Partial Differential Equations
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Module No # 06
Lecture No # 26
Wave Equation – A Mathematical for Vibrating Strings

So welcome to this lecture on wave equations we are going to start our study of wave equation starting from this lecture onwards. In this lecture we are going to derive an equation which governs the transfer's vibrations of a vibrating string.

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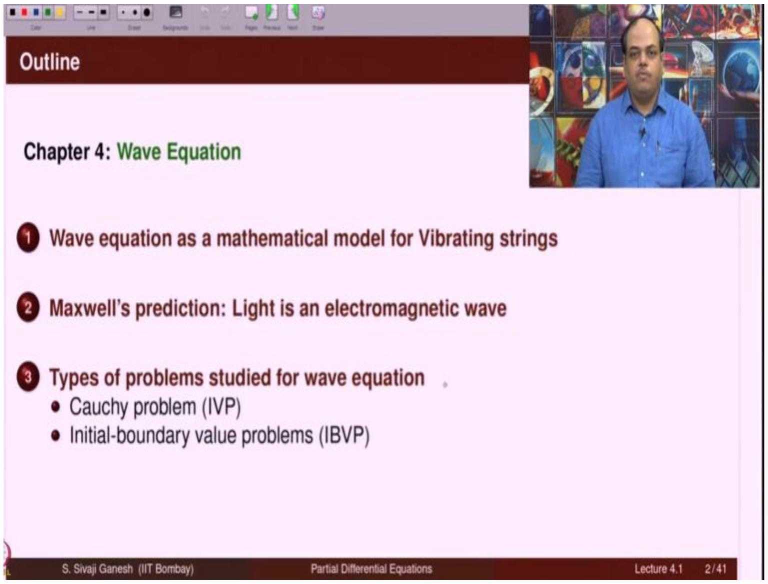

The outline for today's lecture is first we will derive wave equation as mathematical model for transfer's vibration of strings. Then we present them Maxwell's prediction that light is an electromagnetic wave it is based on wave equation actually. And we introduce 2 problems that we are going to study for the wave equation, one of them is called Cauchy problem or initial value problem and the second problem is Initial boundary value problems.

These are the 2 kinds of problems that we are going to study for wave equation mostly for 1 dimensional wave equation.

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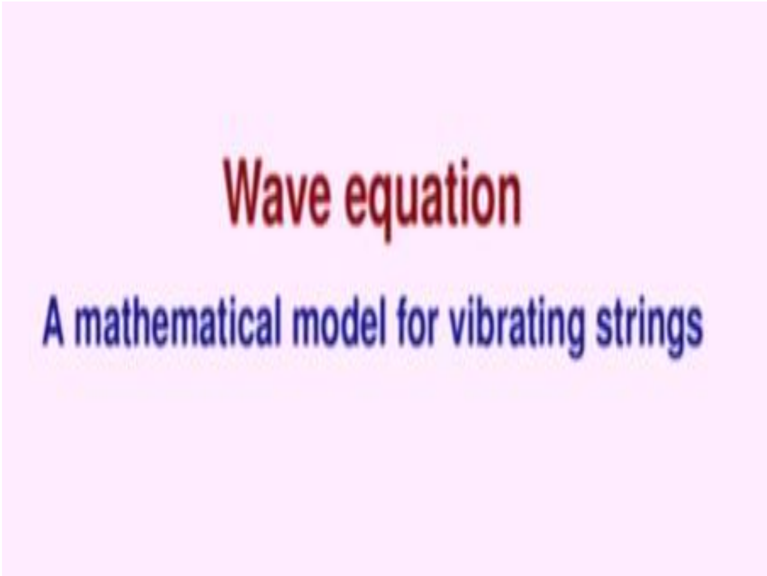

So wave equation as a mathematical model for vibrating strings.

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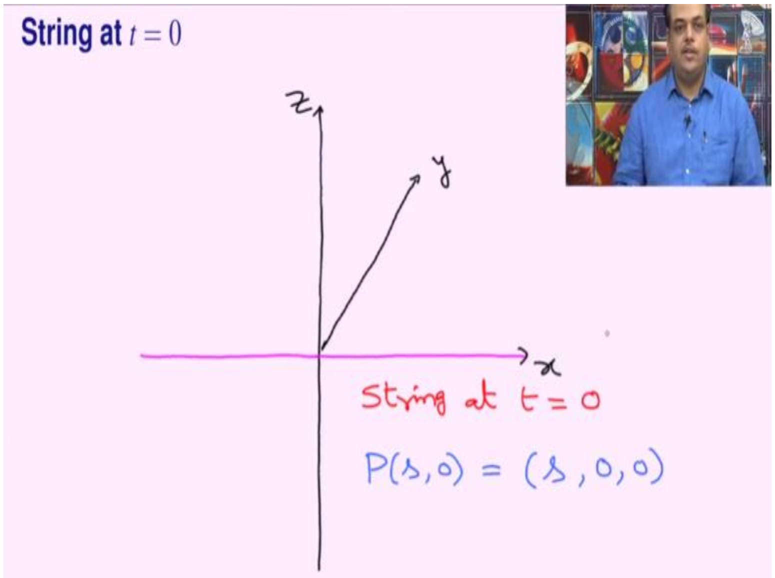

So this is the picture of the string is line along the x axis at time t = 0 so therefore the position vector P of s, 0 is s, 0, 0 that means x = s y = 0 z = 0. At this point of time you can imagine that this string is infinity length or finite length that does not matter.

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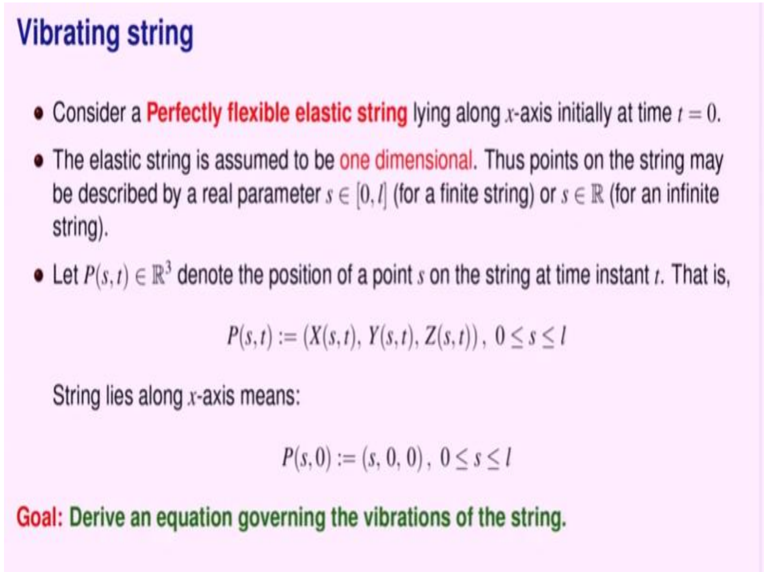

So consider a perfectly flexible elastic string lying along x axis at time t = 0. The elastic string is assumed to be 1 dimensional thus points on the string maybe described by real parameter s belongs to finite interval 0, l that the string is finite or s, in R (()) (02:16) infinite string or even s in 0, infinity both are same actually. As far as this modeling goes but as far as the problems that are being it will be post s belongs to R is different from s belongs to 0, infinity that we will see in the discussion of initial boundary value problems.

So let P of s, t it is a vector in R3 denote the position of a point s on the string as we have introduced here points on the string are identified with s. So it is a points s on the string x = s and at the time instant t. So P of s, t is given by X of s, t Y of s, t Z of s, t and here and here we are writing a finite string. String lies along x axis means that P of s, 0 is s, 0, 0 these are we all have identified points on the string at time t = 0 we said s, 0, 0 is represents points on the string.

And when we s is between 0 and l it means we are considering a finite string so goal is to derive an equation which governs the vibrations of the string.

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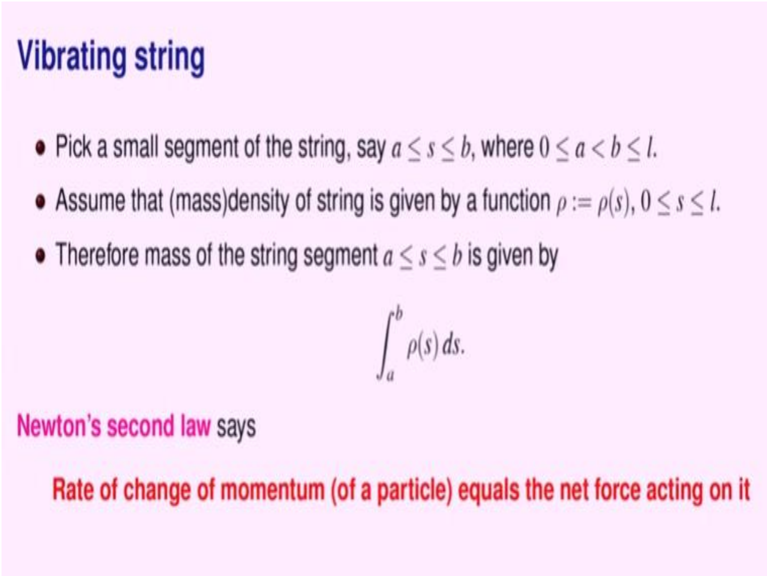

So how do we do that? Pick a small segment of the string let us say for s between a, and b. Of course a, and b itself is between 0 and l assume that it is density when we say density the usually in the first course in physics density is mass density. But in mathematical modeling will see there are lots of densities that are why we stress here it is a mass density of the string is given by a function rho of s between 0 and l.

So therefore if you want to consider the mass what is the mass of the string between a, and b it is by integrating the density you get. Density as the dimension of a mass per unit volume and you are multiplying integrating over volume therefore you will get mass that is, understanding this. So whenever you have density we integrate you get that quantity. In this case you get the mass so Newton's second law it says that rate of change of momentum equals the net force acting on it.

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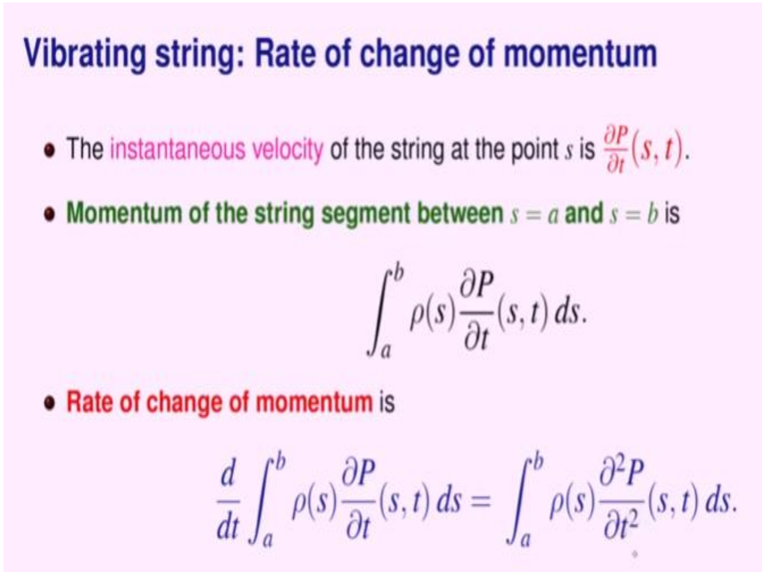

So we would like to apply this Newton's second law so to apply Newton's second law we need know what is this rate of change of momentum and what are the forces acting? So momentum is related to the velocity so the instantaneous velocity of the string of the point is dau P by dau t at S, t. These are the time instant t momentum of string between s = a and s = b momentum is mass into velocity right.

Per mass we have a mass density we multiply the density with dau P by dau t and integrate we get the moment of the string segment between s = a, and s = b is integral a to b rho of s dau P by dau t of s, t ds. So what is rate of change of momentum d by dt of this so d by dt of this quantity that is the rate of change of momentum. So we have got one side of the Newton's second law so we need to see what is the other side?

Namely the forces acting on the segment a, b this can further be expanded d by dt you can go inside and dau P by dau t become dau 2 P by dau t square.

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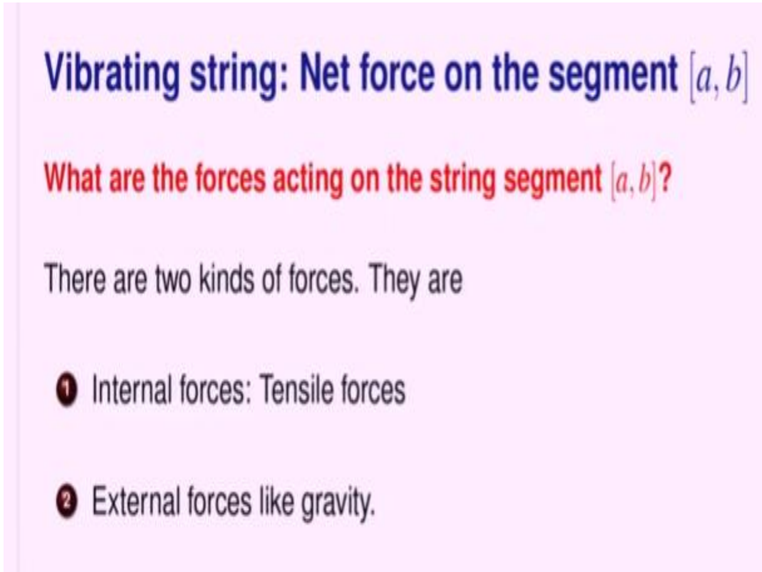

So net force acting on the segment a, b so there are 2 kinds of forces acting on the segment a, b what are those? First one is internal forces these are the tensile forces due to the tension in the string. And second is external forces like gravity and many other forces.

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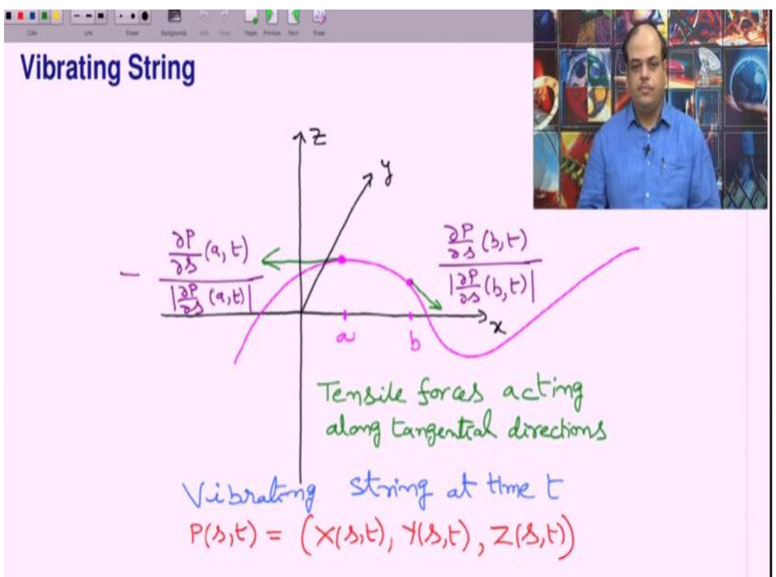

So this is the picture of the vibrating string these are time instant t vibrating string at time instant t this is a position P s, t is X s, t, Y s, t, Z s, t. So this is the string that we are this is the segment a, b that we are considering. So at b there is somebody who is pulling in this direction but the rest of the string and at a this is this side of the string which is pulling in this direction so tensile force acting along tangential directions.

So therefore the tangential direction at this point is dau P by dau s b, t and dividing that with its length will give us unit tangent vector in this direction is this. So at this point a it is in the opposite direction opposite means it is at this direction so that we model with minus sign here. Otherwise it is exactly the time initial direction.

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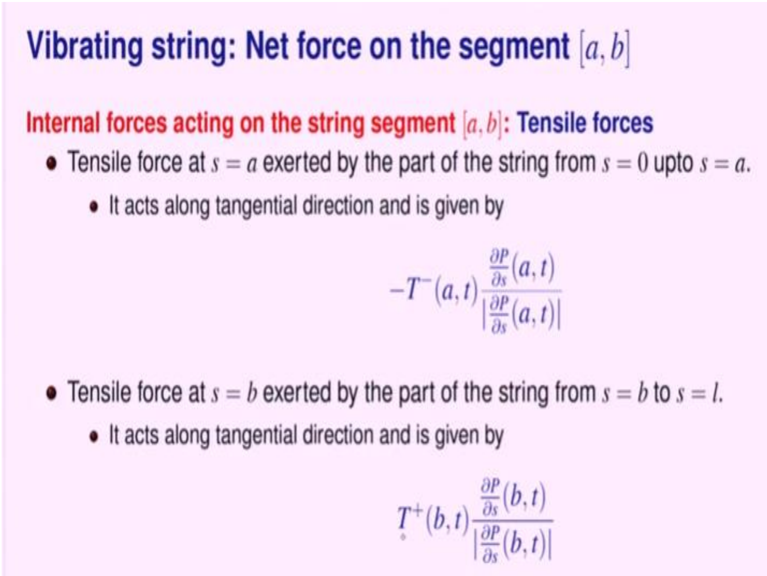

So let us see what are internal forces which are acting on the string segment? What are we saw in the picture now we are going to write it down. So tensile force at s = a is the segment a, b this is at one of the end at s = a. It is exerted by the part of the string from s = 0 to s = a. It acts along tangential direction and it is given by this is the tangential direction and this direction that we have chosen minus because we are at end point a, the left hand point in this interval and there is a number T – a, t.

This is what is a tensile and tensile force at s = b similarly is a exerted by the part of the string from s = b to l and it acts along tangential direction and is given by this is the tangential at the point b and T + b, t. So this minus and plus denote that this something coming from the left side on the string and this is coming from the right side of the string. We will soon see that they are the same at any point you do not have 2 values t – a, t and t + a, t no both are same we will see that soon.

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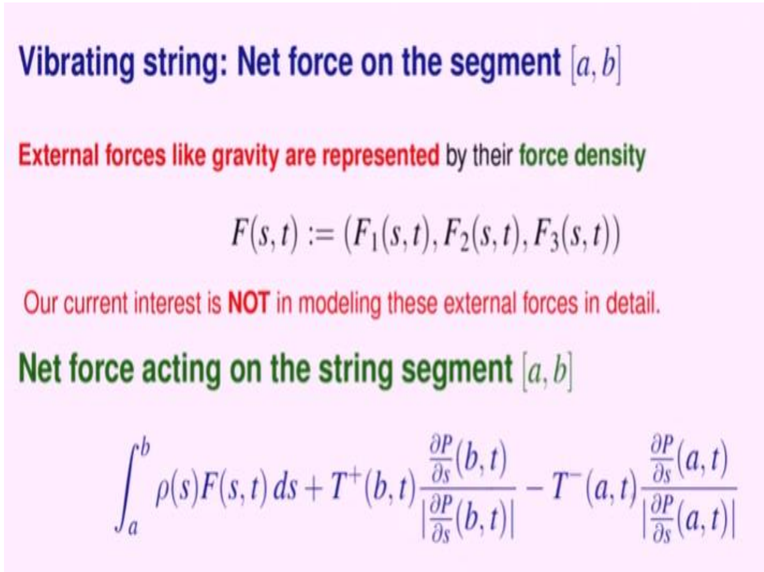

So now how do we model external forces their represented by the force density F s, t = F1, F2, F3. Our current interest is not in modeling these external forces suppose you decide you want to affects some gravity then you would like to incorporate explicitly how the gravity forces apply on the string. So we are not interested in that we are generally taking any external force. So we are not going to model we are not going to consider external forces and what are the resultant F here we are not going to do that.

So net force acting on the string segment is here a to b rho s F s, t ds is a force these are force these are mass density this one is the force in this a, b and this is the internal forces.

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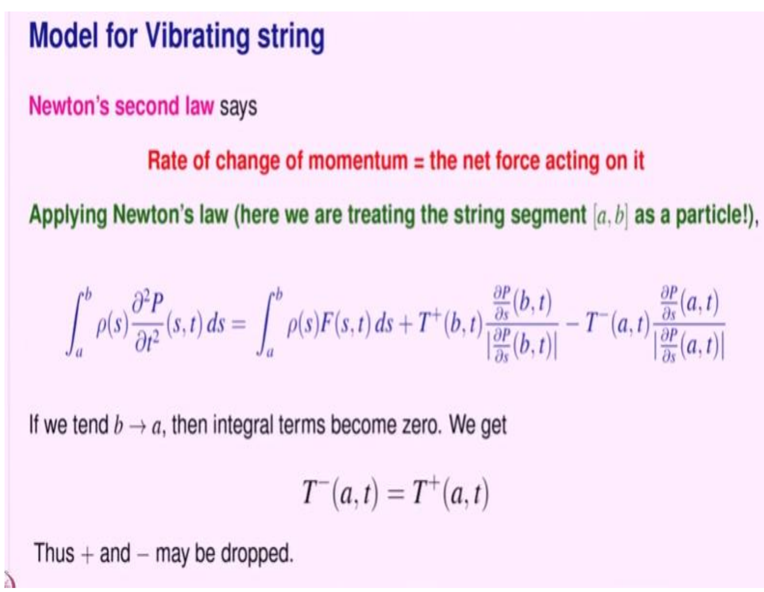

So one is external one is internal so Newton's second law says that rate of change of momentum is equal to the net force on it. So applying Newton's law so here we are treating the string segment as a particle so this is the rate of change of momentum and that equals the net force which is here. So this is the equation that we have got so far now suppose we tend b = a in other words we are trying to as close to a particle as possible by taking this very small.

Imagine a = b we are indeed particle so we tend b to a, and what will happen to this integrals? If inside integrals are reasonable this integral will be 0 this integral will also be 0. And when b goes to a this term goes to T + a, t dau b by dau s a, t by modulus dau P by modulus a, t. And this any way is a that equal to 0 and that will give us the T – is same as T +. Therefore we draw plus and minus in the equation here so let us drop that.

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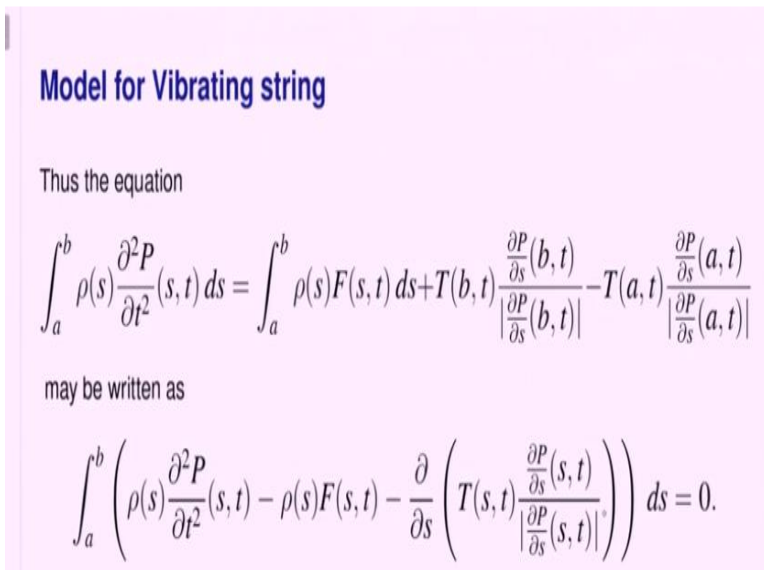

So after dropping this is the equation that we have now we would like to write this also as an integral. This looks like some quantity evaluated b – same quantity evaluated at a so this looks like integral from a, to b of certain derivative of some point of time. So that is exactly this here this is dau by dau s of this into ds a, to b that will give you that this evaluated b minus this evaluated at a, which is exactly this so we can write this.

Now advantage is that all the terms are converted into integrals and we have this expression now this expression equal to 0 this equation holds no matter what a, and b is? Therefore the integrant must be 0 we have discussed this kind of issues while dealing with conservation loss.

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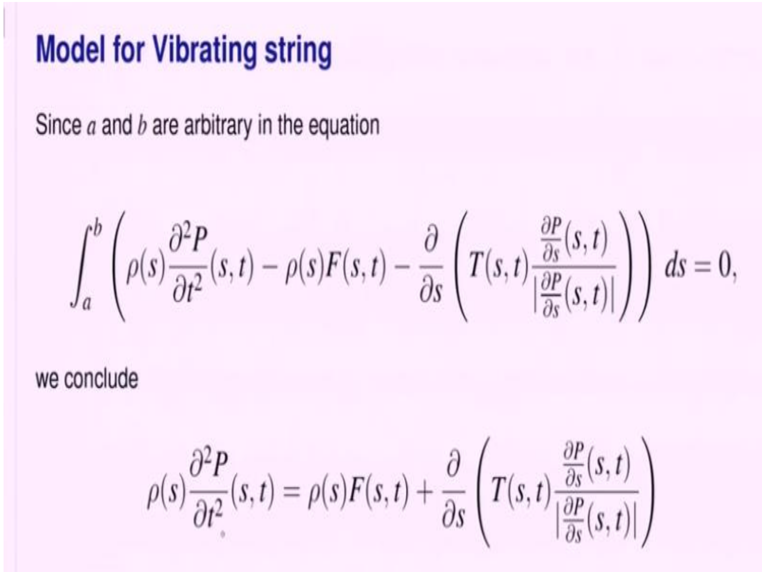

So since a and b are arbitrary in this equation the integrant must be 0 so this is an equation that we are got where P we have a second derivative with respect to t this is external force and P is here and then t there is something which is unknown. Otherwise that is if t is known this will be an equation for P a second order equation in T.

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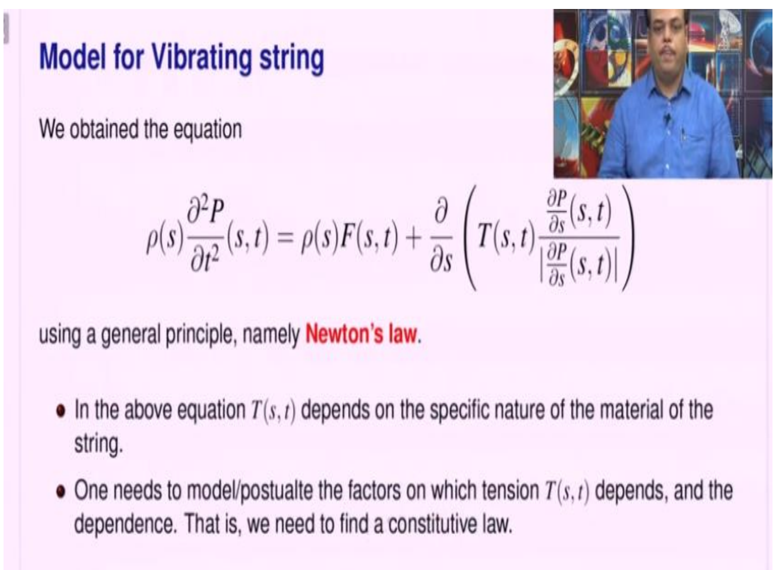

So we have obtained this equation using a general principle which is Newton's second law and in the above equation T s, t is what the tension how the string is what material it is made up of it will depend on that. So 1 is to model or postulate the factors on which the tension depends and the manner in which it depends. So we need to find a constitutive law. If you recall the modeling that we did for the traffic problem this is exactly that this is like modeling the road.

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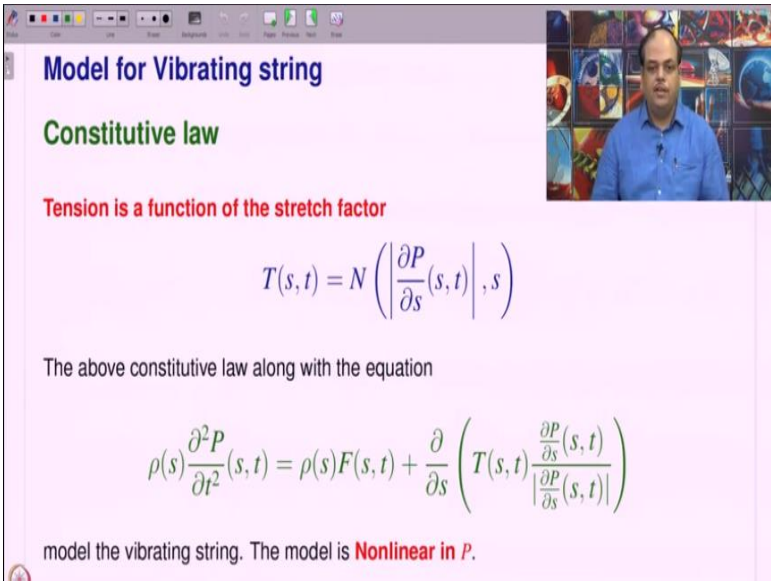

Tension is a function of the stretch factor this is an assumption so it is an function N of mod dau P by dau s s, t, s and the above constitutive law along with this differential equal which we got that they model a vibrating string. Of course it is a non-linear equation in P.

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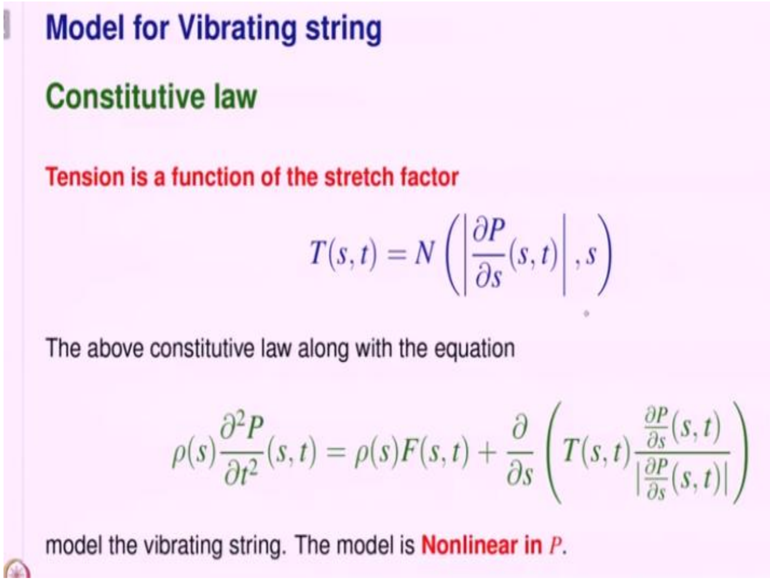

So the model for a vibrating string turns out to be highly non-linear so we would like to understand using simpler equations where we can actually perhaps solve the problem. So one looks for model which approximates this model of course you need to assume more conditions on the nature of vibrations then we can get more simplified model.

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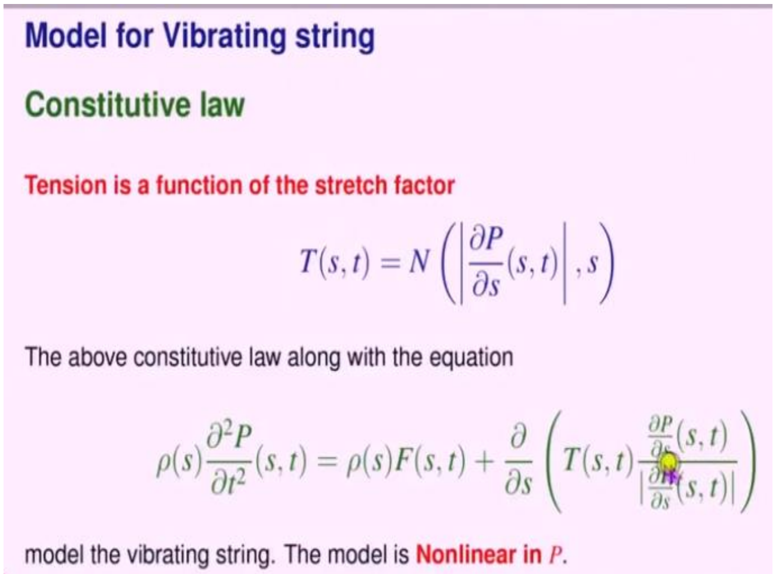

And then one as to see whether this model approximates the original model well these are the approximation go in practice. So in order to simplify the model we make a new assumption on the vibrations what is that assumption? Vibrations are small how do we model this vibrations are small? The X s, t Y s, t Z s, t at time t = 0 X was s and Y and Z are 0 so we assume that it is a small chain from there.

So s + epsilon x of s, t this epsilon y of s, t epsilon z of s, t and tension also we model like this T naught background tension T 0 + a small variation of that epsilon T 1 s, t and F s, t is also a small force epsilon f s, t. Now we want to go and substitute in the equation that we have obtained here. Therefore we need to compute what dau P by dau s is.

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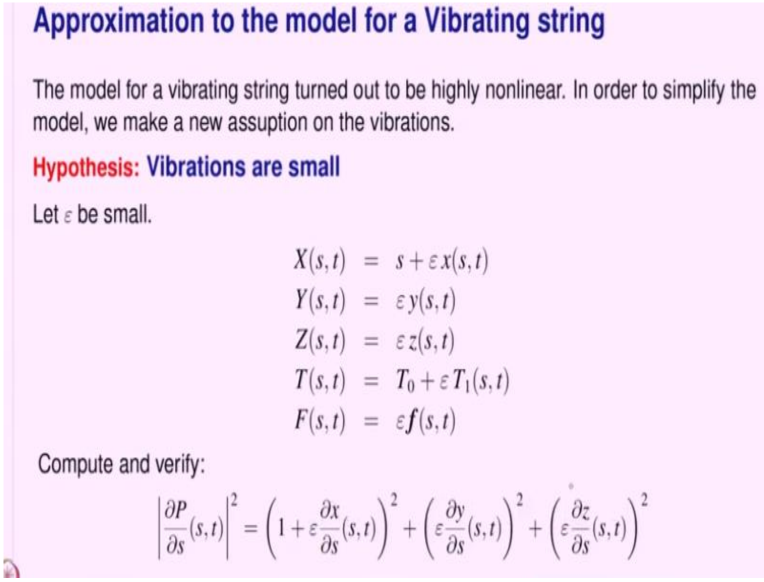

So we will do those competition now so dau P by dau s square is derivative of this with respect to s is 1 + epsilon dau x by dau s that is what you see and similarly we get the other 2 terms.

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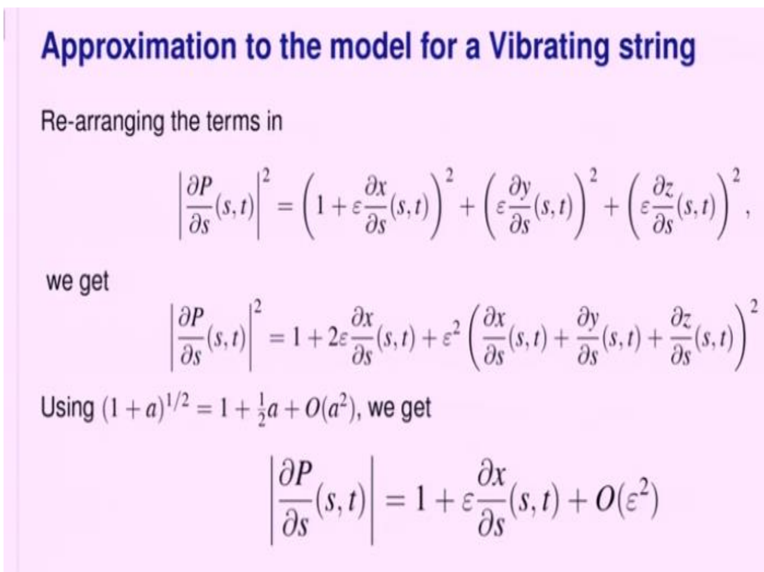

So rearranging the terms we get this expression now here we use this approximation 1 + a square root is 1 + half a + order of a square. So this is 1 + this entire thing is a have a square here so I want to compute dau p by dau s. Therefore I need to square root on the both sides of this equation then I have square root of 1 + something square root is given by 1 + a by 2 plus order of a square. So that will give us this expression please pause for a while and do all this computation by yourself make sure that computations are correct so please do by yourself.

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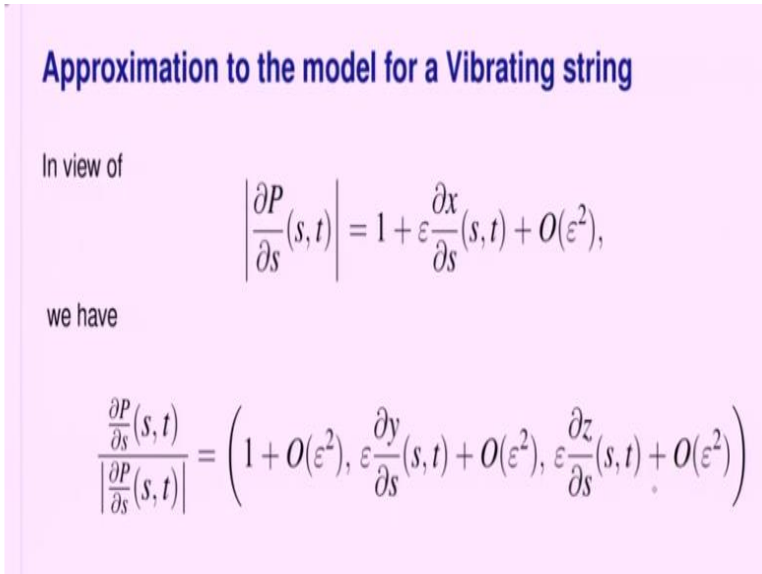

Now we move of this expression we need to compute to this right this is what appears in the equation dau P by dau s by modulus of that. So that will turn out to be a quantity of this type.

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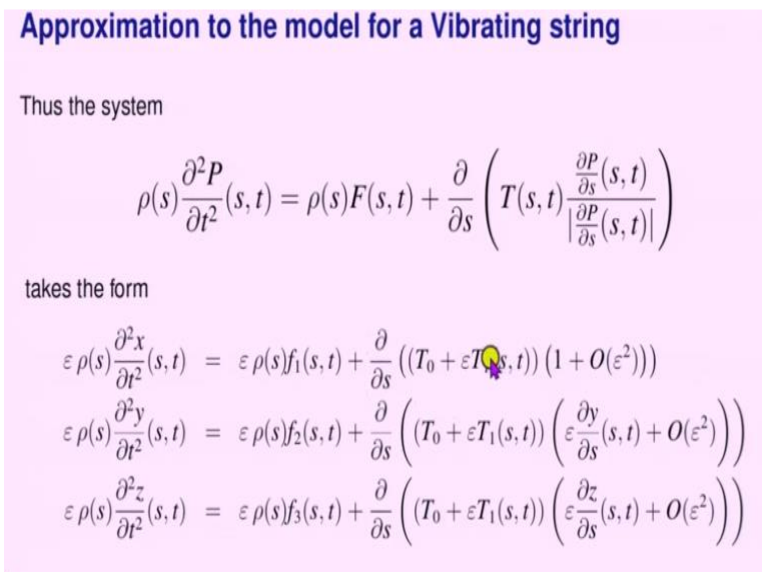

Now so this system now becomes this system so we have epsilon here there are epsilons square terms there are terms without epsilon. So what we do now is equate the coefficients of epsilon from LHS and RHS. Of course LHS coefficient of epsilon is simply these quantities rho dau 2 x by dau t square rho dau 2 y by dau t square, rho dau 2 z by dau t square. And here the first term will be rho f1 let us look at the first equation.

So coefficient of epsilon I mean that one which multiplies epsilon is this and here it is this. Now here we have to find out with t naught no here we get anything with epsilon you get 1. So you dau by dau s T1 you get.

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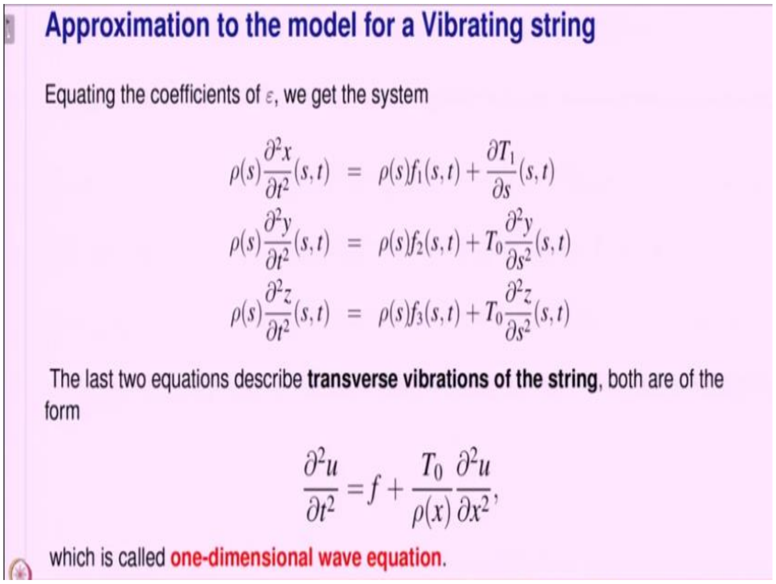

So this is the first equation similar you can see that you get the other 2 equations. Now if you observe this the second and third equation they look similar they describe the transverse vibration of the string whereas the first one describes the longitudinal vibration of the string. So both look like this dau 2 u by dau t square you can put u = y or z you get these 2 equations equal to f + T naught by rho dau 2 u by dau x square. So this is called the, 1 dimensional wave equations.

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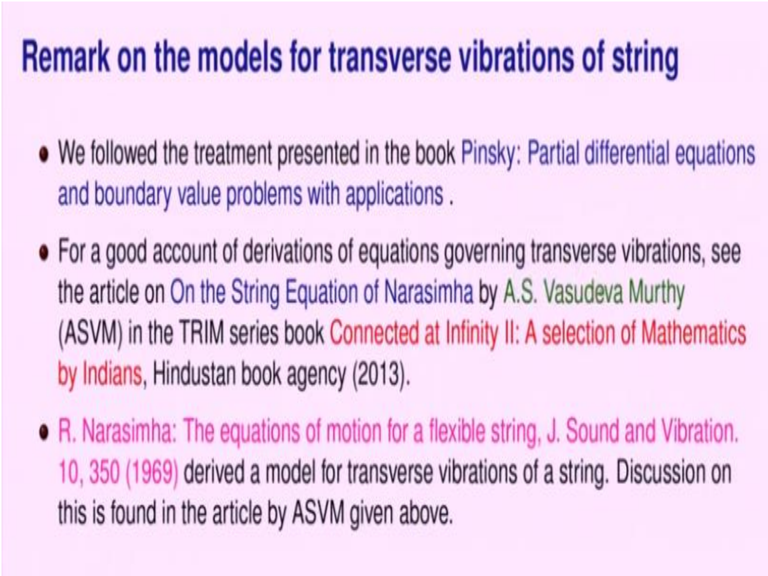

So small remark on models for transverse vibration of string so, we followed the treatment presented in the book by Pinsky on partial differential equations and boundary value column with applications this is the title of the book of Pinsky. For a good account of derivations of equations governing transverse; vibrations of that matter vibration of strings. See the article on the string equation by of Narasimha by A. S, Vasudeva Murthy in this book connected at infinity a selection of mathematics by Indians.

It is TRIM series book published by Hindustan book agency that is where we describe an equation of a Narasimha and this is actually the reference to the original paper of Narasimha. Derived a model for transverse vibrations of a string the different between a various derivations of transverse vibrations of a string and these which I have quoted here is that. While deriving transverse vibrations people assume that there are no longitudinal vibration which affect on that.

Whereas these models take into that account and then further simplify therefore you see you start with a very correct physical assumptions derive a model and then you know what assumptions you are making to get the equation that you get. How the equations get simplified under more assumptions? So that is the remark on this and discussion on this equation is what is there in this particle by ASVM.

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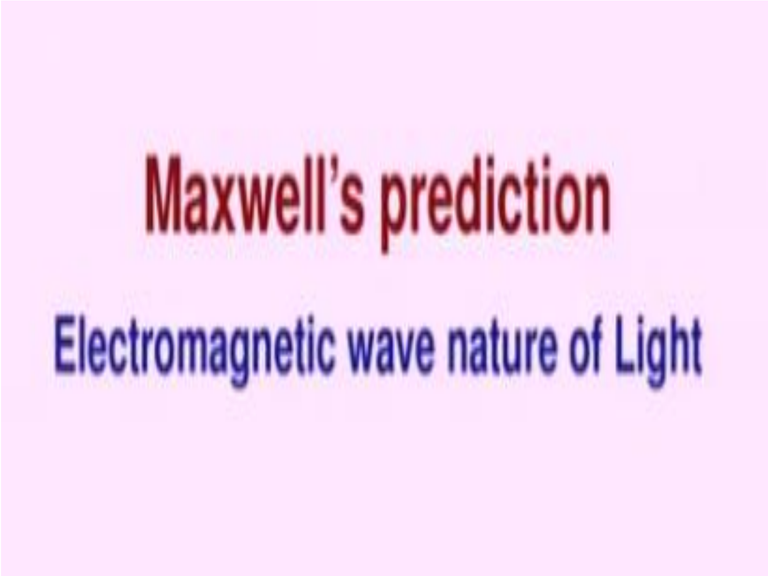

Now let us turn our attention to Maxwell's prediction that electromagnetic wave nature of light.

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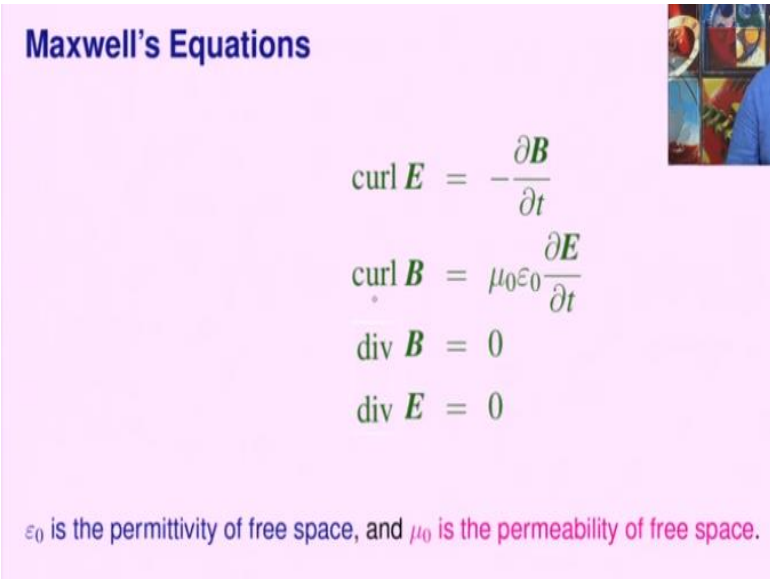

So these are what are called Maxwell's equations I will not read out is electric field is magnetic field u naught and epsilon naught are permeability and permittivity irrespectively. So divergence of E = 0 divergence of B = 0 and there are equations for curl. So this is a first order system of equations.

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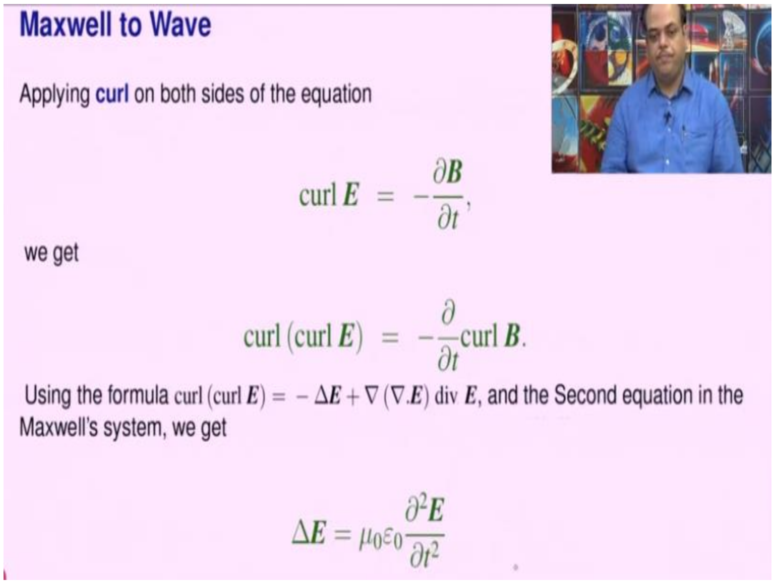

Now applying curl to both sides of the first equation so we get curl-curl E = - dau by dau t of curl B. Now we have a formula on vector calculus curl-curl is nothing but minus Laplacian this is dau 2 E by dau x square + dau t by dau y square + dau t by da z square those are these is Laplacian. And this is, gradient of divergence into divergence and the second equation in the Maxwell system which is for curl B that is a second equation use this equations and we end up this gets simplified to this equation.

Laplacian E = mu naught epsilon naught dau 2 E by dau t square note this is system of 3 equations even E2, E3 of course they are all same.

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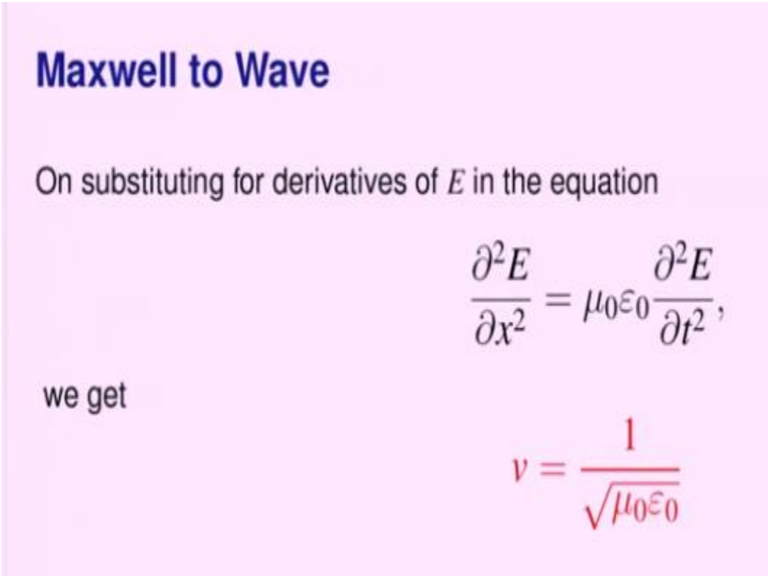

Now when d = 1 let us try these equations as solution of this type so now it is not dau Laplacian E this simply dau 2 E by dau x square. So we try a solution of this form substitute in this equation here wave velocity is mu and a wave length is lambda. Now just compute what is dau 2 E by dau t square this what you get sin becomes Cos, Cos becomes minus sin that is why you get minus every time we differentiate with respect to t you pick up minus mu by 2 by lambda so square.

Similarly dau 2 E by dau square you can compute go back to substitute in this equation you get that velocity is 1 by root mu naught epsilon naught.

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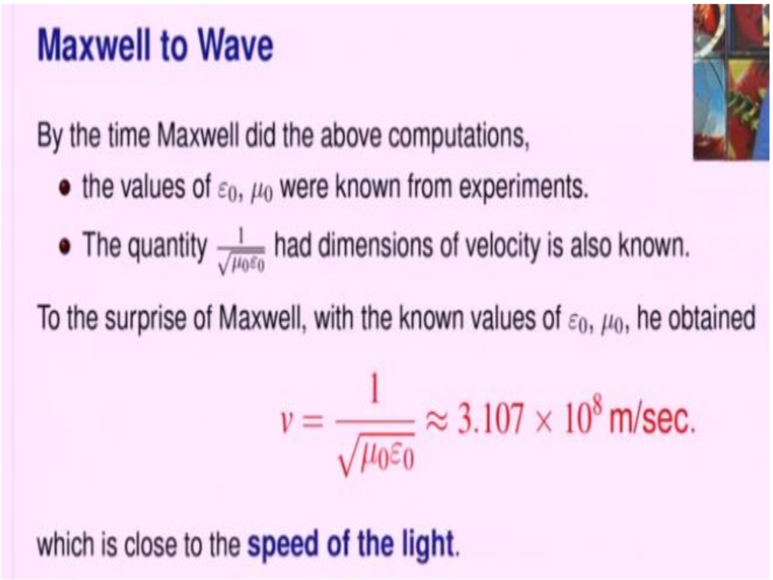

Now by the time Maxwell did the above computations the values of epsilon 0 mu 0 are known from experiments and this quantity 1 by root mu naught epsilon naught it is known to have the dimensions of velocity that is also known. But thanks to the computations of Maxwell with these known values of epsilon naught mu naught he obtained that this velocity is approximately equal to 3.107 into 10 to the power 8 meters per second which is close to the speed of light.

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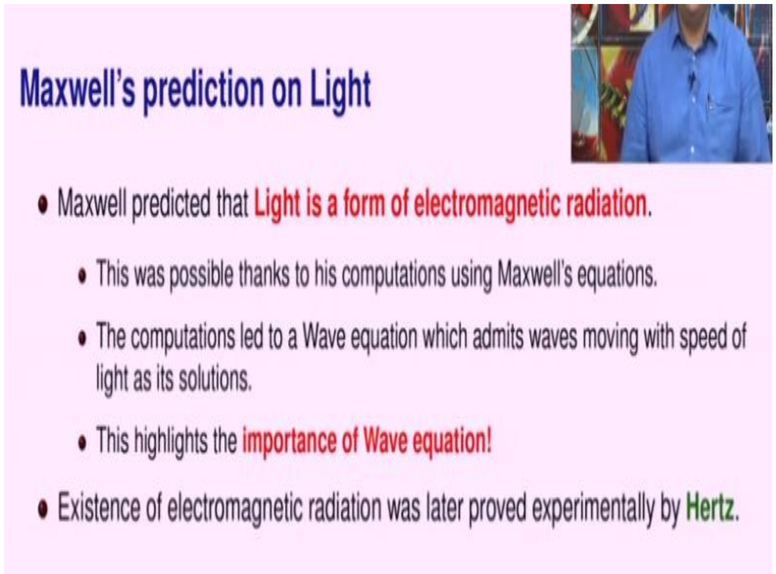

So Maxwell predicted that light is a form of electromagnetic radiation of course this was possible thanks to its computation using Maxwell's equations. The computations led to wave equation which admits wave moving with speed of light as its solutions. This highlights the importance of wave equation of course existence of electromagnetic radiation was later proved experimentally by Hertz.

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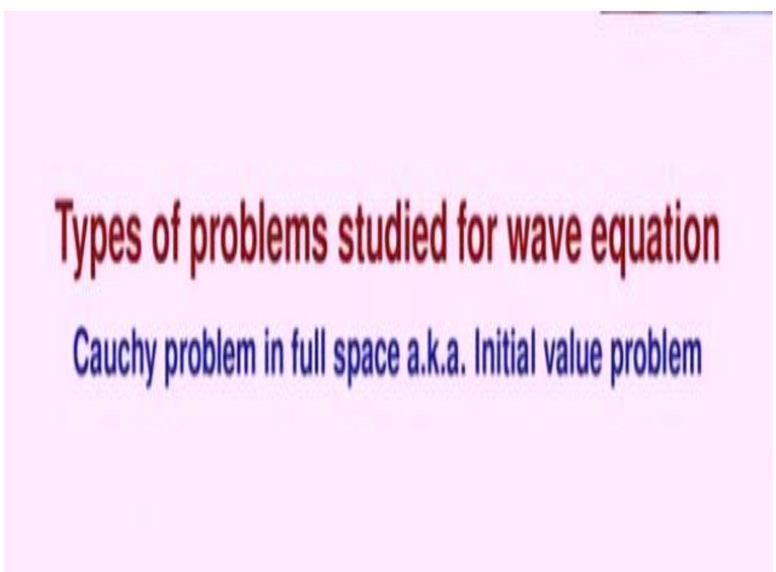

Now let us discuss briefly the types of problems studied for wave equation the first problem is Cauchy problem in full space it is also known as initial value problem.

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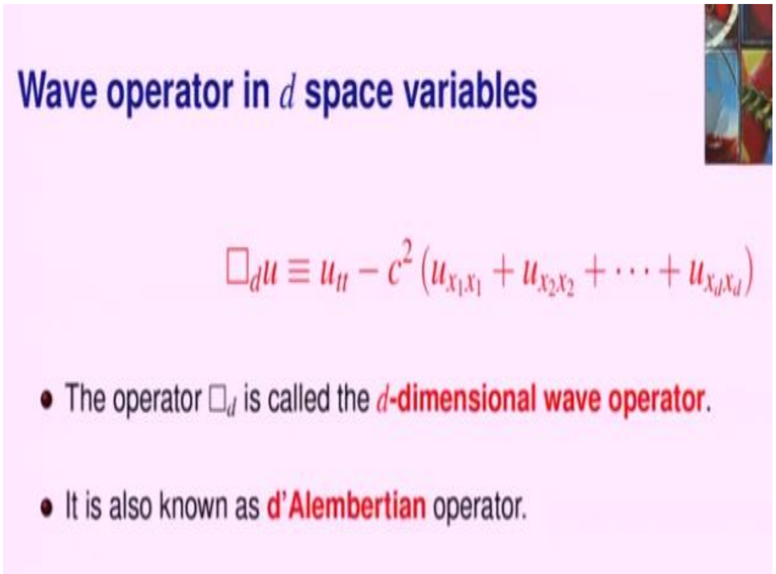

So wave operators in d space variables so let us introduce wave equations of course we have derived wave equations in one space dimension using strings. But you can write wave operator in d space where it is which is nothing but U t t – C square into Laplacian u is precisely this dau 2 u by dau x 1 square + dau 2 u by dau x2 square up to dau 2 u by dau x d square. This is the wave operator these called the d dimensional wave operator when we say refer to the word dimension what we mean here is the space dimension x1 up to xd.

So the equation for the string was a 1 dimensional wave operator so this square d this is the notation used square d stands for this operator. It is also known as d'Alebertian operator.

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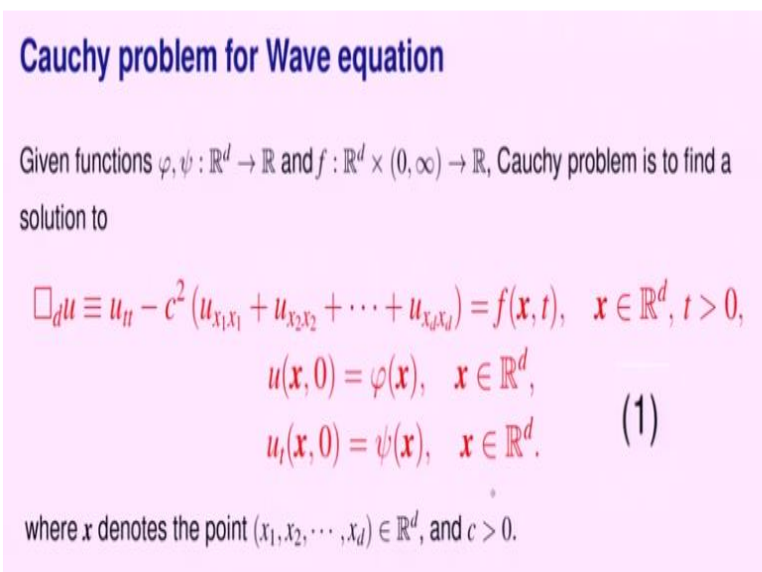

So what is Cauchy problem for wave equation so given functions phi psi and f suitably? Cauchy problem is to find a solution to find a solution to this equation what is this equation? This is a wave operator equal to f and you are given u x 0 to be phi and u t of x 0 to be psi. So f phi psi are supplied find a solution u satisfying all the 3 equations the bold phase x stands for x1, x2, xd which is an Rd and c is positive number.

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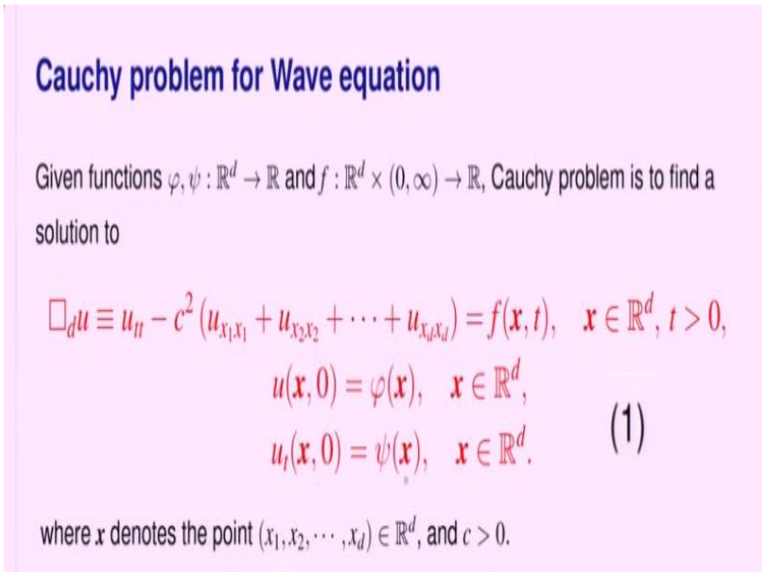

So we need to define what is the meaning of the solution? Solution how you should be a function you should be as many times differentiable as you see the derivatives here 2 derivatives so I want to the function to be 2 times continuously differentiable and I want this condition to be satisfied these 2 conditions and that is it. But if you see here this equation is valid in Rd Cos 0 infinity so, the function must be defined Rd cross open 0 infinity.

But here I am asking the value at 0 should make sense that means you should be continuous on Rd cross closed 0, infinity and similarly u t must be continuous Rd cross closed 0 comma infinity. So that this makes sense and then we can ask it should be equal to the given sin that is all.

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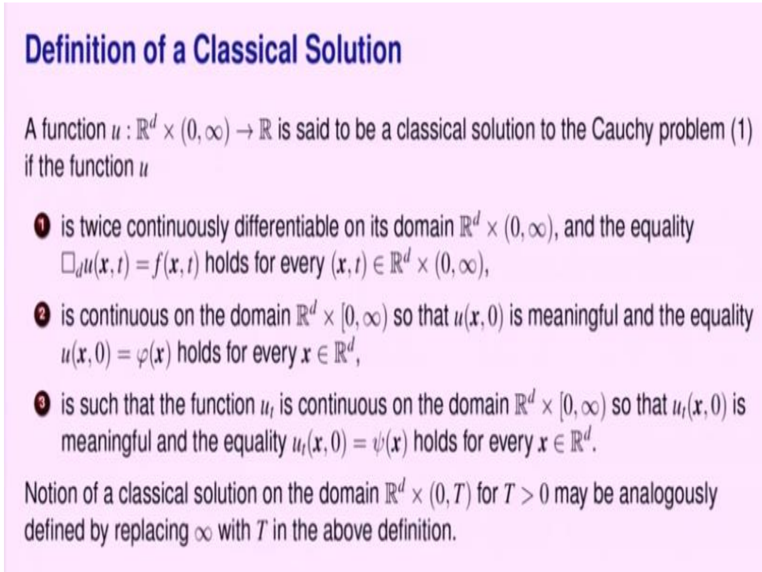

So a function is said to be a classical solution to the Cauchy problem if the function is twice continuously differentiable on its domain. And the equality holds at every points in Rd + 0 infinity and it is continuous on this domain note here I have included a closed 0. Whereas here we have only so both are different conditions. This does not imply this continuity on open said does not mean it is continuous up to it is closure.

So that u x, 0 is meaningful and then the equality is equal to phi x holds for every x in Rd. Similarly the u t should be continuous on this domain so that u t of any x comma 0 makes sense and it is equal to psi x and that holds. We can modify by replacing infinity with the T what you mean by classical solution on domain Rd cross 0, T.

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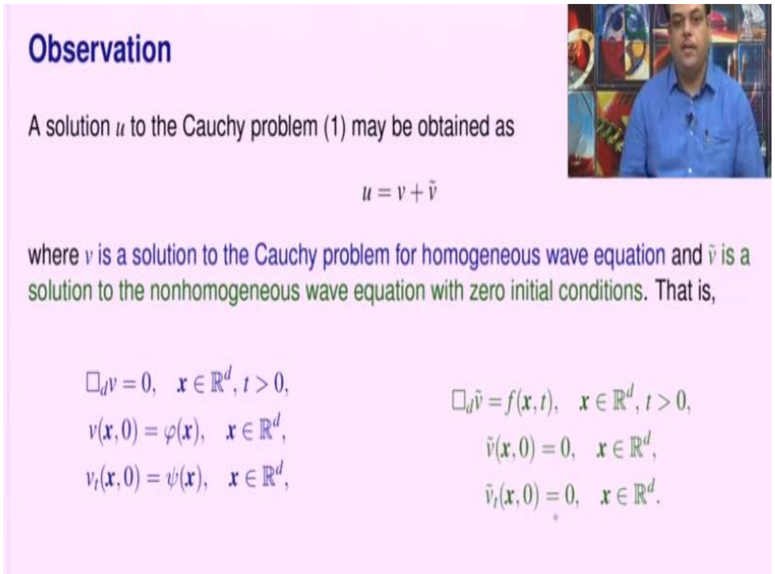

Observation is that the d'Alembertian operator is a linear operator that means you apply a square d to u + v you get square d u + square d v. Thus a solution u to the Cauchy problem were stated earlier on the previous lights that may be obtained as sum of 2 things v + v tilde what are the properties? v solves the Cauchy problem for homogenous wave equation that means the right hand side f is 0 but phi and psi are there.

And v tilde solves non-homogenous equation that means f is here but 0 initial conditions that mean phi and psi are 0. So that is precisely this v solves this problem notice here the right hand side in equation is switched off it is becomes 0 whereas here f is represent by the initial condition is switched off the Cauchy problem. If we add v + v tilde because the linearity square d + v + v tilde will be f because it will be 0 + f that will be f. And x 0 will be phi + 0 it will be phi and v t will be v + v tilde of t will be psi + 0 which is psi.

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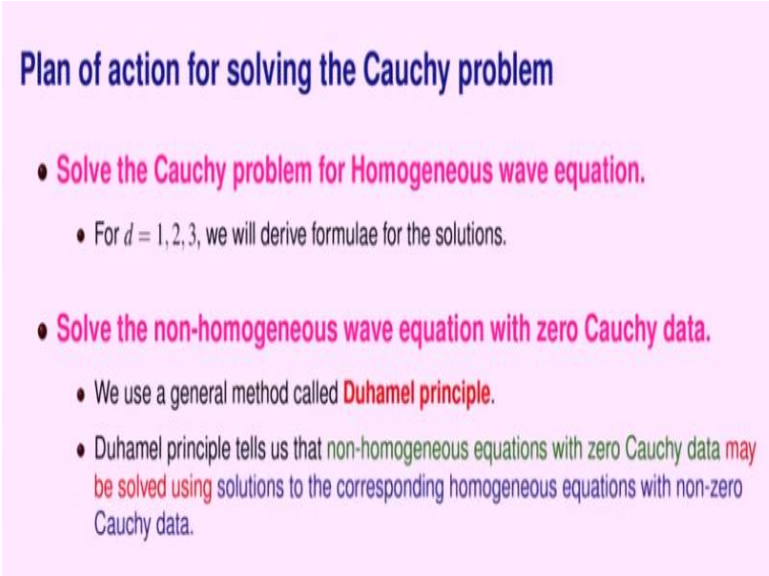

So therefore plan of action for solving the Cauchy problem is solve it for a homogenous wave equation first. That is what we will d = 1, 2, 3 we will derive actually the formulae which are known by various names and we solve the non-homogenous wave equation with 0 Cauchy data. For that we use the general method called Duhamel principle. That Duhamel principle tells us that non-homogenous equations can be solved using solutions of homogenous equations.

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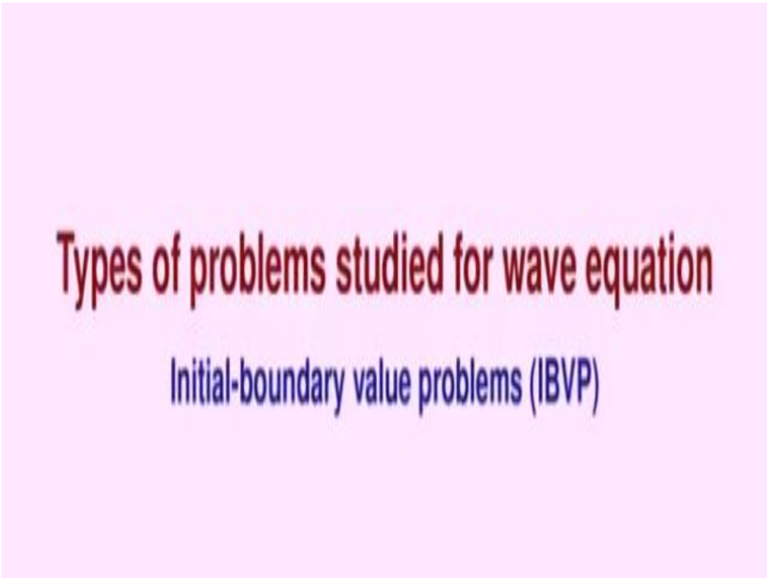

And the second type of problem is initial boundary value problems.

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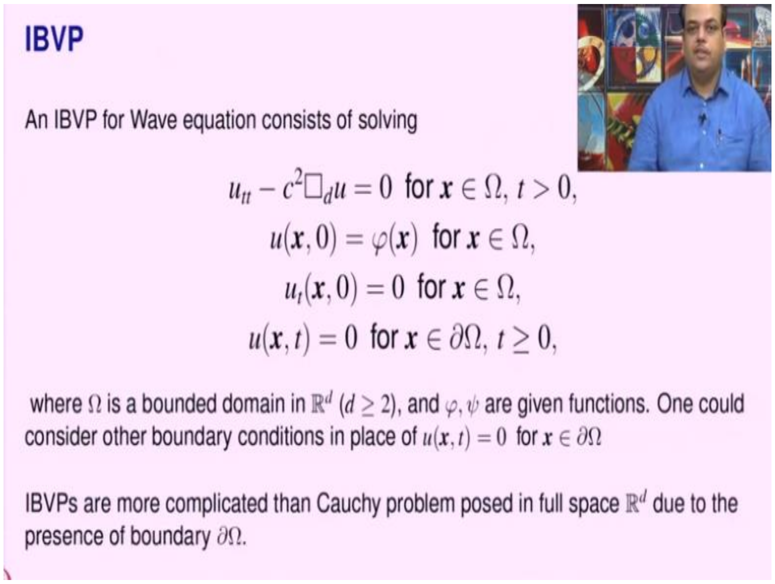

This is applicable when the equation is posed not x in Rd but x in omega is a bounded domain Rd you want to equations to be satisfied on omega that there could be a right hand side f here no problem there could be a right hand side f here. Similarly there could be u t, x, 0 = psi x this is just an example of IBVP so these are the initial condition the first 2 conditions here are initial conditions equation initial condition and this is a boundary condition. u = 0 on the boundary of omega for all times.

So these are the initial boundary value problems and we will also solve initial boundary value problem when omega is a subset of r 1 r and interval. Of course one could consider more general boundary conditions than u for example one could describe the normal derivative of u on the boundary. Or you can prescribe a combination of u and (()) (30:17) derivative is nothing but the normal derivative dau u by dau n grad u dot n on the boundary.

So IBVP's are more complicated than Cauchy problems which are posed on full space due to the presence of the boundary omega.

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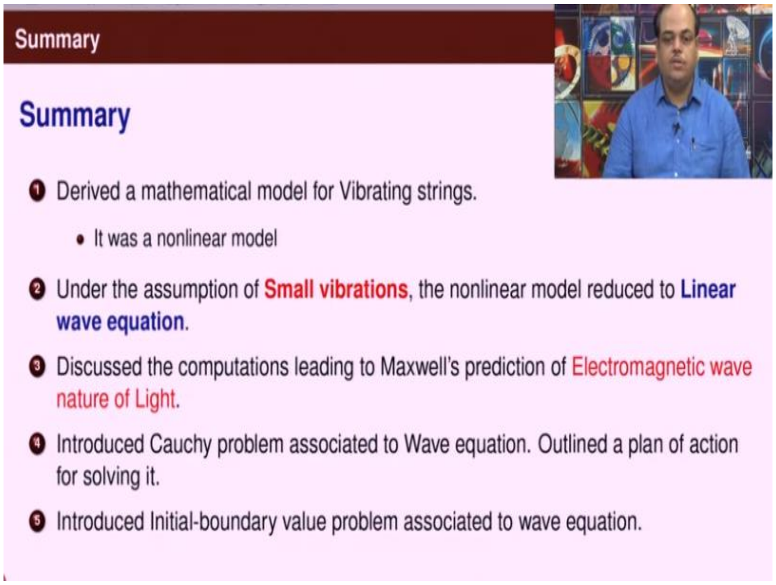

So let us summarize what we did? We derived a mathematical model for vibrating strings it was a non-linear model. Under the assumption of small vibrations the non-linear model reduced to linear wave equation and we discussed the computation which led Maxwell to predict electromagnetic wave nature of the light. And we introduced a Cauchy problem associated to wave equation outline a planned of action for solving it. And we introduce initial boundary value problem associated to wave equation thank you.