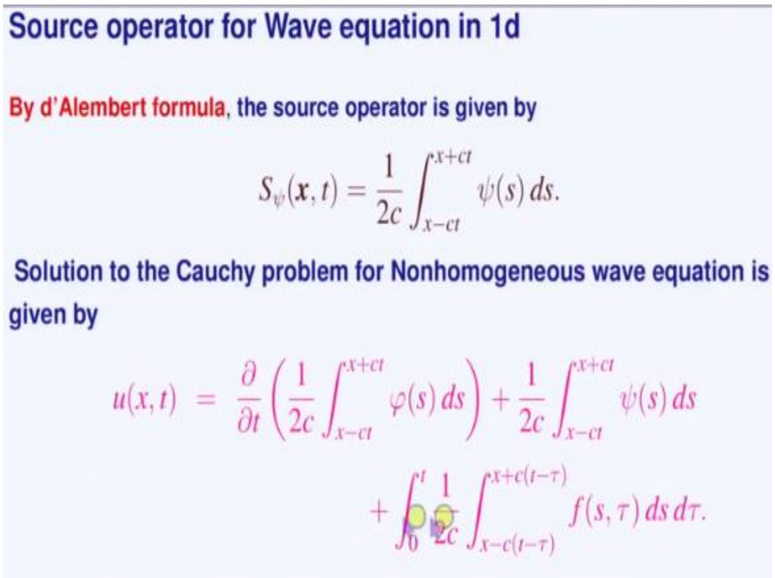

We can simply this further and we get this expression because here it looks a very complicated expression we can use the Leibnitz rule and get this expression.

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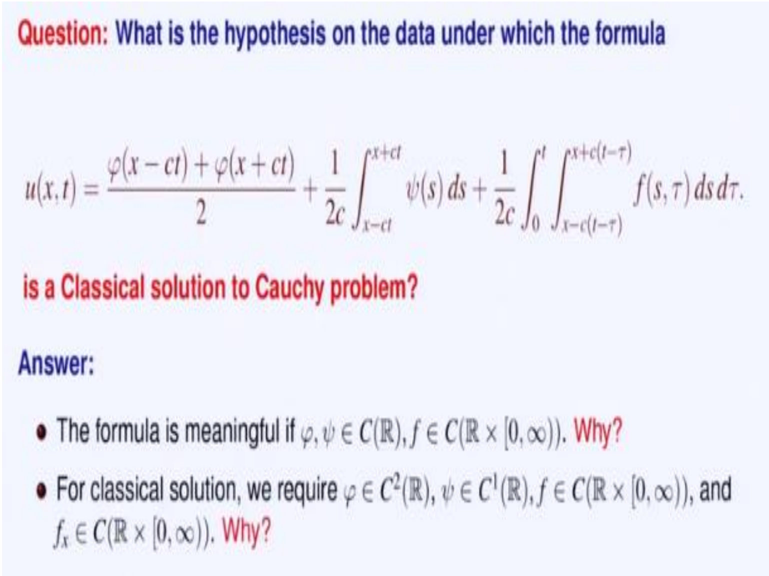

So if you notice the first 2 terms is precisely the d'Alembert formula that means when f is 0 phi and psi that is solution of the homogenous wave equation and the Cauchy problem for that. And this is the new term which we have added to get solution to the non-homogenous Cauchy problem. Now of course we have to ask the question when is it classical solution what are the assumption needed?

Of course if you look at the formula it is meaningful phi let us say continuous integral we need therefore psi should be continuous again some integral so f should be continuous that is not enough. For classical solution we require phi in c2 psi in c1 we have already noted down this. When we discuss the homogenous wave equation problem for the one dimensional wave equation now what about this?

Remember this f this term what we will get the f went into initial velocity in the definition of source operator therefore we need some first derivative of f to be continuous. So f should be continuous so that this makes sense in addition you need first derivative of, f with respect to x to be continuous. Because that is what we needed to make sure that the source operator is well defined with this that the initial vecloty done.

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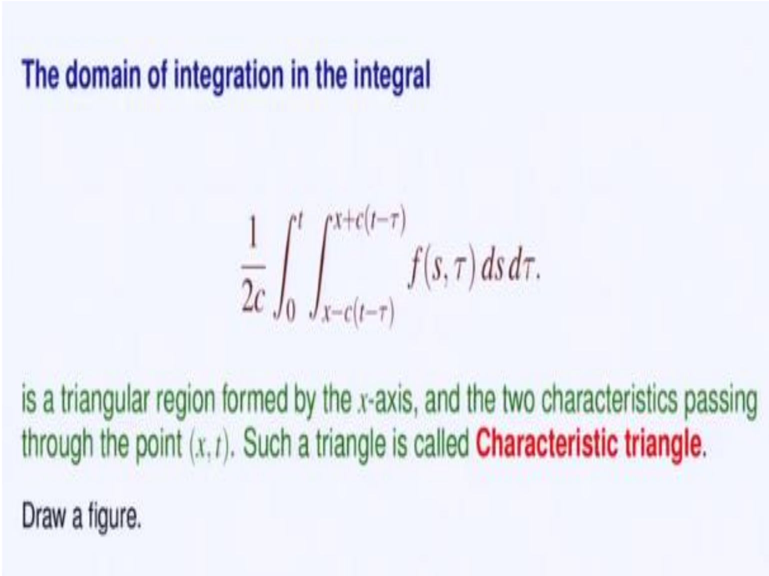

Now the comment on the domain of integration in this integral what is this? It is a triangular region formed by x axis on the 2 characteristics which are passing through the point x, t. That is if you look at this it take a point like this is the point x, t so this is a line which are going to through this is x – we should have used x naught t naught then this is x - ct = x naught –ct naught that is the line and this line is x + ct = x naught ct naught and this is x axis so this is the triangular region.

If you see this is where we plot x and t, here so here this formula you just read with x, 0 t, 0 that is what I am reading x, 0 t, 0 this is also t, 0. So this point is t, 0 you fix anything in between let us say t = tau then you are going from this point to this point this is precisely the 2 points here this and this. So this triangle is called characteristic triangle.

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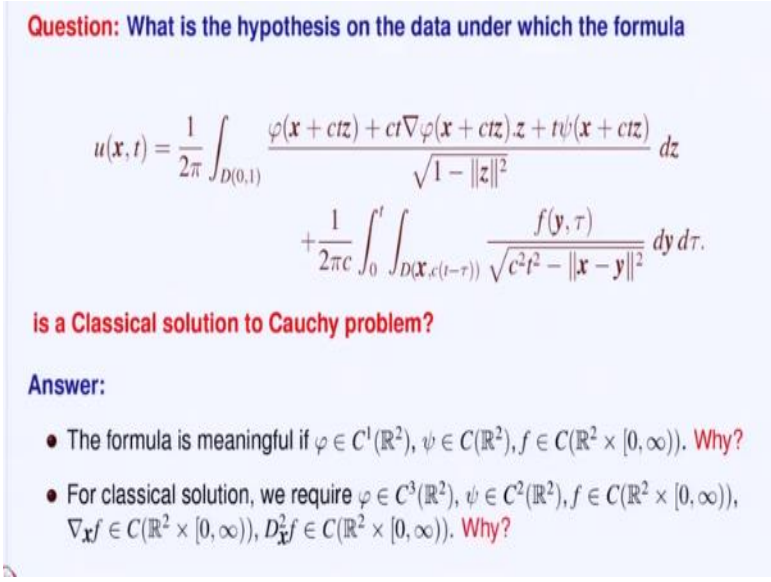

Now let us move on to the 2 space dimension by Poisson-Kirchhoff formula the source operator is given by this formula and now this is expression dau by dau t of s phi + s psi + that integral. So we get this formula it is in a change of variable it is convenient to write it down disc of radius one under 0 the homogenous equation part. Again same question where under what high process is an classical solution.

No need to look at the formula phi integration phi is involved so phi continuous grand phi integral is there that is what grand phi continuous psi continuous f continuous. We can put such conditions but then we want a solution for which we need higher smoothness phi should be c3 psi should be c2 and f should be continuous and also gradient should continuous with respect to x and second derivative should also be continues with respect to f.

Reason is exactly the same as continuous per d = 1 this f is going to shift in according to the Duhamel principle this f the source term will go to the initial velocity and for initial velocity v need c 2 (()) (29:02) that is why this condition.

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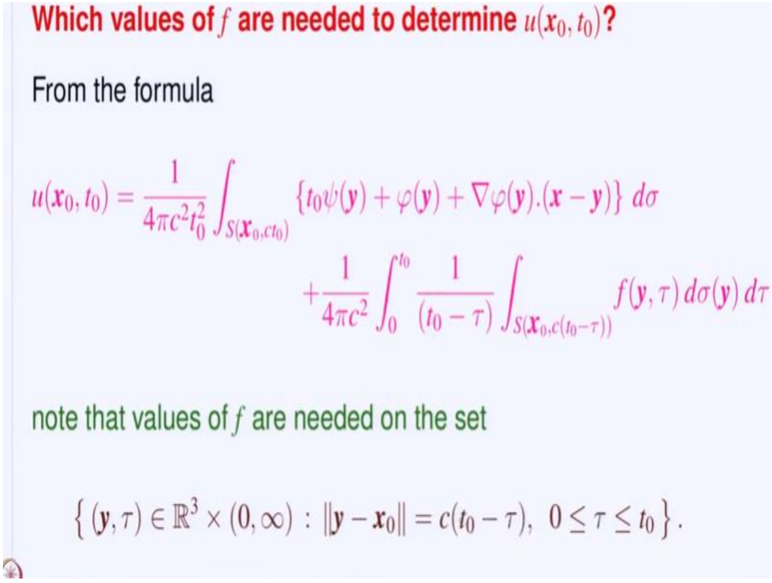

Let us move on to 3 space dimensions again from Poisson Kirchhoff formula in 3 dimensions s psi as this expression. And therefore by Duhamel principle u as this expression now which values of f are needed to determine u at the point x, 0 t, 0 that is the question we ask. So we write the expression for u of x, 0 t, 0 of course f is not coming here but if you ask what values of psi and phi are needed it is clear that we need the values on the sphere on radius ct naught with center x naught .

What are the values of f which are needed? That is what we are asking here so set of all phi tau such that tau is here y is here and tau between 0 and t naught is here y is here and tau between 0 and t naught. So they are needed on this set now what is y is here? y – x naught now is c times t naught – tau c into t naught – tau exactly that y should belong to that and tau should be belong to 0 to t naught that is exactly what I have written here. But what is the set this is the subset of r4, r3 cross 0 infinity.

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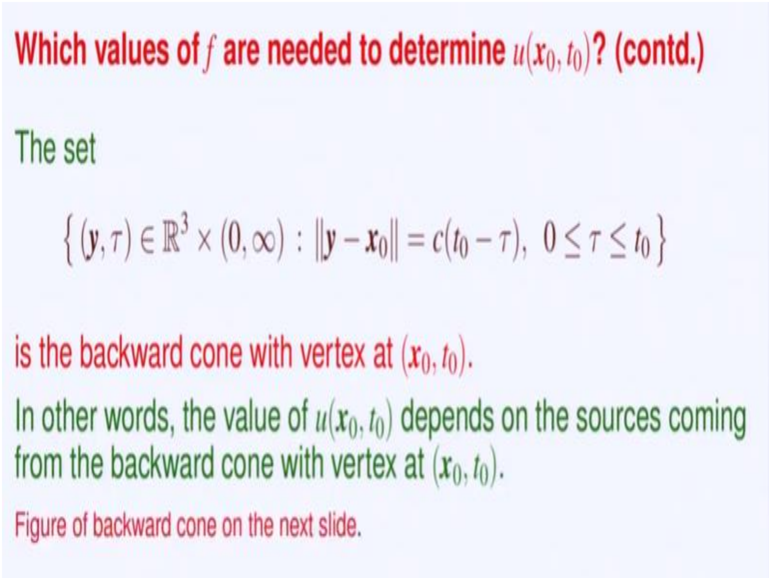

This is a back ward cone with vertex at x, 0 t, 0 we know the cone in 3 dimensions right so this is a something similar we have to imagine in 4 dimensions I will show a picture. In other words the value of u at x naught t naught depends on the sources coming from the backward done with the vertex at x, 0 t, 0.

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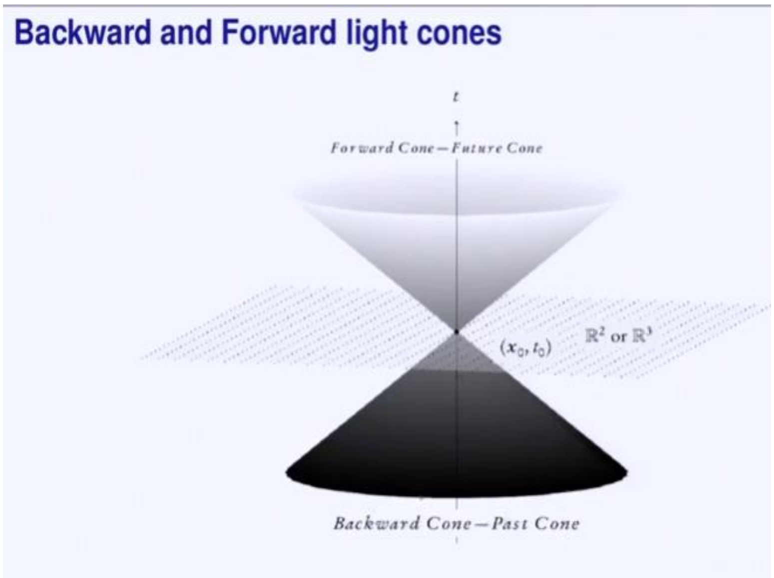

So figure of a backward cone is on the next line it is here this is a point x, 0 t, 0 what we are used to when it is r2 in the plane and you have a cone this is called backward cone or past cone this is called forward cone or further cone. We will discuss about this in a future lecture more on this but this is what is needed the values of f for needed on this cone.

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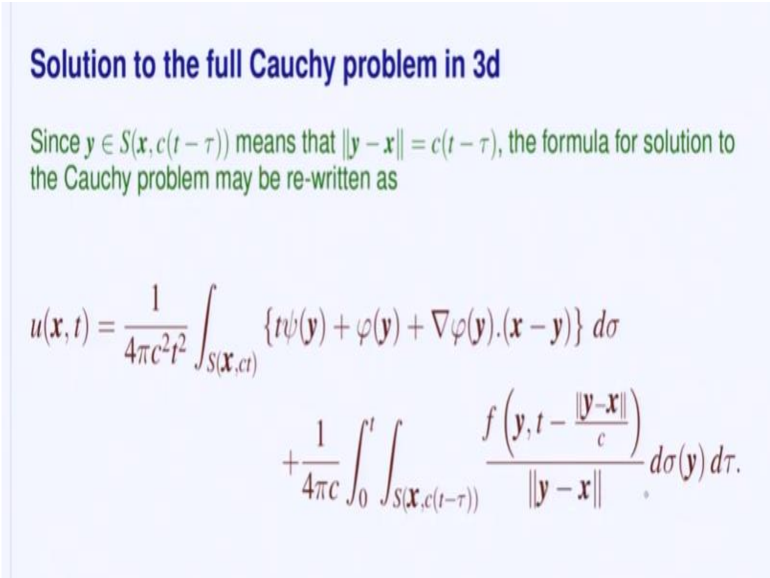

Therefore since y is in this fear that means the norm y – x is t into t – tau the formula may be written like this. Now t – nau – x y by c and now – x earlier it was written t – tau but now that we can convert in terms of y and x.

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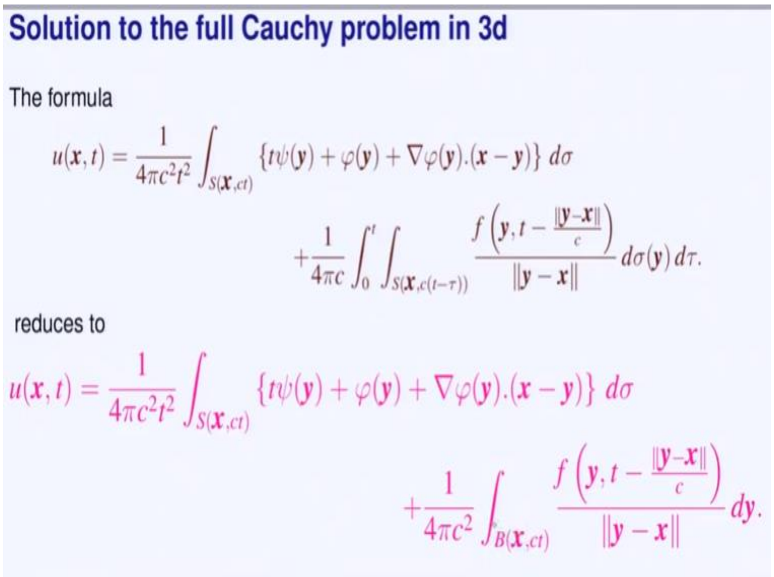

And if you just combine what you get here is precisely the ball of radius ct with center at x. So the ball is written as a lot of spheres which are union or so that is what you have. For each fixed tau you have this fear and then take the union as tau various from union to t we get this.

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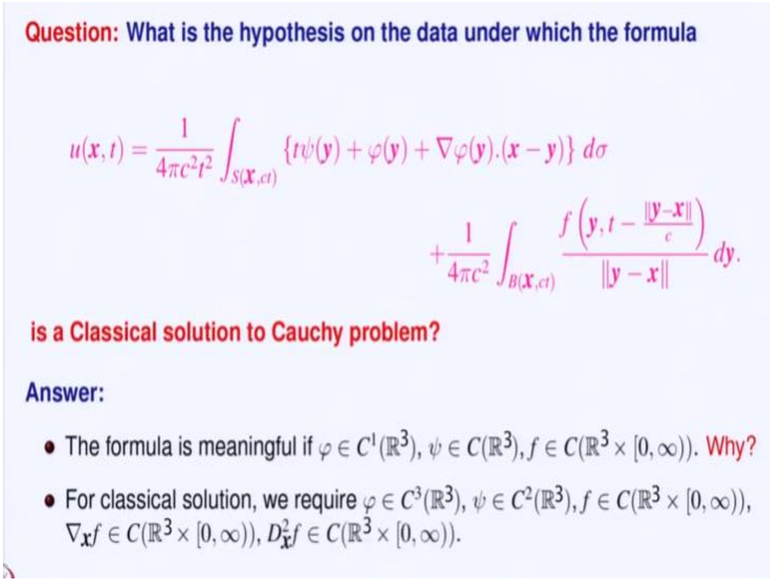

So again the same question when is this the classical solution to the Cauchy problem of course the formula is meaningful under some set of assumptions and phi, psi and f. But if you want a classical solution you need phi to be c 3 psi to be c2 and f to be here continuous gradient to be continuous and second order derivative to the continuous on all 3 Cos 0 infinity.

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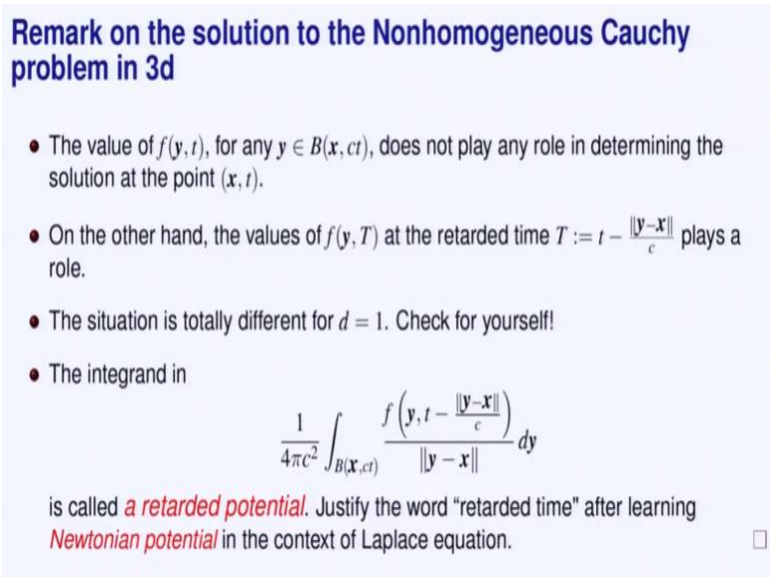

So the value of f of y, t for any y in this ball of x, ct does not play any role in determining the solution at the point x, t. On the other hand the values of f of y, t at the retarded time t which is t – nau – x by c. That is what they place a role if you see the formula and this situation is totally different for d = 1 I will request you to check what is that different from 3d in 1d and this integrand is called the retarded potential.

Because instead of t we have t – nau y – x by c so justify the word retarded time after you learn about Newtonian potential in the context of Laplacian equation which will be done later on.

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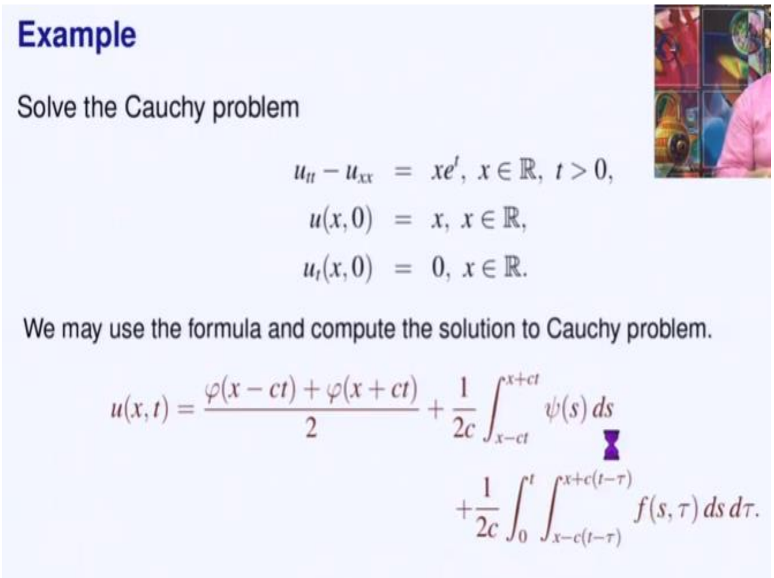

So let us solve in example solve a Cauchy problem non-homogenous Cauchy problem I have taken simple initial data because I want the computations to be simpler. Of course we have already said this is the formula f is smooth function no problem so this will represent classical solution because phi is smooth psi is 0 and f is definitely smooth. You can compute directly by applying in the values we do not do that.

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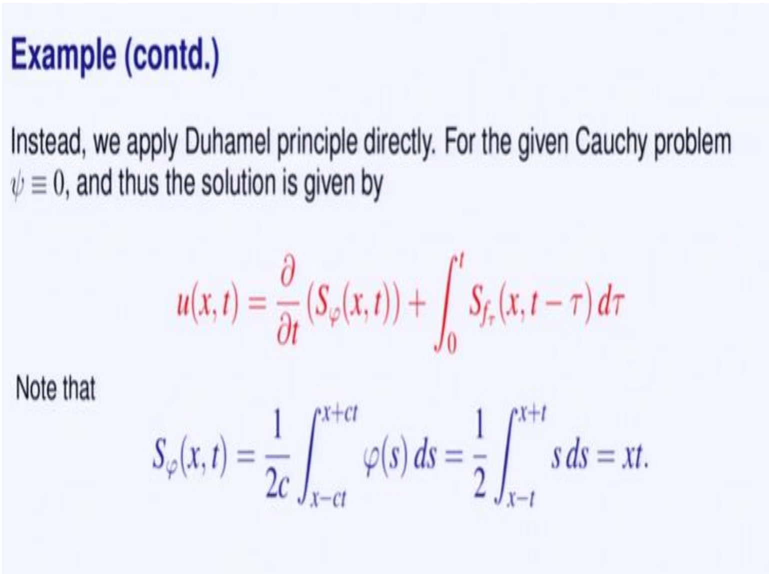

What we do is? We apply Duhamel principle directly so for the given Cauchy problem psi is 0 and therefore the solution in terms of the source operator is given like this. Now we have to find the source operator and that is given by this d'Alembert formula which we can compute that is x, t s phi of x, t is x t.

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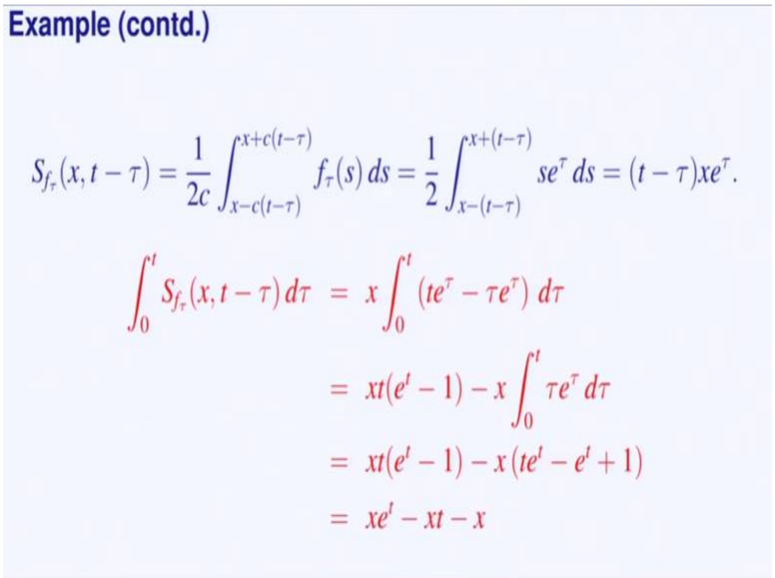

Now let us compute what is S of tau b definition it is this given by this formula and compute what is f tau as s that is s into e power tau compute this integrals you get t – tau into x e power tau. Now find this integral because this is what will give you solution to the non-homogenous wave equation and that is given by this integral and simplification we get this one x e power t – xt – x.

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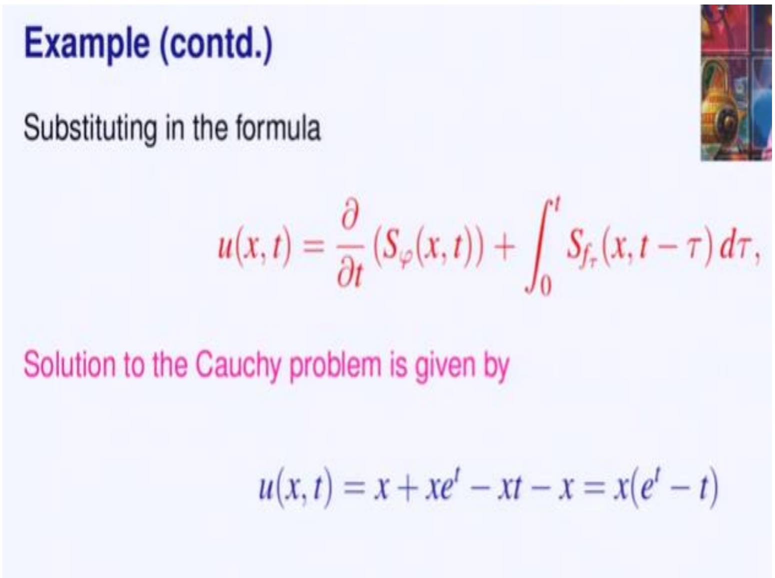

Now we have to add this 2 dau by dau t of s phi is required s phi was x t so dau by dau t will be simply x + this we will plug in from the last slide. Solution is x into e power t – t.

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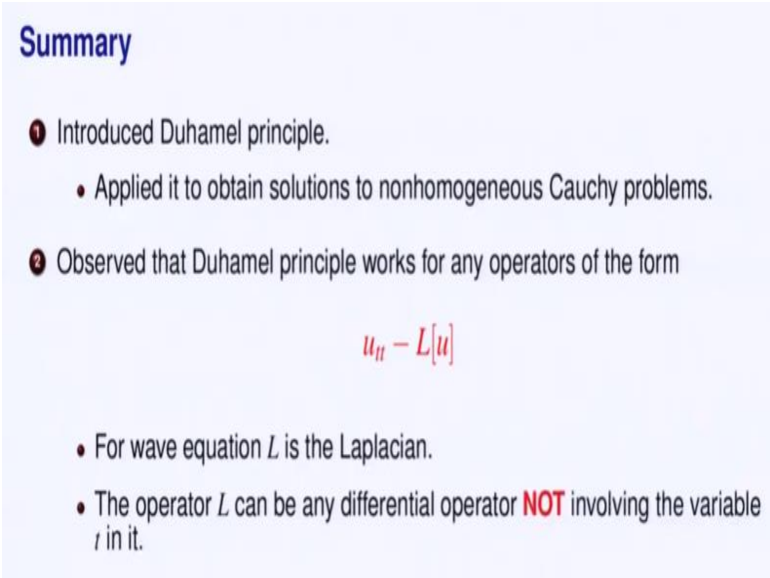

So we have introduced Duhamel principle applied it to obtain solutions to non-homogenous Cauchy problems we observed that the Duhamel principle works for any operators of the form u t, t – l u. For wave equation L is Laplacian for what operator it will work is the operator Laplacian can be any differential operator not involved in the variable t in it thank you.