Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Science, Bombay

Lecture – 25
Second Order Partial Differential Equations Canonical Forms for Constant Coefficient PDEs

Welcome to this lecture. In this lecture, we are going to discuss about canonical forms for constant coefficient partial differential equations of course in more than two input variables.

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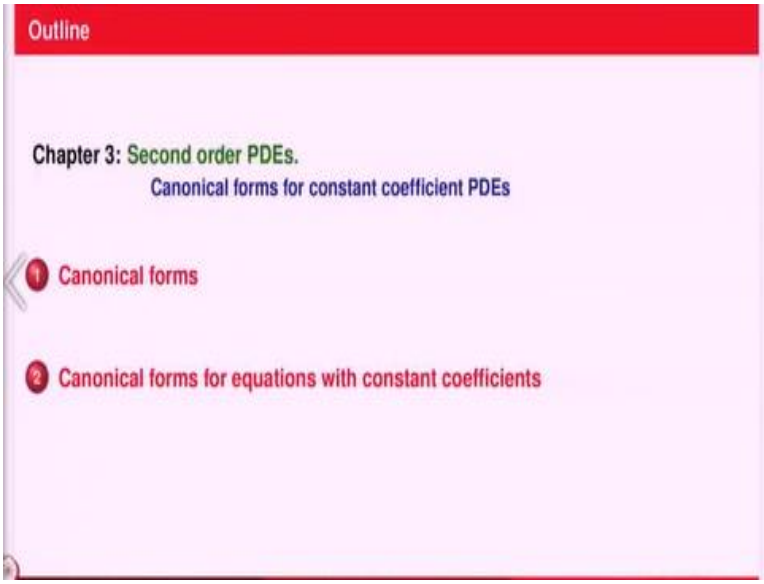

The outline for today's lecture is we start discussing about canonical forms and then we actually do canonical forms for equations with constant coefficients.

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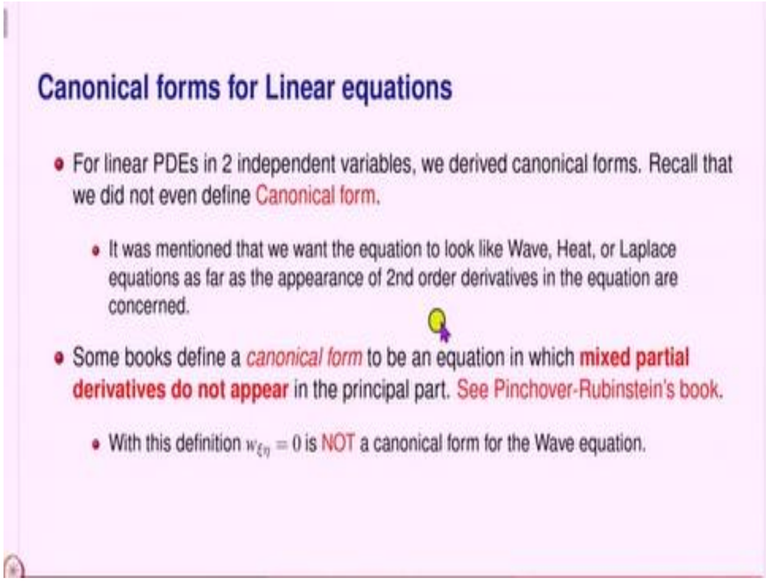

Canonical forms for linear equations. For linear PDEs into independent variables we derived canonical forms. Recall that we did not even define what does the word canonical form stands for. It was mentioned that we want the equation to look like wave equation, heat equation or Laplace equation as far as the appearance of this second order partial derivatives are concerned. Some books define what is the canonical form.

They say that in equation is in canonical form if mixed partial derivatives do not appear, that is a definition in the principal part. Because principal part secondary linear PDE in the principal part there will be only secondary partial derivatives. See Pinchover Rubinstein's book PDEs, they hint this kind of definition. But if you adopt that definition then w psi eta = 0 which is a canonical form for a wave equation will no longer will canonical form.

If you adopt this definition, saying mixer partial derivatives do not appear in the principal part and somehow, we do not like this. Why is that? Because this equation w psi eta = 0 is far more easier to find solutions than let us say u tt - t square u xx = 0, u tt - t squared x x = 0 is a wave equation. And there are no mixed partial derivatives. How do you solve it? I do not know. But when you convert that into w psi eta = 0 with psi and eta as the coordinate transformations.

Which are coming by solving characteristic equation things are nice. So, it is not a canonical form, if you follow this definition as in this book.

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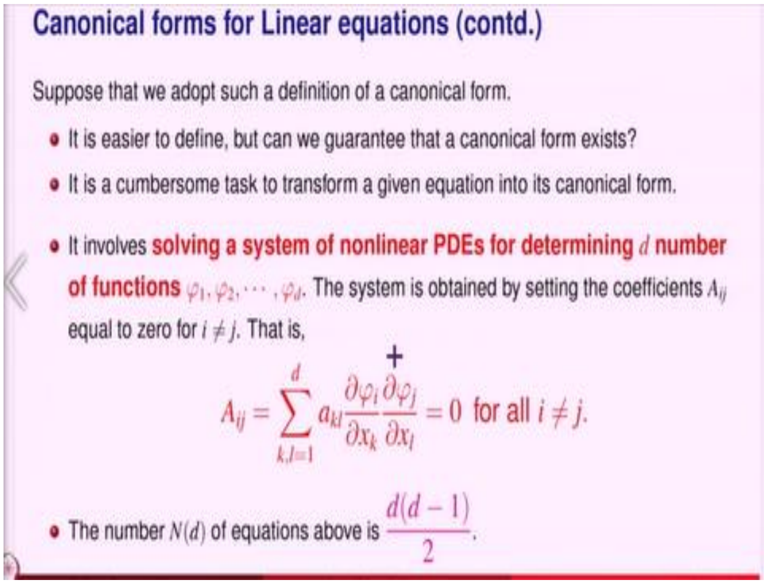

Now suppose we adopt such a definition. Let us say that okay does not matter. My definition of canonical form is that equation in which no mix partial derivatives appears. Of course, very easy to define anything, it is very easy to define, no problem. But can we guarantee that the canonical form exists is an actual question. Otherwise, there is no use of this definition. So, can we guarantee that the canonical form exists?

It is a cumbersome task to transform a given equation into its canonical form. Remember, what do you need to do? We have to make sure that in a new change of coordinate system which we are defined, no mix partial data appears. So, that is a system of nonlinear PDEs for determining phi 1, phi 2, phi d which go on to define the coordinate chain transformation. And the system to be satisfied is this summation k, l = 1 to d a k l dou phi by dou x k dou phi j by dou x l = 0.

And this should happen whenever i is different from j. Because A ij is the coefficient of doh to w by dou eta i dou eta j, you do not want mix partial derivatives to appear. Therefore, when i is not equal to j, you want A ij to be 0. How many equations are there? There are d into d - 1 by 2.

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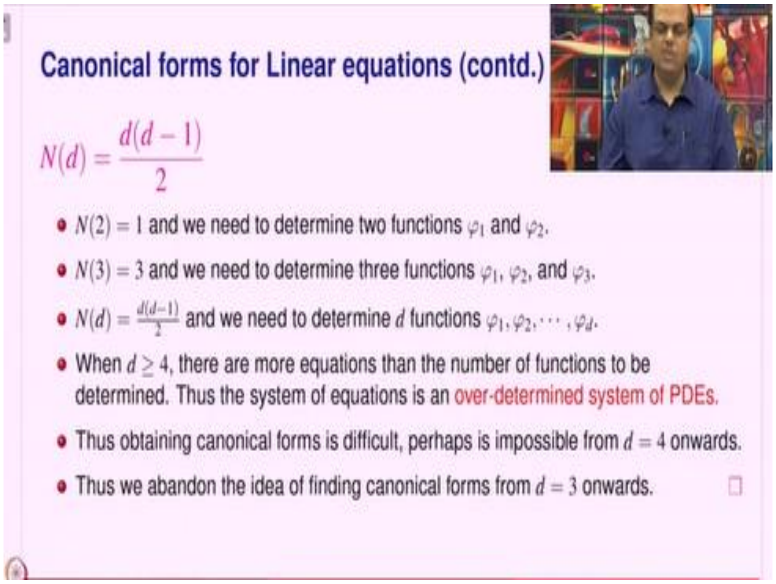

So, when d = 2, N of 2 is 1 we need to determine two functions phi 1 and phi 2, fine no problem. Two functions one condition looks good. N of 3 is 3. Three equations, three unknowns nonlinear equations that is anyway there. But at least three relations three conditions and three functions maybe look reasonable. But d into d - 1 by 2 equations to determine d functions. This number is much more than d if d is bigger.

When d is greater than or equal to 4, number of equations is more number of constraints to find the function phi 1, phi 2 to phi d is bigger than the more equations then the number of functions to be determined. So, therefore, the system is what is called over determined system of PDEs. More restrictions than what you need to find, a number of things that you need to find. In such cases, the natural thing to believe is that perhaps there are no solutions unless a magic happens you monitor solutions.

Even if our solutions finding them is not easier, because if d is bigger N of d is also big number. So, obtaining canonical forms is difficult, perhaps impossible, we do not know from the equal 4 onwards. Therefore, we abandon this idea of finding canonical forms, if the number of independent variables is three or more, we do not do that.

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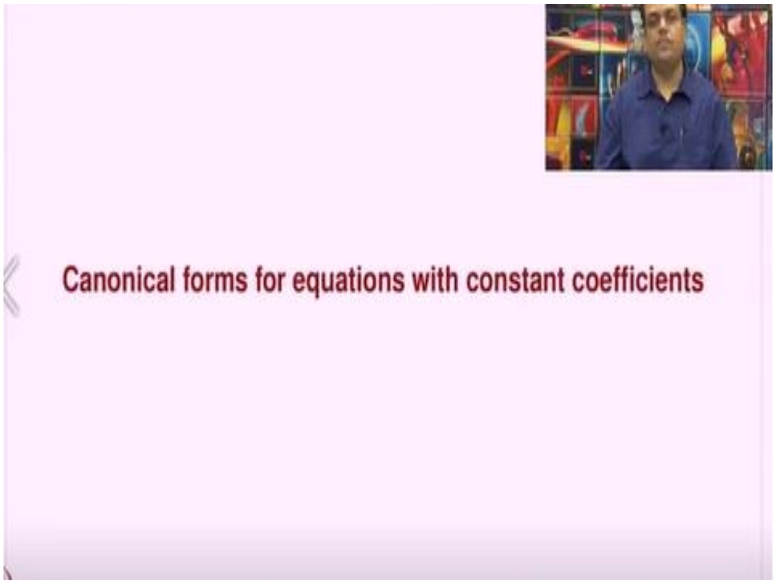

Now why are we discussing for constant coefficient case. When the equation is with constant coefficients a miracle happens things are easy, you can find.

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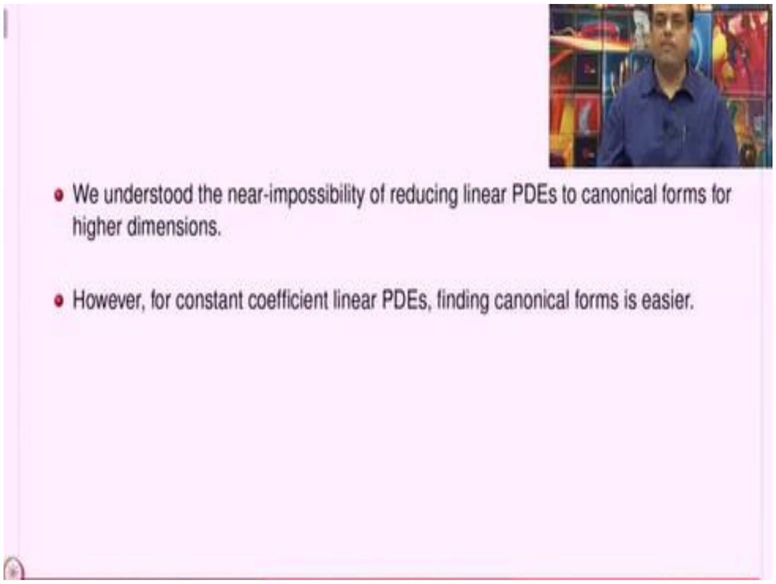

So, we understood the near impossibility of reducing linear PDE to canonical forms in higher dimensions. However, for constant coefficient PDEs, please finding canonical forms is easier.

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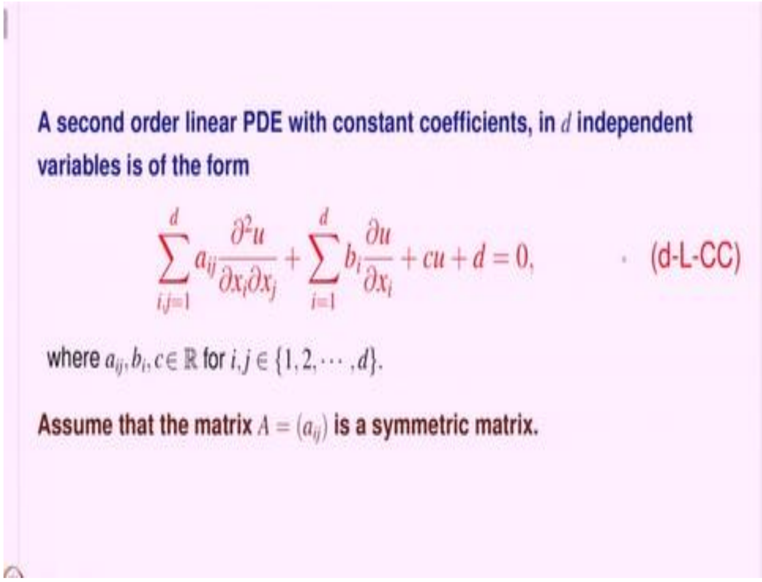

A second order linear PDE with constant coefficients in d independent variables is of this form. Exactly as earlier d-L but now it is d-L-CC therefore, the coefficients a ij, bi, c and d are constants. In fact, this d may not be constant does not matter because everything depends only on this. So, these are numbers that are stated anywhere we are assuming constant coefficients now assume everything is constant d mean and we constantly d can be a function of x.

It can be on the right-hand side does not matter; it does not change the discussion. So, assume that the matrix is symmetric as before.

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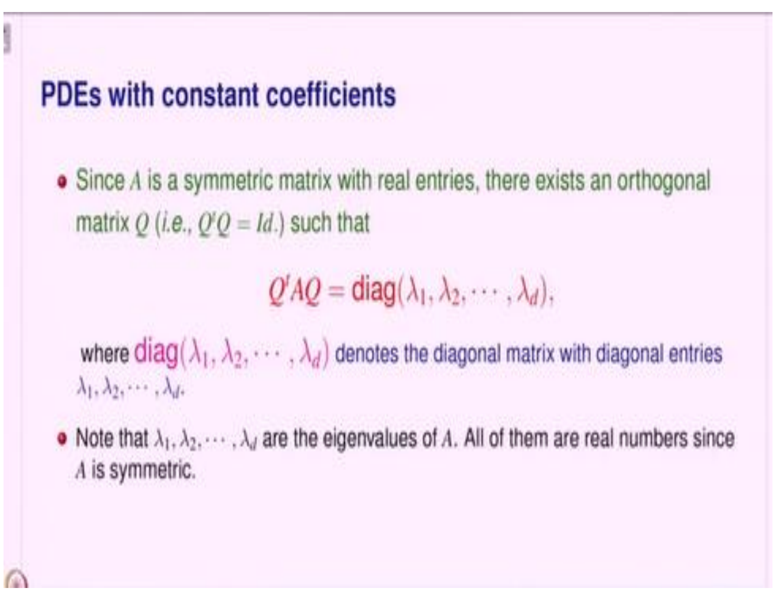

Now, since A is the symmetric matrix with real entries, there is an orthogonal matrix Q. What is the orthogonal matrix? Q transpose Q is identity. This transfer stands for transpose. With what property? Q transpose A Q is a diagonal matrix with entries lambda 1 lambda 2 lambda d. So, when this lambda 1 lambda d just stands for lambda 1, lambda 2 on the diagonal integer d by d matrix everywhere else it is 0 matrix that is what it stands for.

So, symmetric matrix is diagnosable with an orthogonal matrix as this transformation similarity transformation. How to get their Q? We know we have to put the columns of Q as eigen vectors of the matrix A that is from linear algebra. So, lambda 1, lambda 2 lambda d are eigen values. All of them are real numbers because the matrix is symmetric matrix.

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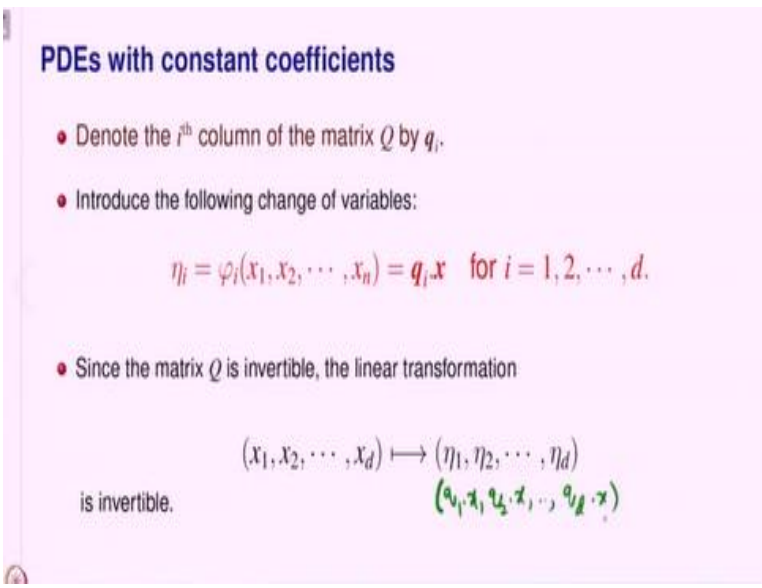

In general, for matrix with real entries, eigen value can be a complex number. But if the matrix is symmetric eigenvalues must be real numbers. Denote the if ith column of the matrix Q by q i. So, if this is ith column that means, it will be q 1i, q 2 i, etcetera q d i, this is what is called q i, ith column matrix. Define phi i of x equal to q i dot x this is the dot product, dot is hidden inside this. It is q i it is a vector dot x q i dot x. So, we are defining eta 1, eta 2, eta d.

So, since the matrix Q is invertible this linear transformation is also invertible x 1, x 2, x d going to q i dot x, q 1 dot x, q 2 dot x up to q d dot x that is invertible. This map is invertible.

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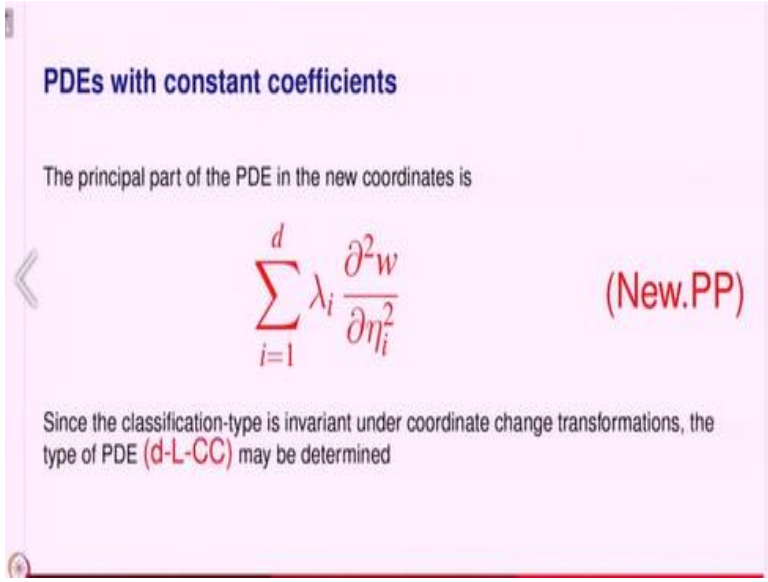

The principal part of the PDE in the new coordinates is this summation lambda i dou 2 w by dou eta i square. Let us call this as new principal part new PP i = 1 to d. Since the classification type is invariant under coordinate chain transformations that type of the PDE d-L-CC which is given to us maybe determined from using a New dot PP.

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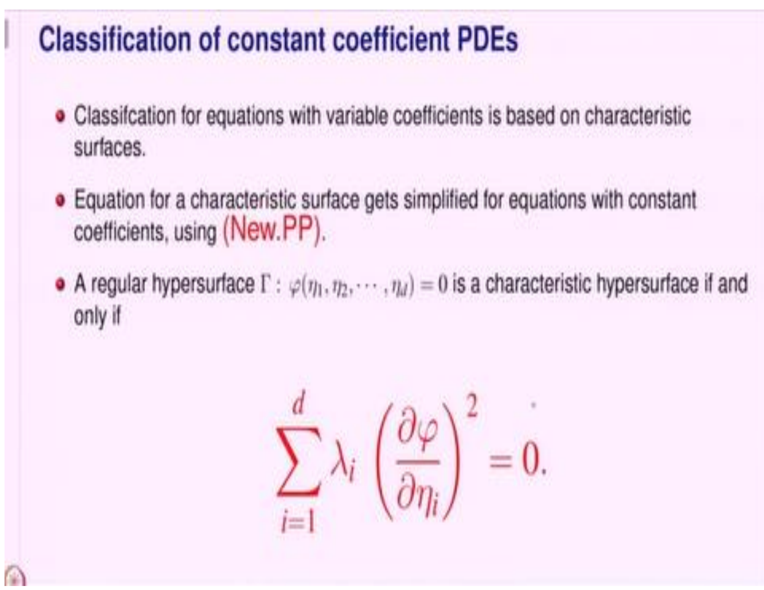

So, classification for equations is variable coefficients is based on characteristic surfaces. Now, we will see what characteristic surface become when the questions are constant things become much easy and much more easy if you are in this new coordinate system eta i which is defined on the previous slide of this. Eta i = q i x things are much simpler. So, equation for a characteristic surface gets simplified for equations with constant coefficient using new PP.

So, regular surface eta 1, eta 2, eta d = 0 is a characteristic surface if and only if this is 0. Once you know the gamma in terms of eta i is you can always write down in terms of x 1, x 2, x 3.

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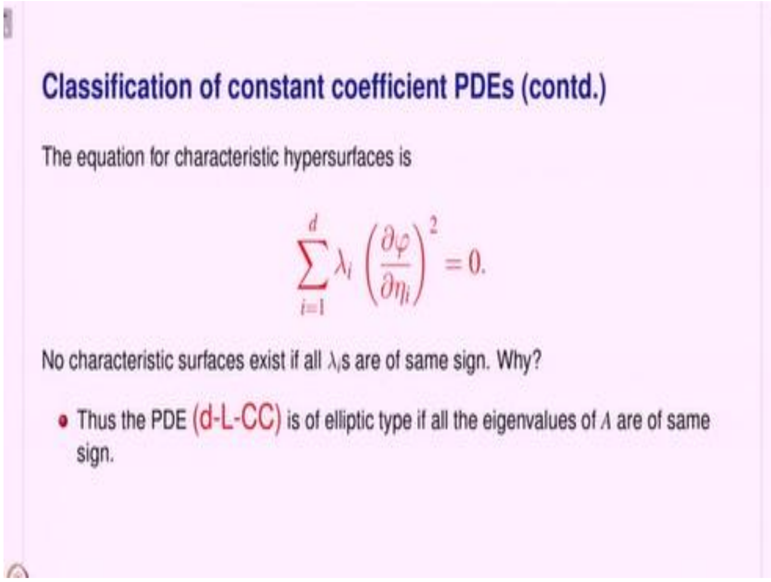

So, this is the equation for a characteristic hyper surface, phi = 0 is a characteristic hypersurface if and only if this equation is satisfied. Now, if you observe if all lambda i are the same sign non 0 on same sign that means psi already means non 0 then this will either be positive or negative. So, it will never be 0 which means there are no characteristic surfaces if all lambdas are the same sign. So, therefore the PDE d - L - CC is of elliptic type.

If all the eigenvalues what are lambda is their eigenvalues of A? If all eigenvalues of your same time then d - L- CC is of elliptical type.

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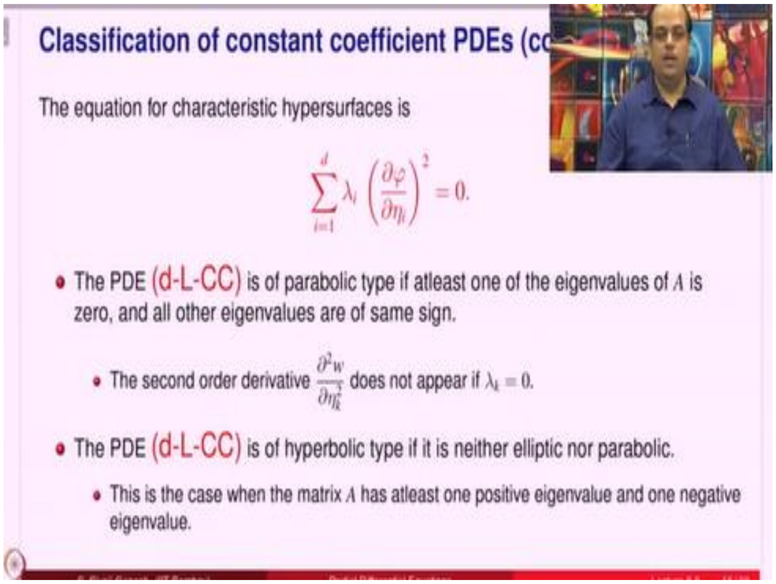

Now how about parabolic type? Well, it is parabolic type. Parabolic type what is the definition it says one of the independent variables should be missing in the principal part. At least one of eigenvalues is 0 if lambda k is 0 then in the equation dou 2 w by dou k squared does not appear. So, the PDE d – L - CC is a parabolic type if at least one of our eigenvalues have A is 0 on all other eigenvalues must be at the same sign.

So, the second order derivative dou 2 w by dou eta square it does not appear in new PP if lambda k is 0. Therefore, d - L- CC is a hyperbolic type. Now we are to say it is hyperbolic type, it is not elliptic, it is not parabolic that is a definition. Now, what does that translate to in terms of the eigenvalues of this. This is a case in the matrix A has at least one positive eigenvalue and one negative eigenvalue.

Not elliptic means, elliptic means what all eigenvalues of same psi. Not elliptic means, eigenvalues of different signs, not parabolic means no eigenvalue at least one eigenvalue is 0 that is a parabolic. So, what is what do you mean when not parabolic that also if you input you will get this case this will happen. When this will happen, then it is hyperbolic.

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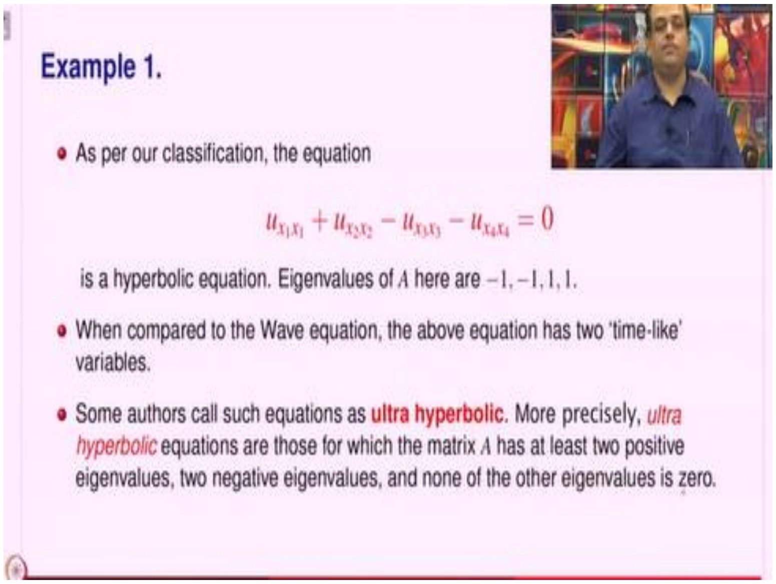

So, as per our classification this equation which is here is a hyperbolic equation. We are asked what are the eigenvalues are, they are minus 1, minus 1,1, 1 because the matrix A is 1, 1, minus 1, minus 1 diagonal matrix. So, when compared to the wave equation, the above equation has two time like variables. Because we have the wave equation, I can really say like 1, 1 minus 1 or minus 1, minus 1, 1 that is all.

But I here there are two eigenvalues which are negative two eigenvalues are positive. So, we may say it has to time like variables. So, some authors call such equations as ultra hyperbolic. So, more precisely the definition of ultra hyperbolic is the following. Some equation is called ultra hyperbolic for which the matrix A has at least two positive eigenvalues and two negative Eigen values and none of the other Eigen values is 0.

Our definition of hyperbolic will allow some eigenvalue to be 0. But ultra hyperbolic by definition, there are at least two positive eigenvalues, at least two negative Eigen values and none of the other eigenvalues is 0. That is what is called ultra hyperbolic. It is just for a definition sake.

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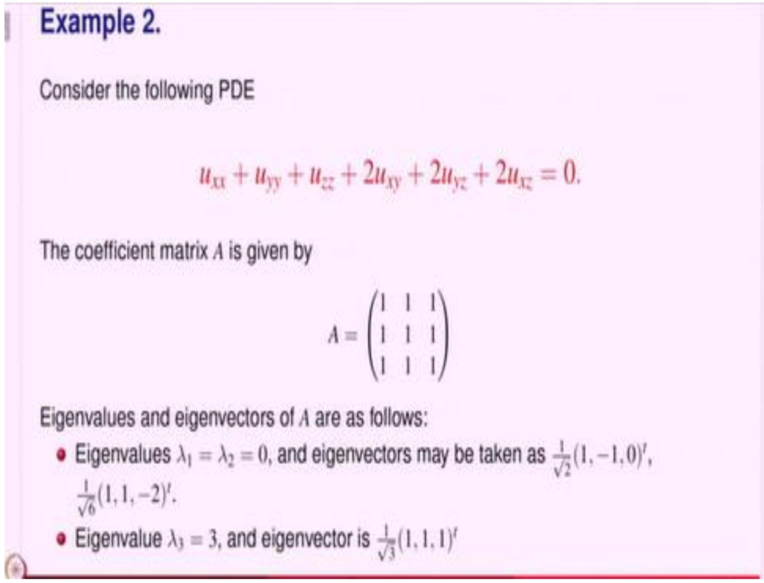

Now, let us look at an example and determine its type. u xx + u yy + u zz + 2u xy + 2u yz + 2u xz = 0. So, what is the matrix here? The diagonal entries are 111 and these are off diagonals, they are also 111. So, here is the coefficient matrix. And we have to ask what are eigenvalues? These are very well-known matrix by now, everybody knows it is eigenvalues. This matrix is clearly singular, rank is 1, the formality is 2 which means 0 is an eigenvalue of multiplicity 2.

And three is another eigenvalue which is a sum if you notice some of each row is actually the same constant. Therefore, if you look at 111 that even eigenvector with eigenvalue 3, so we know everything very clearly here. So, the eigenvalues are lambda 1, lambda 2 is 0. These are two Eigen values and we need to take eigenvectors which are if you look at carefully, I am putting a factor of 1 by root 2 and 1 by root 6 here to make the length to be 1.

It is of unit length; it is of unit length and they are orthogonal to each other. This dot product with this vector is 0, because Q is orthogonal matrix are to construct that is fine. And the other eigenvalues 3 and eigenvector is 111. I put 1 by root 3 to get that length is 1. And of course, this is orthogonal to both of them.

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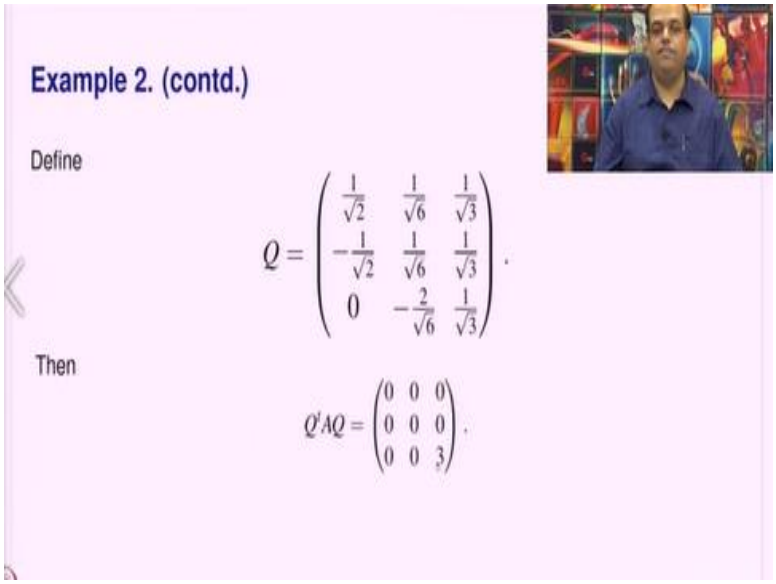

Now, I am going to consider Q like this, the first column is actually one of the Eigen vectors for 0 Eigen value, the second Eigen vector that we have written down for 0 Eigen value. This eigenvector for the eigenvalue 3. Now, Q transpose AQ will become the diagonal matrix 0 0 3.

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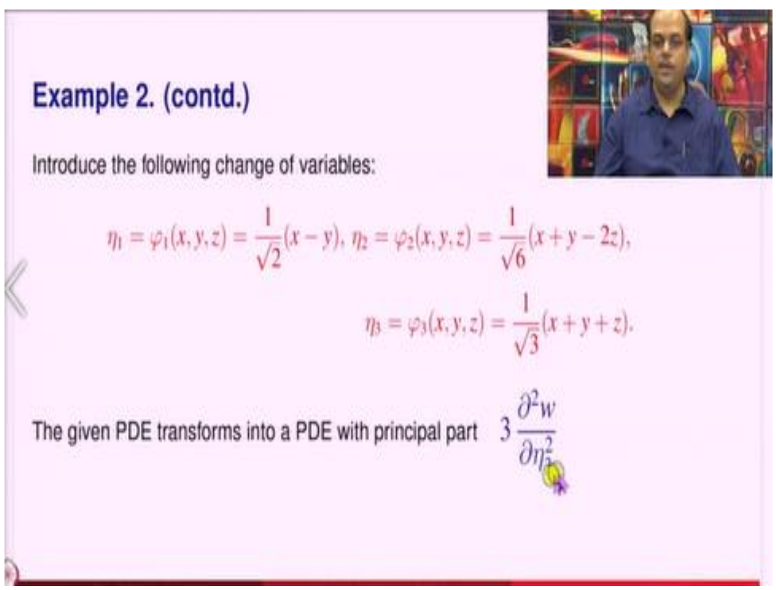

So, these are the change of variables phi 1 = x - y by root 2, phi 2 is x + y - 2 square root 6 and phi 3 is x + y plus that by root 3, then the PDE transforms to this. This is what exactly we saw in the theory. Take AQ as a Q transpose Q is the diag lambda 1, lambda 2, lambda d then introduce new variable eta equal to q i dot x exactly this keyword dot x. In this case I have returned on x, y, z so, q i dot x, y, z is precisely this. So, you will end up with lambda 3 and dou 2 w by dou eta 3 square.

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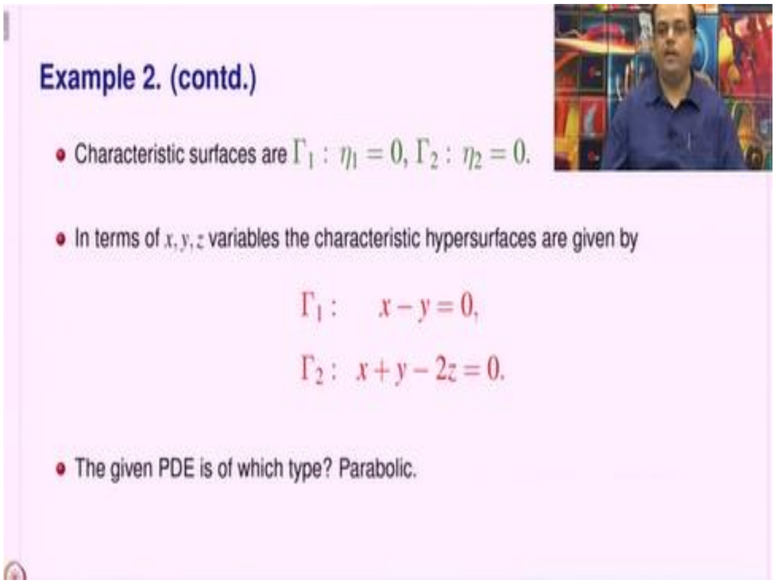

Characteristic surfaces are eta 1 = 0 and eta 2 = 0 because they do not appear in the new PP. What is gamma 1 0? It amounts in x y coordinates to x - y = 0 and this is x + y - 2 z is = 0. The given feed is of which type? Parabolic.

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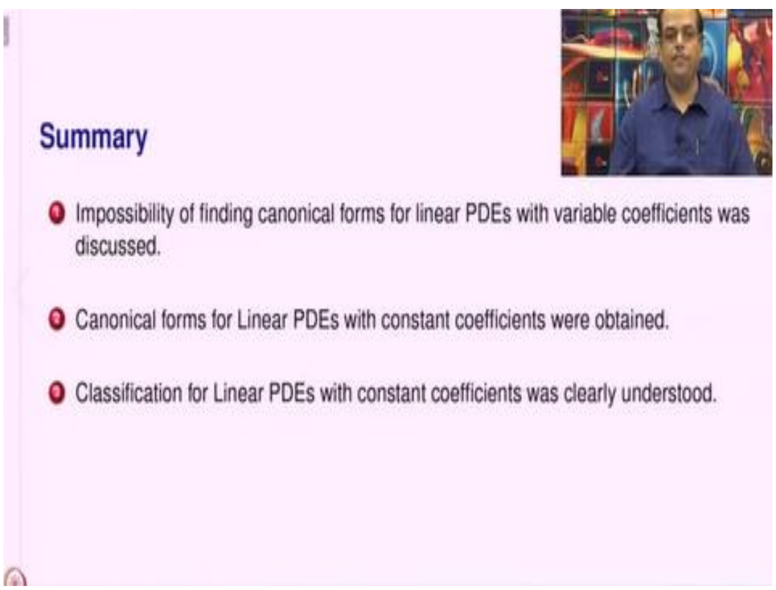

So, let us summarize what we did in this lecture. Impossibility of finding canonical forms for linear PDEs with variable coefficient was discussed impossibilities or near impossibility.

Canonical form for linear PDEs with constant coefficient were obtained. Classification for linear PDEs with constant coefficient was understood more clearly. Thank you.