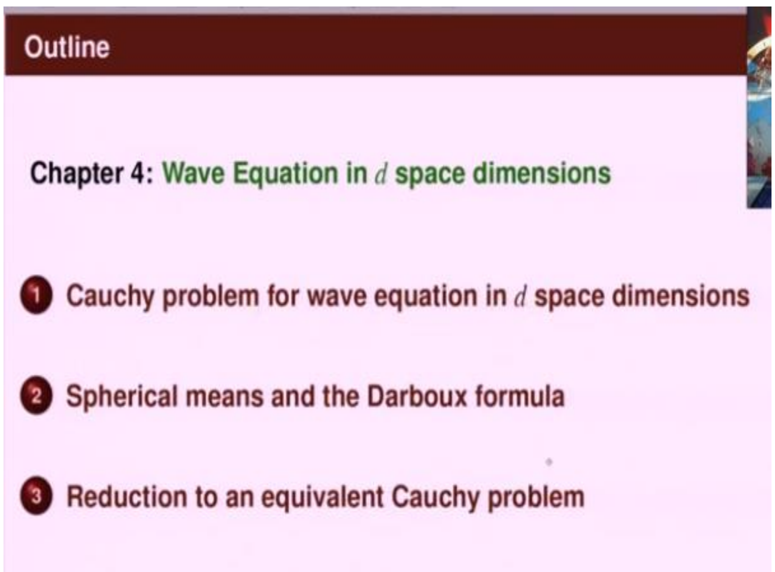
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology - Bombay

Module No # 06
Lecture No # 29
Wave Equation is d space dimensions
Equivalent Cauchy problems via Spherical means

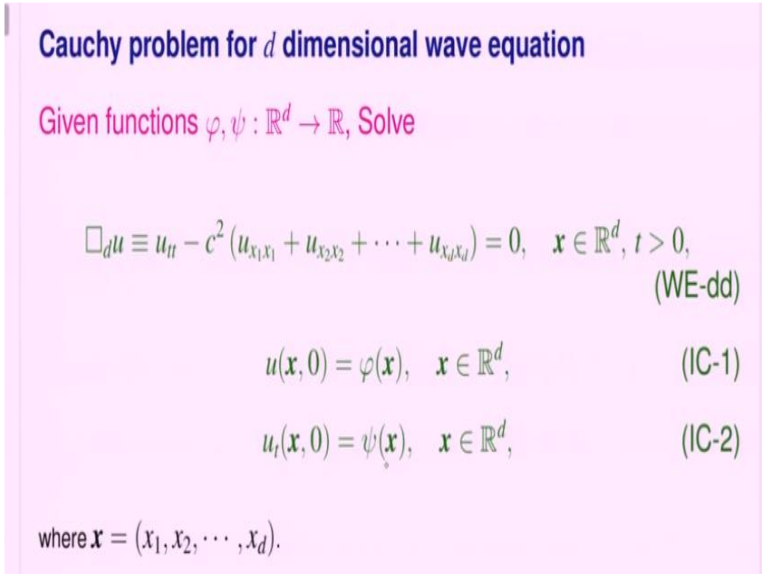
Welcome in this lecture we are going to discuss Cauchy problem for wave equation in d space dimensions in fact we are not going to discuss fully what we are going to do is? We are going to reduce this Cauchy problem in d space dimensions to a Cauchy problem for PD which is having just 2 independent variables that we will do bias spherical means.

(Refer Slide Time: 0:46)



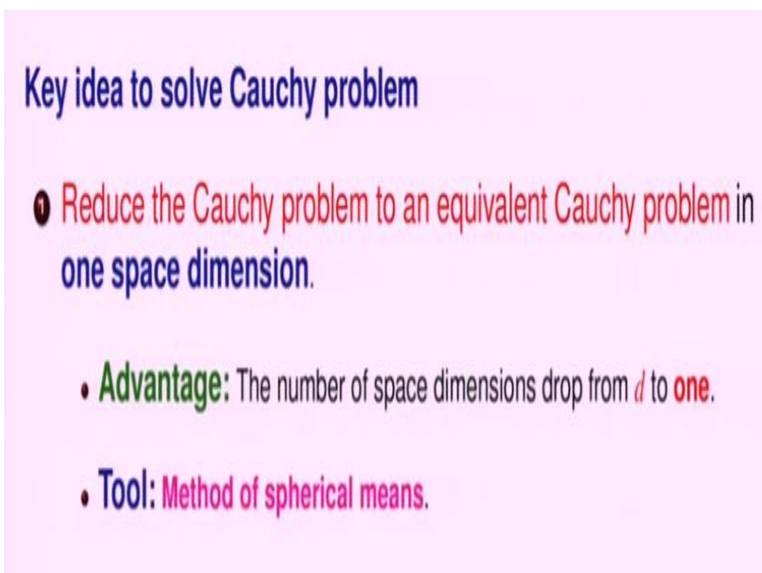
Outline for today's lecture is we start with statement of Cauchy problem for the wave equation in d space dimension. Then we introduce spherical means and a formula known Darboux formula associated to that then we reduce the Cauchy problem for the wave equation to an equivalent Cauchy problem.

(Refer Slide Time: 01:09)



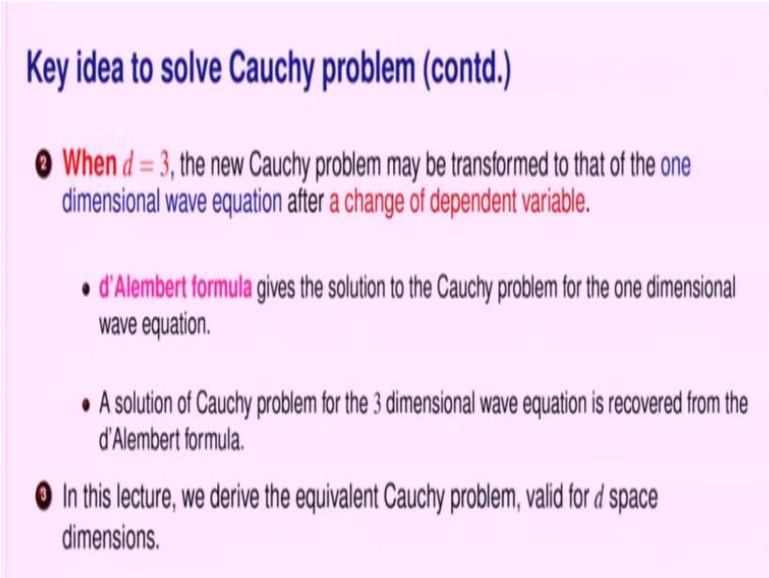
So Cauchy problem for d dimensional wave equation reads as follows given functions phi and psi solve the homogenous wave equation in d space dimension have you see x1, x2, xd are the space dimensions are the space variables. And u of x o is phi s u t of x; 0 equal to psi x. We are not yet stating what is regularity or the smoothness of the functions phi and psi required. We will see them later on.

(Refer Slide Time: 01:42)



So key idea to solve this Cauchy problem is reduce the Cauchy problem to an equivalent Cauchy problem in one space dimension advantage the number of space dimension drop from d to 1. And the tool is method of spherical means.

(Refer Slide Time: 02:05)



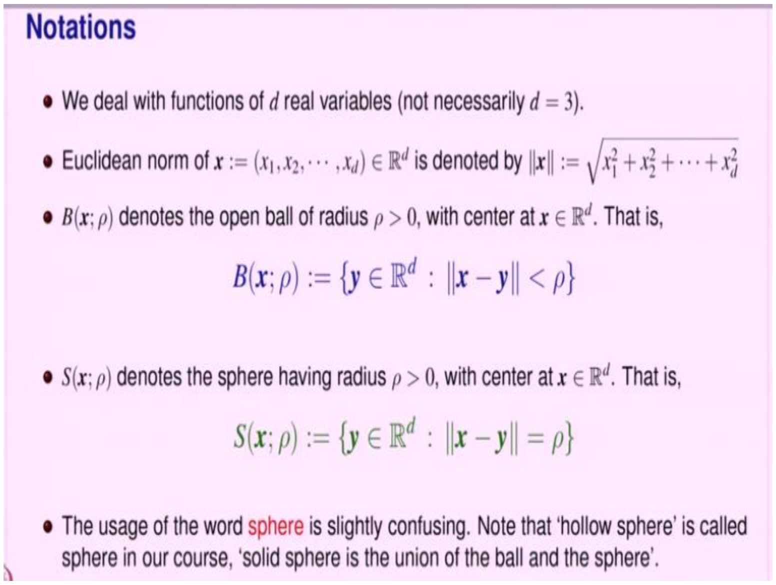
So when d = 3 the new Cauchy problem which we have obtained it can be transformed to, one dimensional wave equation. Cauchy problem for one dimensional wave equation of course after a change dependent variable which is straight forward. And then we are on with the d'Alembert formula which uses the solution to the Cauchy problem for the one dimensional wave equation. And retrieve the solution to the 3 dimensional wave equation; the Cauchy problem solution from the d'Alembert formula that we get here.

So in this lecture we confined ourselves to deriving the equivalent Cauchy problem valid for all space dimensions.

(Refer Slide Time: 02:56)

So let us see a few computations and notations involving multiple integrals.

(Refer Slide Time: 03:03)

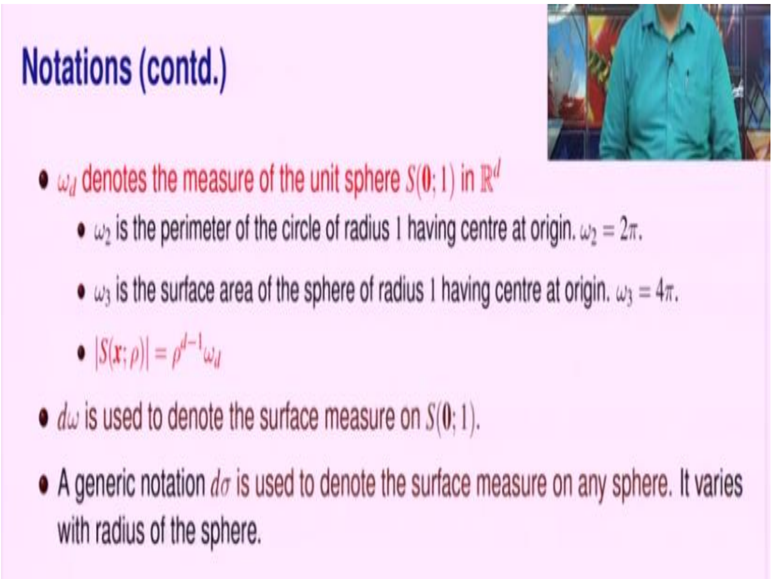


We will start with notations so we deal with functions of d real variables not necessarily d = 3. Euclidean norm of a vector x is denoted by norm x that is notation and that is given by square root of x1 square + x 2 square + up to +xd square. The usual length of the vector x B of x, rho it denotes the open ball of the radius rho of course radius as a positive and with center at x. So B of x, rho is set of all y in Rd whose distance from x is at most rho that is norm x – y is less than rho.

S of x, rho denotes this sphere having radius rho and center at x that is S of x rho is y in Rd such that norm x – y equal to rho. The usage of the word sphere is slightly confusing because from our high school by sphere what we mean is a sphere which is in this case actually the ball union this sphere the boundary of the ball is this sphere both of them you should take the union you get the so called solid is sphere.

So what we mean by sphere here is the hollow sphere is called sphere in our course in fact that is shown in any advanced mathematical courses. And solid sphere is the union of the ball under this sphere.

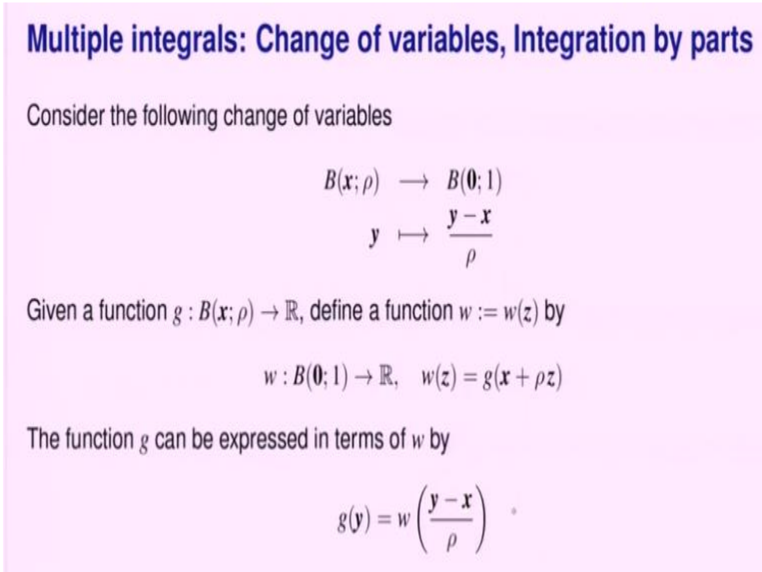
(Refer Slide Time: 04:45)



So you should not get confused now omega d demotes the measure of the units sphere S of 0, 1 in Rd. Omega 2 for example d = 2 omega 2 is a perimeter of the circle of radius 1 therefore it is 2 pi. Omega 3 is surface area of sphere of radius 1 and that is 4 pi we also have this relation which connects the surface area of the unit sphere to surface area of sphere of any radius. So this is a surface area of the sphere of radius rho that is rho power d – 1 times the surface area of the unit sphere.

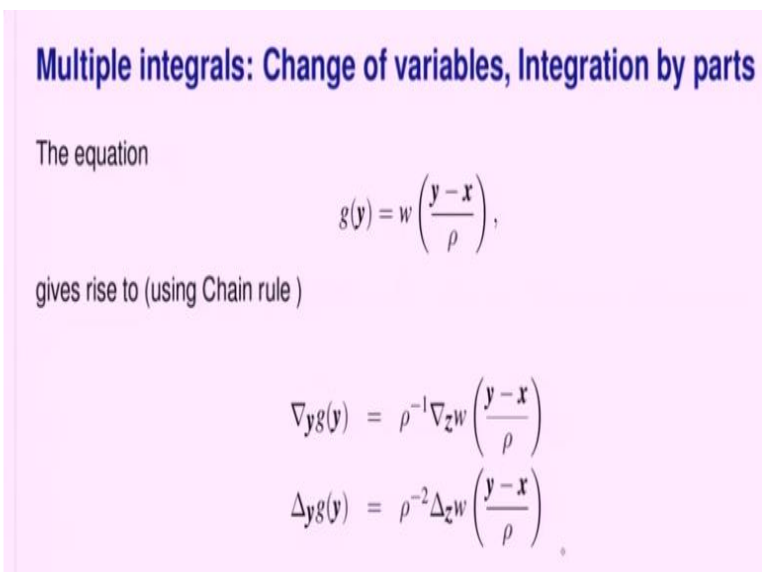
d omega is used to denote the surface measure on S, 0 1 we are going to deal with varies integrals as S 0, 1. So we write d omega denotes the surface measure now we use a generate notation d sigma to denote surface measure on any hyper surface in our context since we are dealing only with spheres it is a d sigma is used to denote the surface measure on any sphere. Of course it varies from sphere to sphere in fact it varies with the radius of the sphere.

(Refer Slide Time: 06:06)



Now let us the change of variables from so it is a mapping from B of x rho to B 0, 1 y in B of x, rho map to y – x by rho. So the ball of radius rho x will be going to the ball of radius 1 and center 0 under this mapping. It is of course it is an inevasible mapping so given a function g defined on B of x rho we can define a function which is define on B of 0, 1 that is 2 of z for z in B 0, 1 by g of x + rho z.So a function define here can be transferred to function here and vice versa so g of y is given by w of y – x by rho in terms of w.

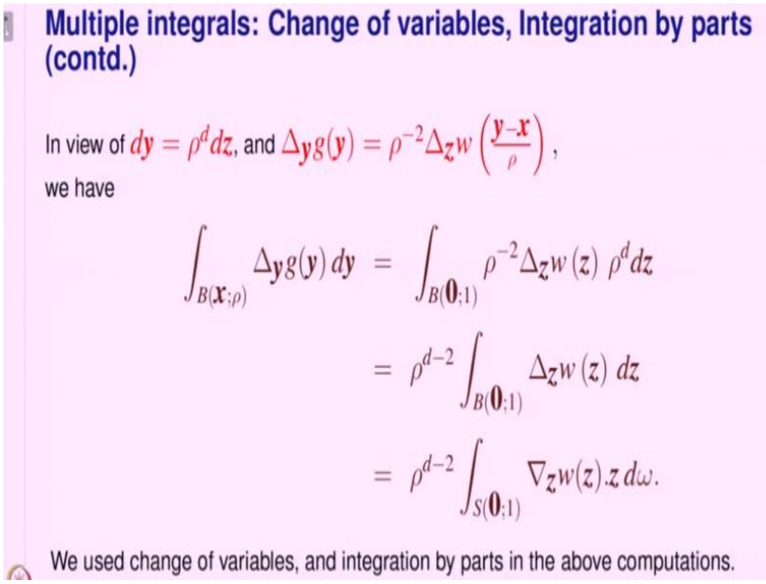
(Refer Slide Time: 07:01)



So this equation gives rise to the following equations of course chain rule so we are going to differentiate both sides with respect to y. So gradient with respect to y is gradient of w with respect to z and gradient of the inside quantity which is y – x by rho with respect to y that will give you 1 by rho that is why rho inverse. And Laplacian consist of second order derivatives so we get one more rho, inverse.

So we have rho power -2 into Laplacian in z of w evaluated at y – x by rho so we have to be as I told you we have to be comfortable with using a chain rule particularly when there are change of variables involved.

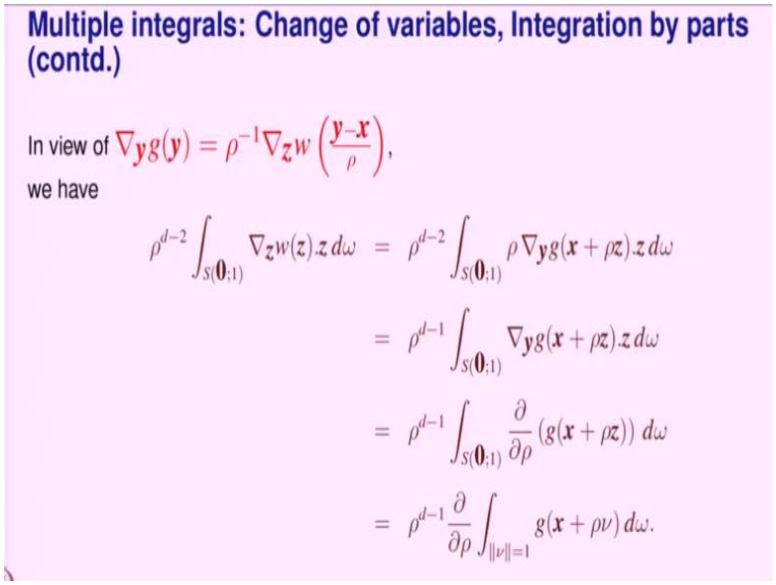
(Refer Slide Time: 07:46)



So the measure dy is given by rho power d into dx in view of that and the equation that we have just derived Laplacian yg in terms of Laplacian zw what we get is this particular integral. Now we are going to do change of variable z = y – x by rho that domain of integration transforms to this b of 0, 1 and the integrant will be rho power -2 into Laplacian z and the dy the measure becomes rho power d dz.

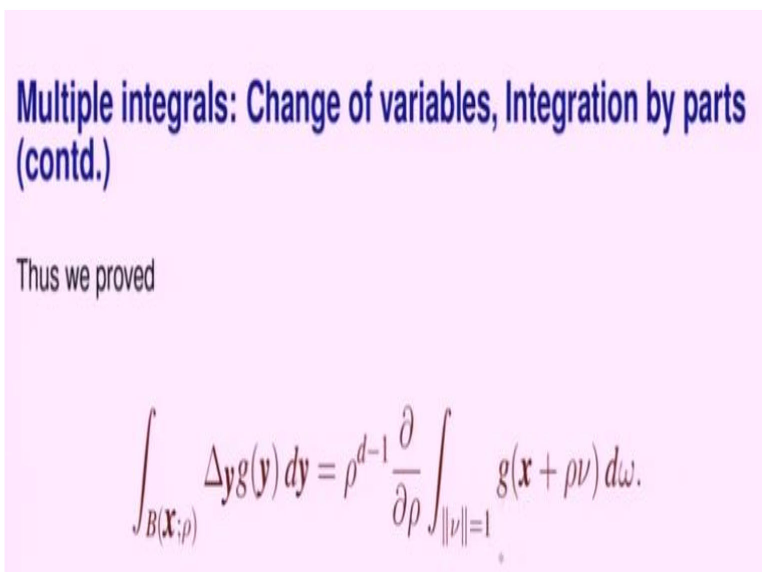
So pulling the rho terms outside what I have is rho power d – 2 into integral over ball of radius 1 center 0 Laplacian at w. So now from here we get this by integration by paths or (()) (08:45) theorem. So we use change of variable or integration by paths in these computations.

(Refer Slide Time: 08:54)



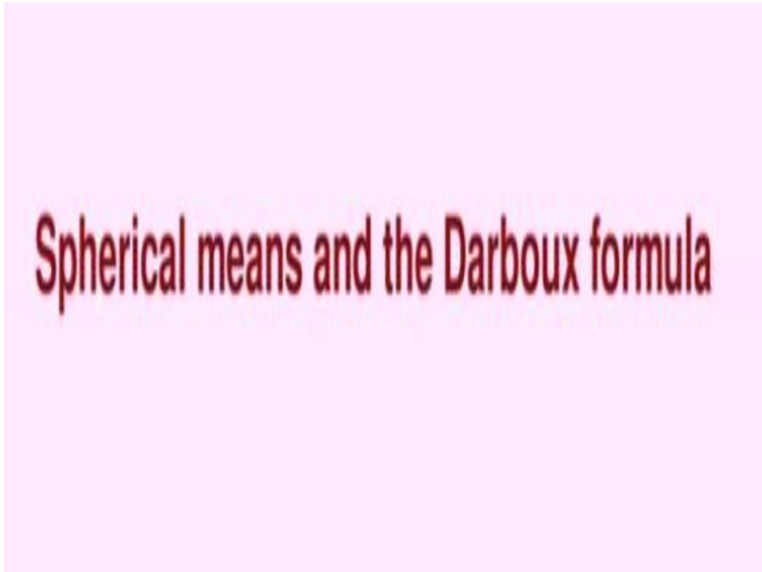
Now we also had one relation for the gradient under change of variables so using that the last equation that we obtained on the previous slide can be simplified expressed in terms of g. So this integrant is precisely dou by dou rho of this quantity and dou by dou rho will come out of the integral now. So this is the expression that we have got.

(Refer Slide Time: 09:32)



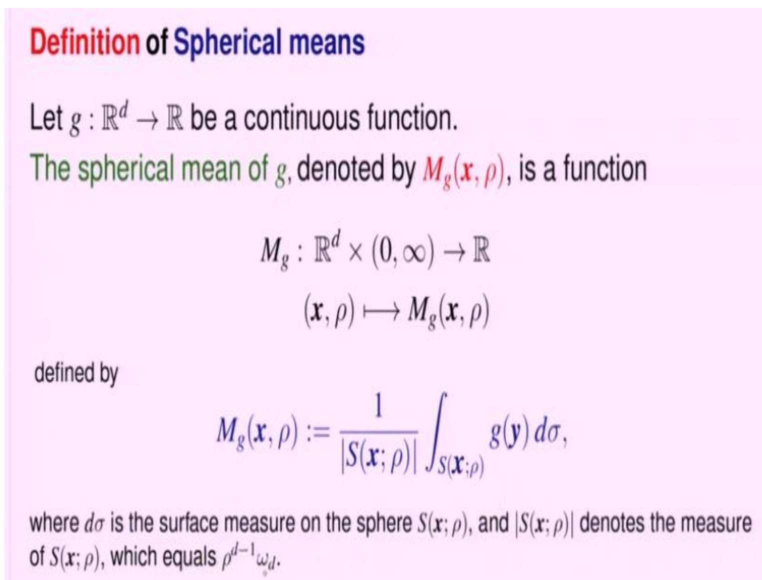
So this what we have proved integral of the Laplacian G over B of x rho is given in terms of rho power d – 1 rho by dou rho del derivative of this quantity the integral on the unit sphere with center origin.

(Refer Slide Time: 09:53)



Now we are in the position to introduce spherical means and the Darboux formula.

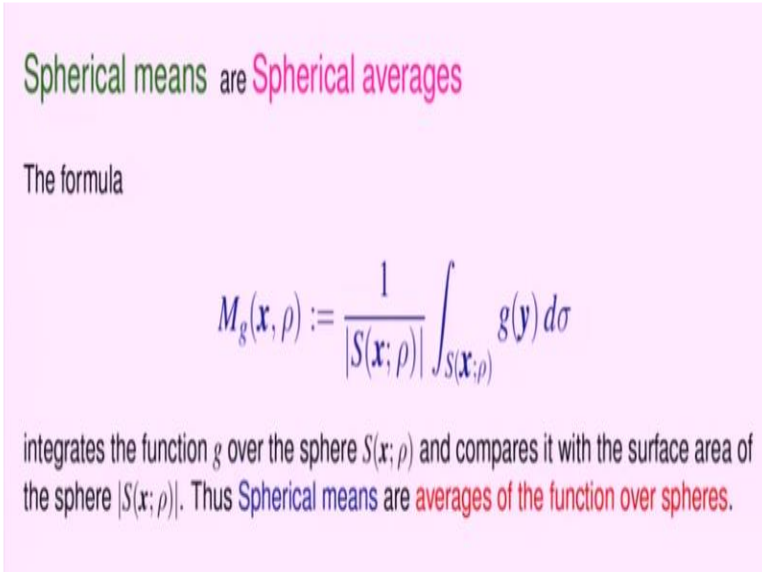
(Refer Slide Time: 10:01)



So take continuous function g defined an Rd to R let them continuous function. The spherical mean of g denoted by Mg M for mean we are talking about mean of the function g. Because later on we are going to deal with means of at least 3 functions 1 is means of the solution u and mean of the initial data phi and psi that is why we introduce g here not to get any confusion later on. So Mg of x, rho it is a function defined an Rd cross infinity taking values in R.

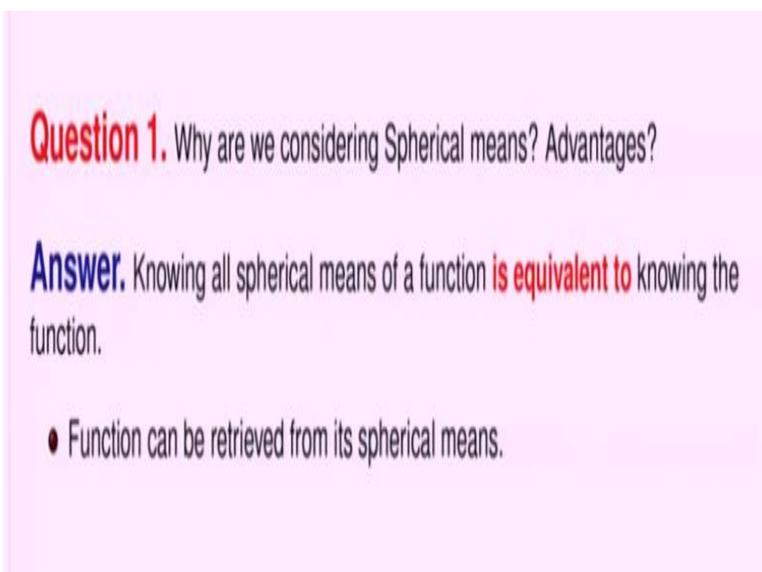
So Mg of x rho is defined by integral g over this sphere S x rho and divide with the measure of S x, rho that is why it is called mean. And d sigma is surface measure on this sphere and what is the measure of S x, rho? That is precisely rho power d – 1 into omega d this we have already stated.

(Refer Slide Time: 11:12)



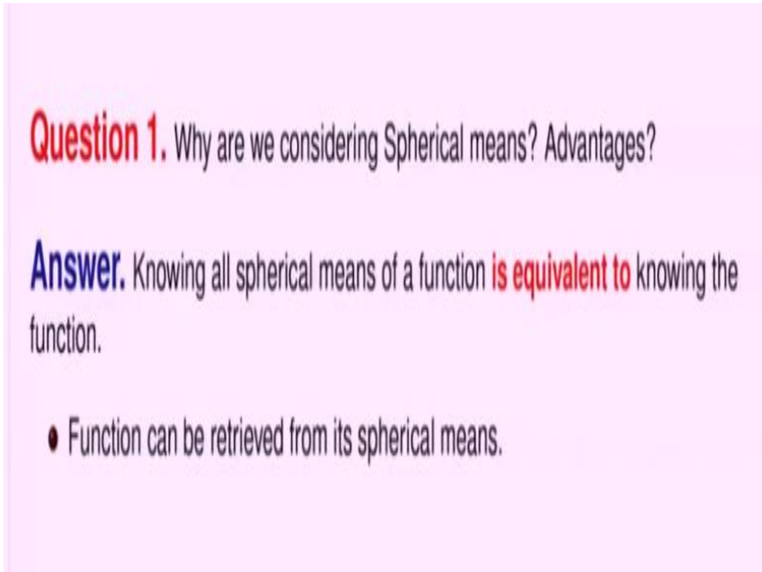
Spherical means mean spherical averages so this formula which defines spherical means is integrating the function g over this sphere and compares it with the surface area of this sphere. Recall how athematic means are defined if this is exactly similar so spherical means are averages of the function over spheres.

(Refer Slide Time: 11:39)



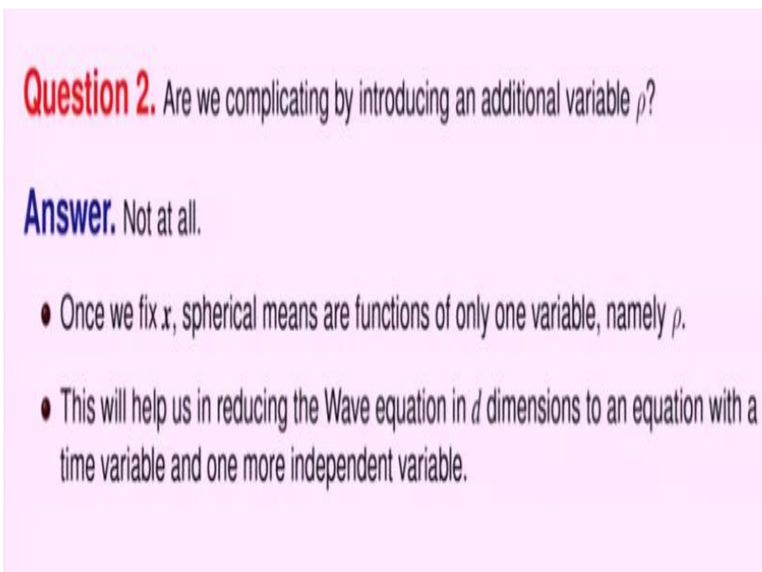
Now we a question why are we considering spherical means? Are there any advantages to be done? Knowing all the spherical means of a function is same as or is equivalent to knowing the function itself. And importantly function can be retrieved from its spherical means that is the idea.

(Refer Slide Time: 12:03)



We will state more precisely results which say which asset function can be retrieved from spherical means we will do that.

(Refer Slide Time: 12:12)



So now another question is are we complicating by introducing an additional variable rho? Not at all once we fix x the spherical means are functions of only one variable namely rho. So this will help us in reducing the wave equation in d dimensions to an equation with a time variable and one more independent variable we will see that.

(Refer Slide Time: 12:38)

$$\begin{aligned}\int_{S(\boldsymbol{x};\rho)} g(\boldsymbol{y})\,d\sigma &= \int_{B(\boldsymbol{x};\rho)} \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left(g(\boldsymbol{y})\,\frac{y_i - x_i}{\rho}\right) d\boldsymbol{y} \\ &= \int_{B(\boldsymbol{0};1)} \sum_{i=1}^{d} \rho^{-1}\frac{\partial}{\partial z_i}\left(g(\boldsymbol{x}+\rho\boldsymbol{z})\,z_i\right)\,\rho^d d\boldsymbol{z} \\ &= \rho^{d-1}\int_{B(\boldsymbol{0};1)} \sum_{i=1}^{d} \frac{\partial}{\partial z_i}\left(g(\boldsymbol{x}+\rho\boldsymbol{z})\,z_i\right)\,d\boldsymbol{z} \\ &= \rho^{d-1}\int_{S(\boldsymbol{0};1)} \sum_{i=1}^{d} g(\boldsymbol{x}+\rho\boldsymbol{z})\,z_i^2\,d\omega \\ &= \rho^{d-1}\int_{S(\boldsymbol{0};1)} g(\boldsymbol{x}+\rho\boldsymbol{z})\,d\omega\end{aligned}$$

Let us look at this computation where we are integrating g over the sphere S x, rho that by Greens theorem is equal to this. So please compute this quantity on the right hand side you will get the quantity on the left hand side. Now why, are we doing this we will see at the end of this computation and the reason for doing this? Now this we are now changing the variable where ball of x rho becoming ball of 0, 1 and the measure, d by natural rho power d dz and the derivative dou by i is rho inverse dou by dou z i.

We have already established these facts so finally we have an integral on ball of 0, 1 now we will write this using once again integration by parts of Greens theorem and express this as an integral on the sphere S 0, 1. So we end up with this expression and if you notice this is does not depend on i so it can come out what I have summation i = 1 to d zi square where is z? z is on this sphere therefore norm z is 1 and summation I = 1 to d zi square is 1.

Therefore we get this expression so we have got integral on S x, rho we have converted that to integral on S 0, 1 so we did change of variables in spheres.

(Refer Slide Time: 14:18)

$$
\begin{aligned}
\int_{S(\boldsymbol{x};\rho)} g(\boldsymbol{y})\, d\sigma &= \int_{B(\boldsymbol{x};\rho)} \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left(g(\boldsymbol{y})\, \frac{y_i - x_i}{\rho} \right) d\boldsymbol{y} \\
&= \int_{B(\boldsymbol{0};1)} \sum_{i=1}^{d} \rho^{-1} \frac{\partial}{\partial z_i} \left(g(\boldsymbol{x} + \rho \boldsymbol{z})\, z_i \right) \rho^{d} d\boldsymbol{z} \\
&= \rho^{d-1} \int_{B(\boldsymbol{0};1)} \sum_{i=1}^{d} \frac{\partial}{\partial z_i} \left(g(\boldsymbol{x} + \rho \boldsymbol{z})\, z_i \right) d\boldsymbol{z} \\
&= \rho^{d-1} \int_{S(\boldsymbol{0};1)} \sum_{i=1}^{d} g(\boldsymbol{x} + \rho \boldsymbol{z})\, z_i^{2}\, d\omega \\
&= \rho^{d-1} \int_{S(\boldsymbol{0};1)} g(\boldsymbol{x} + \rho \boldsymbol{z})\, d\omega
\end{aligned}
$$

So in view of the computation on the previous slide what was it about? We are just answered it is something to do with change on variables on spheres. Usually we learnt change of variable theorem for open sets in Rd so that means we know how to handle open sets in Rd whereas S x, rho is not an open set in Rd. So directly if you write this equal to the last lines there are questions and this; what is establishing that this is indeed true. So we can be here like yes I know how to do change the variables even for spheres.

(Refer Slide Time: 14:53)

Remark on Spherical means

In view of the computation on the previous slide (What was it about?), and since $|S(\boldsymbol{x};\rho)| = \rho^{d-1}\omega_d$, we may write

$$
\begin{aligned}
\frac{1}{|S(\boldsymbol{x};\rho)|} \int_{S(\boldsymbol{x};\rho)} g(\boldsymbol{y})\, d\sigma &= \frac{\rho^{d-1}}{\rho^{d-1}\omega_d} \int_{S(\boldsymbol{0};1)} g(\boldsymbol{x} + \rho \boldsymbol{z})\, d\omega \\
&= \frac{1}{\omega_d} \int_{\|\nu\|=1} g(\boldsymbol{x} + \rho\nu)\, d\omega.
\end{aligned}
$$

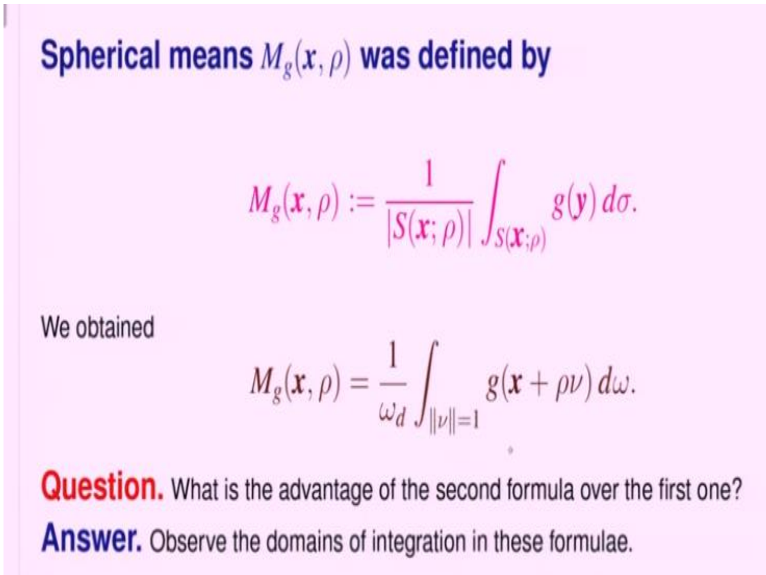
In view of the definition of spherical means $M_g(\boldsymbol{x},\rho)$, we get

$$
M_g(\boldsymbol{x},\rho) = \frac{1}{\omega_d} \int_{\|\nu\|=1} g(\boldsymbol{x} + \rho\nu)\, d\omega. \quad \square
$$

So that is about it how this is the definition of Mg of x , rho now that we have alternate expression which is 1 by omega d integral of g of x + rho mu d omega integral over norm mu =

1. So in view of the definition of the spherical means which is actually Mg equal to this now we have established is equal to this. So we have a new formula for Mg equal to this.

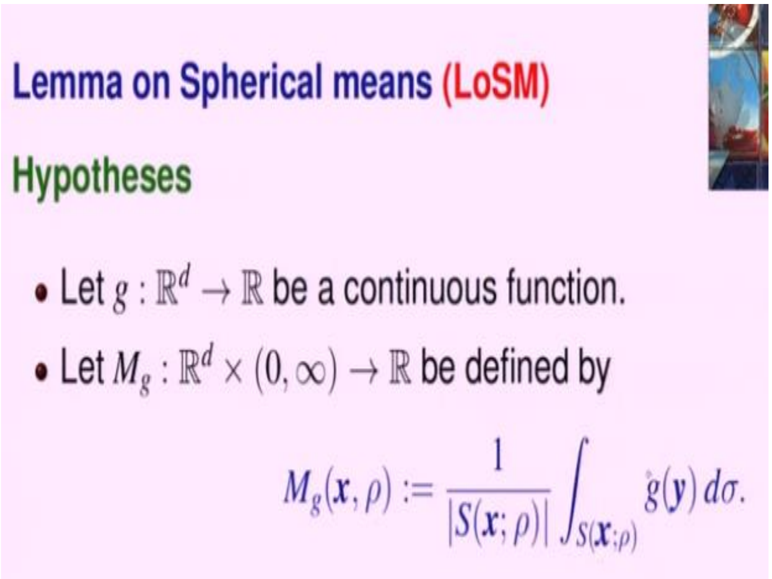
(Refer Slide Time: 15:32)



So spherical means was defined by the formula Mg = 1 by measure of S x rho by integral over S x, rho of g y d sigma and we obtained another expression why are we doing this? What is that advantage of the second formula over the first one the answer is observed the domains of integration in these formulae. What is the domain here? S x rho; what is the domain here? Norm nu equal to 1 what is this function Mg of x, rho so if I want to differentiate either with respect to S x r with respect to rho here it is difficult.

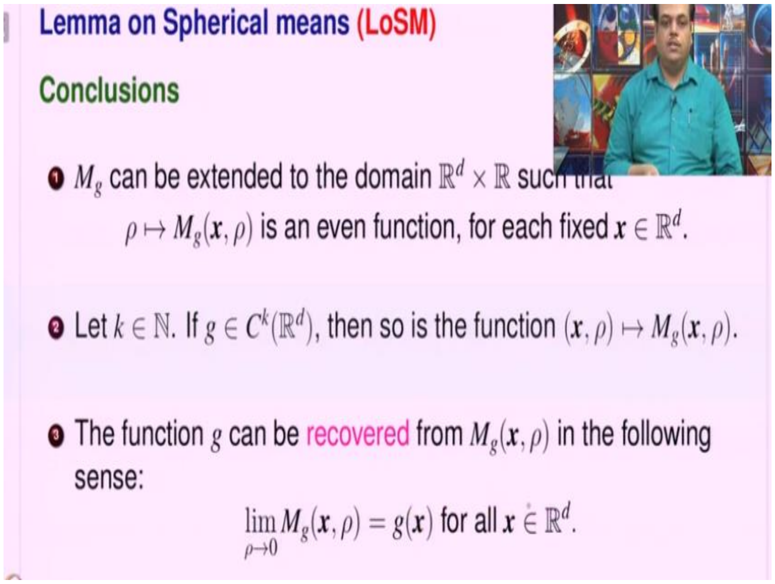
Because the domain itself is varying with x and rho whereas in this expression it does not depend on x it does not depend on rho that is the advantage we have.

(Refer Slide Time: 16:23)



So we have a lemma on spherical means we will often be using, we calling it LoSM lemma on spherical means hypotheses g is the continuous function and let Mg be define by this formula of the spherical means of g.

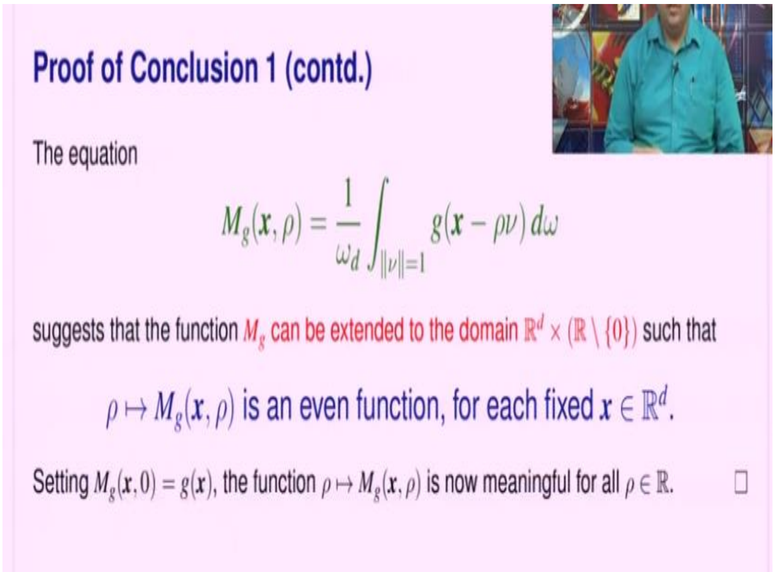
(Refer Slide Time: 16:41)



Conclusion 1 sometime we call LoSM 1 what is it? Mg can be extended to the domain Rd cross R such that for each fixed x as the function of rho it is an even function. Currently it has defined only for positive because we needed the ball to have a positive radius. So let k belongs to N if g is the Ck function in Rd then SO is the function x, rho mapping to Mg of x rho it is function defined on Rd cross, R.

Note by 1 conclusion 1 we already extended the function Mg of x, rho were rho belonging to R and the function g can be recovered from the spherical means this is the most important property. What do you mean by that? You take Mg of x, rho and take limit as rho goes to 0 you get g of x and this happens for every x.

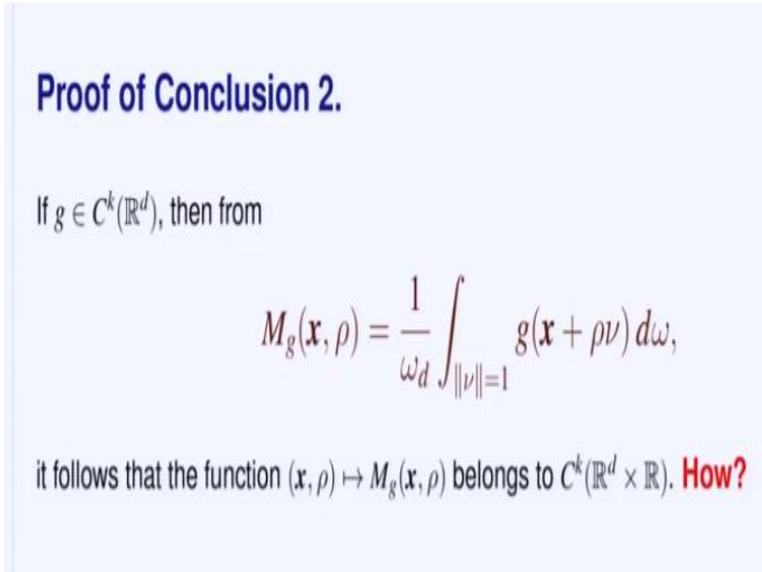
(Refer Slide Time: 17:46)



Now let us prove conclusion 1 observe this equality holds why? Because the units sphere is invariant under the mapping nu going; to minus nu so, therefore these integrals are same therefore Mg which we know is equal to this is now also equal to this quantity. So in other words Mg as the function of rho whether you put plus rho or minus rho it is a same that means the function is even. So we can use the same formula to extend Mg of x rho even for rho negative.

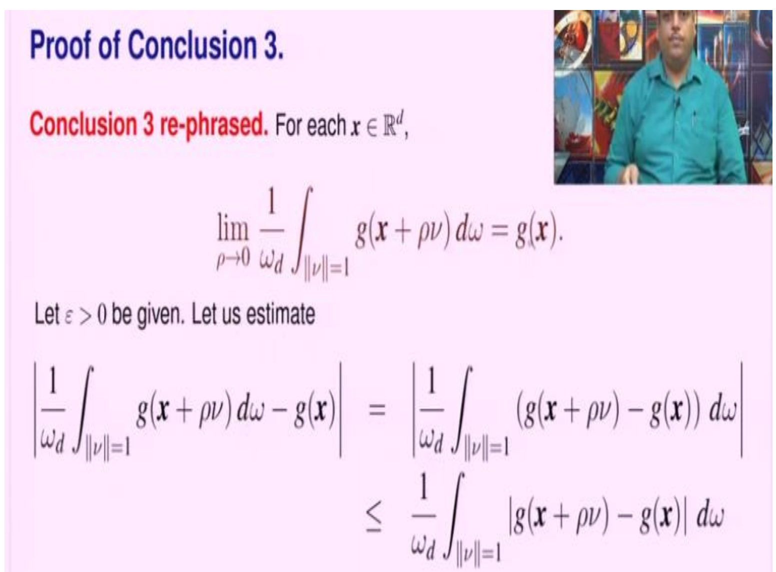
So we can do this for rho negative rho positive we have started defining spherical means for rho positive we have not extended for rho negative what about 0? So 0 we just take it to g x Mg of x, 0 = g x this also consistent with the third assertion of this lemma then it is defined for all rho. So just to summarize we have defined Mg of x rho for rho positive then we observe that if repress rho with minus rho the integral does not change. Therefore we can extend Mg of x rho even for negative rho and for rho equal to 0 we define by Mg of x, 0 = g x.

(Refer Slide Time: 19:21)



So let us go to the proof of conclusion 2 if g is the Ck faction in Rd define an Rd then from this formula Mg of x rho = 1 by omega d integral over norm nu equal 1 g of x + rho nu d omega. Notice that x and rho appears inside the argument of this function g therefore if g CK then Mg of x rho will be Ck also. So it follows that this function x rho going to Mg x rho Ck of Rd cross R how? We have to justify the interchange of derivative on the integration and there is an exercise in an answer.

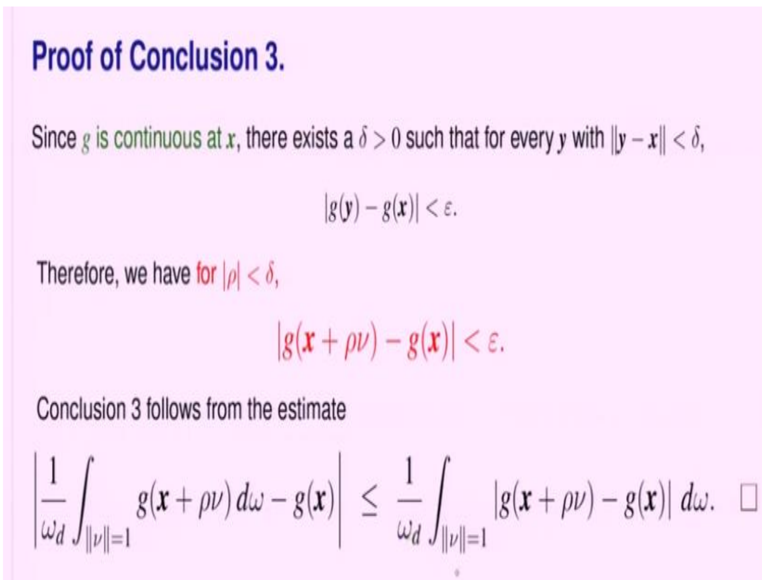
(Refer Slide Time: 20:04)



Now proof of conclusion 3 let us refresh the conclusion this is limit rho goes to 0 this is Mg of x rho. That is equal to gx this is what we want to show so do you show this? Start estimating the difference between this quantity and g x so let epsilon be given positive be given let us estimate this quantity whose limit you want to find you want to show that limited g x therefore estimate the distance. Now g x can be written as 1 by omega d integral norm nu = 1 because omega d is after all the measure of the set norm nu = 1.

Therefore we can do that of course g itself cannot depend on nu so we can put it inside now we take the modulus inside the integral. So modulus of the integral is (()) (21:05) integral of modulus and we get this. Now this suggest something to be used the property of g here g is continuous at x. So therefore this can be made small as a result this can be made small and we achieve what we want to show that is the idea.

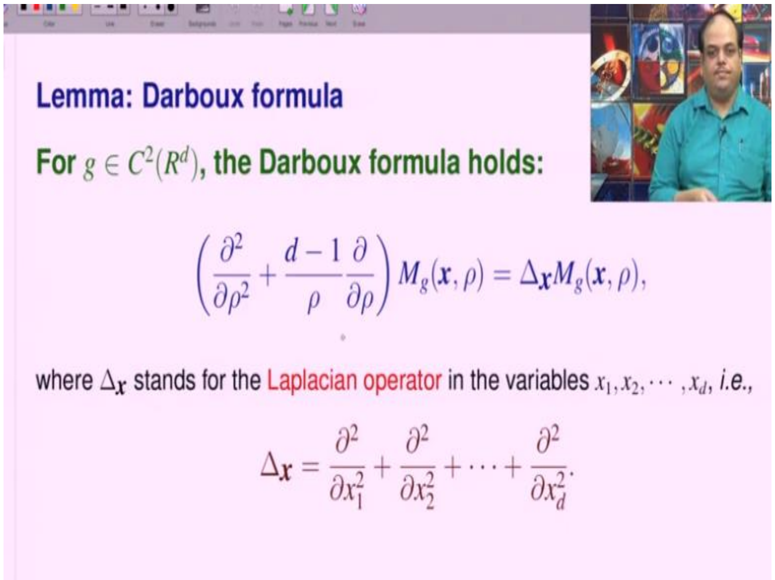
(Refer Slide Time: 21:27)



Since g is continuous at x there is a delta whenever y is delta close to x g y – g x is less than epsilon in modulus. Therefore we are not interested in arbitrary y but we are interested in y which is looking like x + rho nu where nu is having unit norm. So therefore for mod rho less than delta this becomes mod g of x + rho nu – g x becomes less than epsilon. So now conclusion 3 follows from this estimate because this is less than epsilon.

So the RHC is less than or equal to epsilon by omega d integral norm nu =1 d omega which will give you omega d. So omega d- omega d cancels and we get epsilon so the conclusion 3 follows.

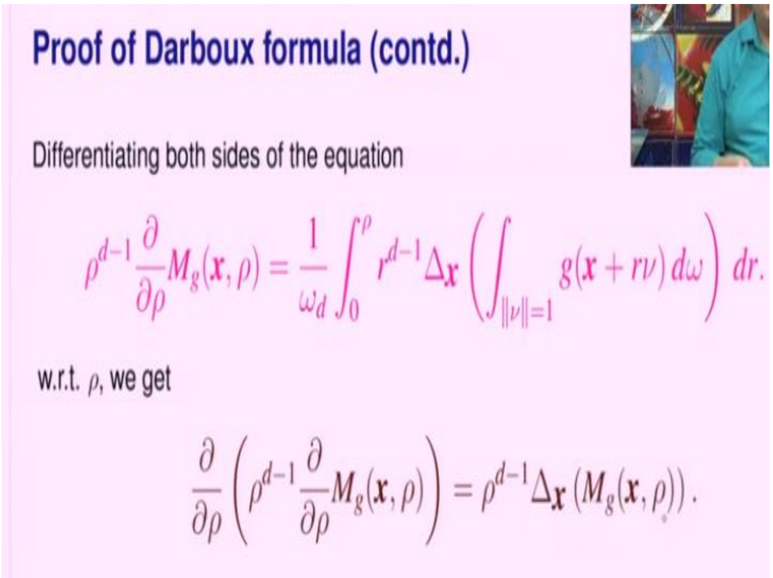
(Refer Slide Time: 22:27)



Now let us prove Darboux lemma for g in C2 of Rd the Darboux formula holds what is the Darboux formula? It is this Laplacian in the x variable of Mg of x rho is this quantity. So the Laplas x sub x stands for Laplacian operator the variables x1 up to xd which is here. Actually if you looking at the Laplacian equation we have not yet looked at Laplas equation and you want to look at solution which depends on only on the radius R then that you end up with this operator.

So this is actually the radial Laplacian other angular parts are ignored I am not looking at solution which depends on angles therefore only the radial part this is that.

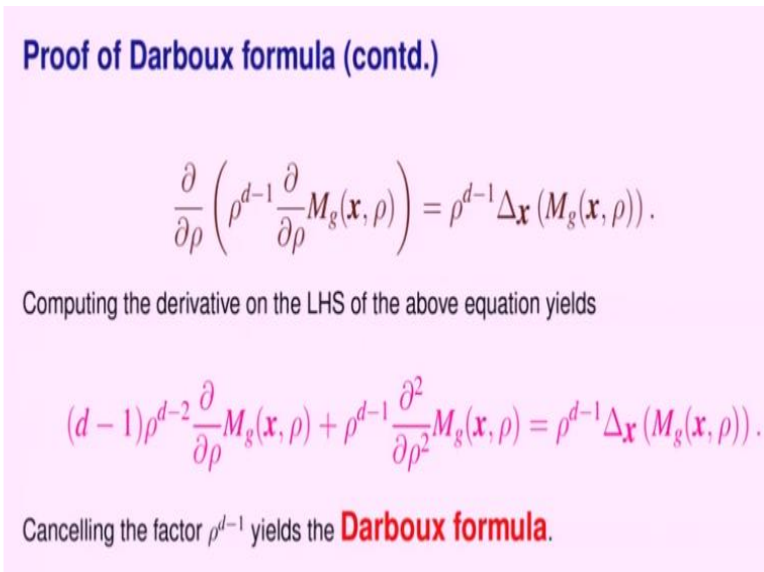
(Refer Slide Time: 23 27)



Recall that we have proved this Laplacian on d of x rho is given by this quantity this we have already proved in the beginning. So this quantity is actually this with a omega d 1 by omega d missing you put omega d so you get this multiply both sides with 1 by omega d you have this equal. Now express this integral on the ball using polar coordinates R and d omega so you have this expression.

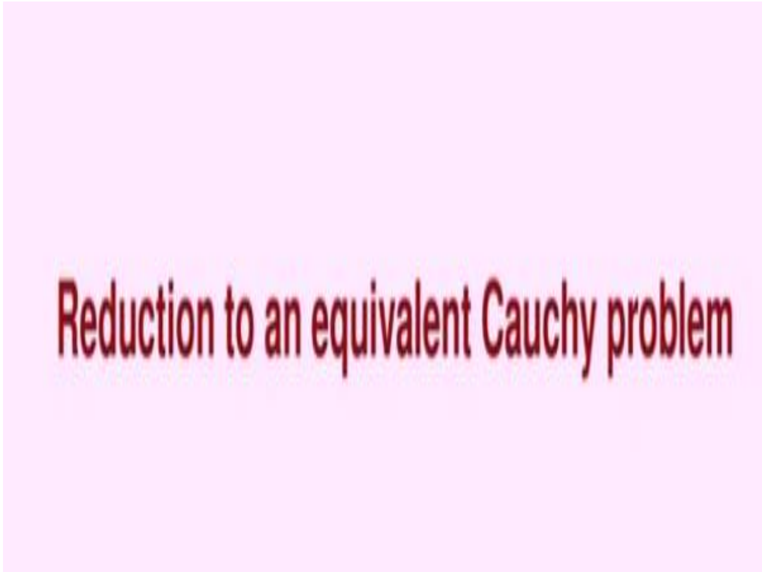
Now differentiate both sides with respect to rho so LHS is simply this and RHS by fundamental theorem of calculus is the integrand evaluated at rho which is this.

(Refer Slide Time: 24:23)



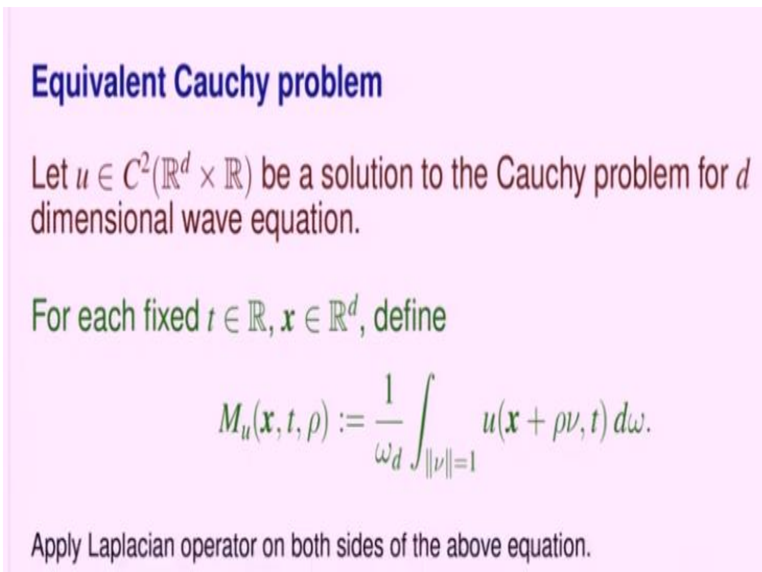
Now we can expand this derivatives which are here and we get this now if you cancel rho power d -1 on both LHS and RHS what we get is precisely the Darboux formula.

(Refer Slide Time: 24:43)



Now let us reduce the Cauchy problem for the wave equation to an equivalent Cauchy problem.

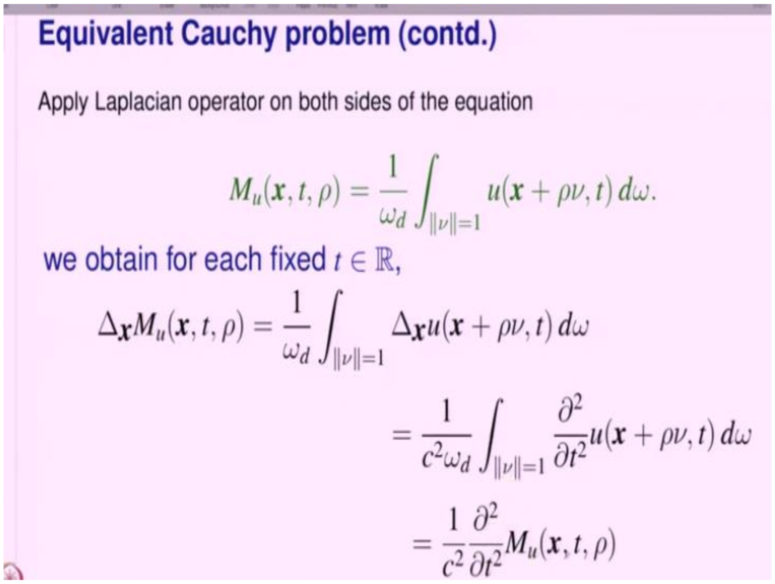
(Refer Slide Time: 24:51)



So for that we have to introduce spherical means associated to the solution or the unknown function u. So let u be a C2 function be a solution to the Cauchy problem for d dimensional wave equation for each fixed t and x define the spherical means. A Mu x, t rho earlier when we were dealing with spherical means there is no variable called t therefore it was missing we had x and rho but now we have to include that because we are trying to solve wave equation now.

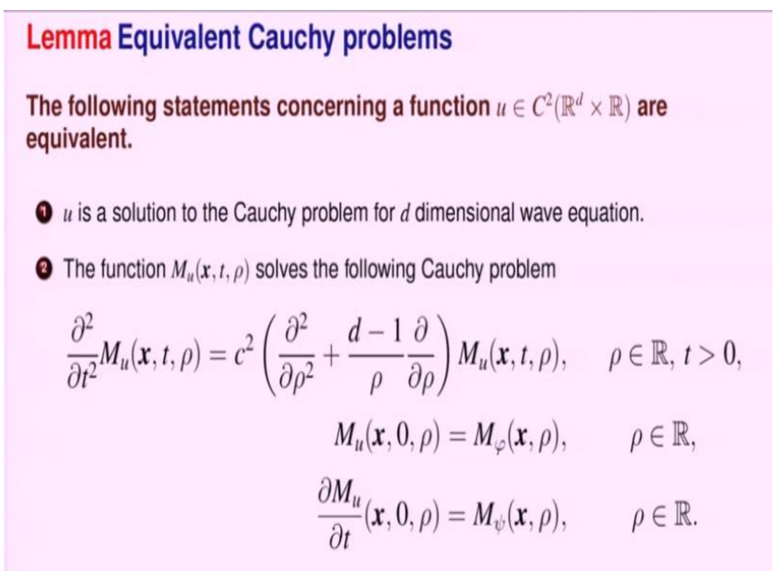
So mu of x t rho is exactly as before now apply Laplacian on both sides of the above equation what do you get?

(Refer Slide Time: 25:37)



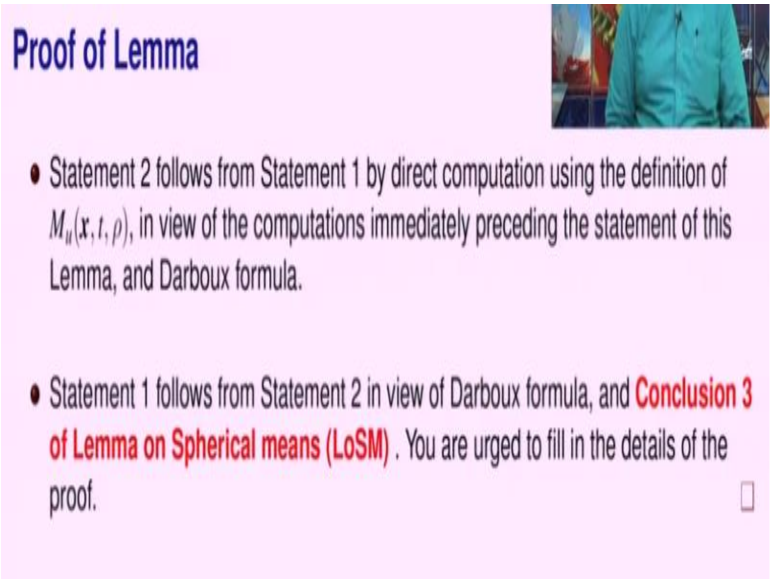
What we get is? This is just an applying Laplacian and writing so we need to really do here the Laplacian has gone inside here that is the change. Now u is the solution right so Laplacian x u is u tt by c square because u a solution to the homogenous wave equation so then we have this. Now bring dou 2 by dou t square outside and we have this so what we have is Laplacian Mu is 1 by c square dou 2 by dou t square Mu.

(Refer Slide Time: 26:25)



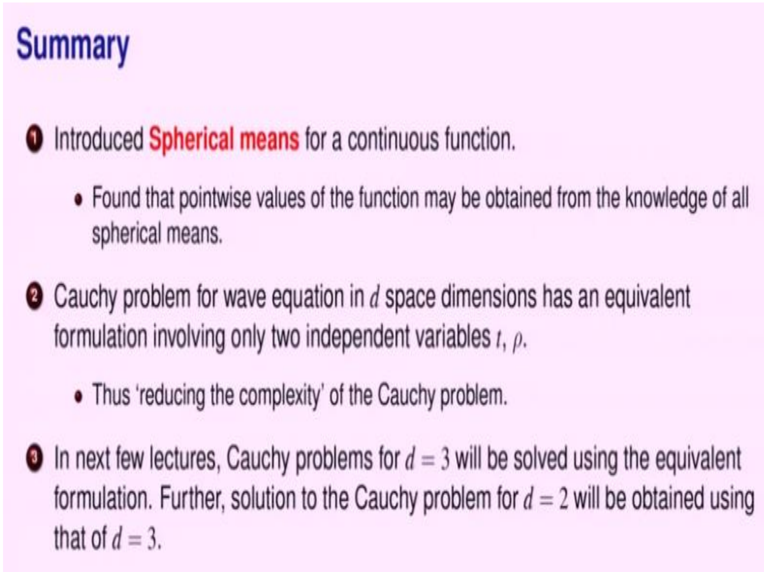
Now we have this statement about the equivalent Cauchy problems so the following statement concerning a function u which is C2 of Rd cross R, are equivalent what are those? u is a solution to the Cauchy problem for d dimensional wave equation and that is same as saying dou 2 by dou t square of Mu is c square into this operator acting on Mu and of course the initial data for Mu and dou by dou t of Nu we will prove this. Very simple proof we have done all the computation I will just indicate how to go about this proof.

(Refer Slide Time: 27:07)



So statement 2 follows from statement 1 but direct computation using the definition of Mu we have done all the computation all the computations immediately preceding the statement of this lemma and Darboux formula. Statement 1 follows from state 2 in view of the above formula and conclusion 3 of LoSM that is the function can be retrieved from its spherical means. So you are asked to fill in the details of the proof very simple details you have to just write down a few lines. All the computations are already there and I have given you how to go about here.

(Refer Slide Time: 27:49)



So let us summarize what we did in this lecture we have introduced spherical means for a continuous function found that point wise values of the function may be obtained from the knowledge of all spherical means. Cauchy problem for wave equation in d space dimensions has a equivalent formulation involving only 2 dependent variables. Thus reducing the complexity of the problem in the few lectures Cauchy problem for d = 3 will be solved using the equivalent formulation.

Further solution to the Cauchy problem for the d = 2 will be obtained using the solution that we get for d = 3 case.