Partial Differential Equations
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Lecture – 24
Second Order Partial Differential Equations Characteristics Surfaces

So, in our discussion on the second order partial differential equations, so for we have considered partial differential equations second order in two independent variables. In this lecture on the next we are going to discuss second order partial differential equations in more than two independent variables. The first concept that we are going to discuss is what is called characteristic surfaces. These are generalization of characteristic curves that we have seen in the discussion of second order equations in two independent variables.

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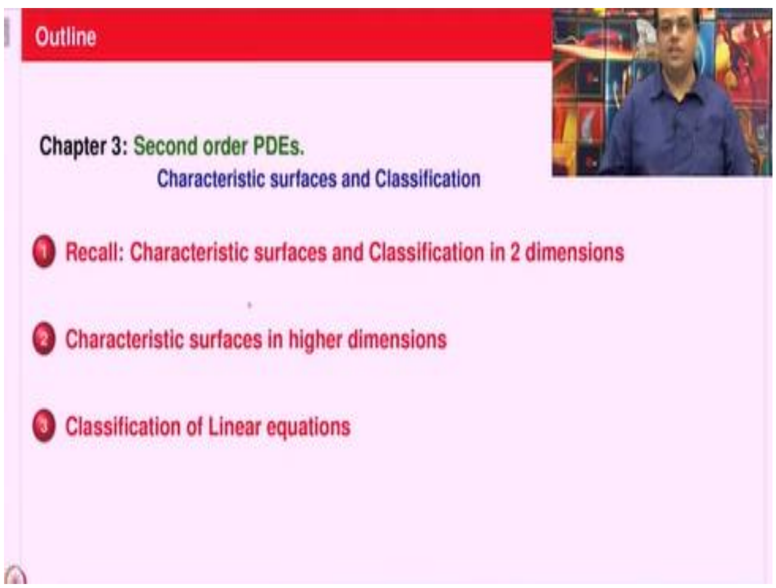

The outline for today's lecture is, we first recall the characteristic surfaces and classification which was done in two dimensions which is same as in two independent variables, whatever we have covered for the partial differential equations. Where we call it was a characteristic curves and we have of course, classified based on characteristic curves. So, we will recall that and then we generalize that to higher dimensions.

And they are now called characteristic surfaces for obvious reasons, because they are going to be one dimensionless. For example, characteristic curve, it is a curve in R 2. Now here we will have to imagine things which are just one dimensionless. So, that is why we will come to that, then we will understand what this means. And then we classify the linear equations in more than two variables.

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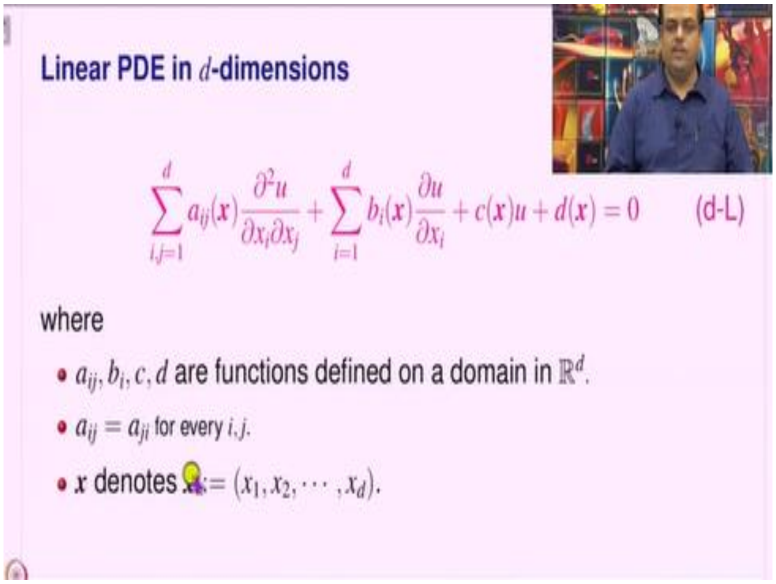

So, linear PDE in d dimensions, this how wecall it is an equation of this form. There is a most general linear partial differential equation in d independent variables. Often, we call that d dimensions PDE in d dimensions means, there are d independent variables. Sigma ij = 1 to d, a ij of x dou 2 u by dou x i dou x j. This is often called what called a principal part of the PDE. So, it is that part of the PDE where the highest order derivative appears.

The highest order derivative appears with power one, coefficients may depend on x u and first order derivatives, does not matter. So, this is what is called principal part. In the context of this d dimensional linear equation, the principal part is precisely this, because this is where the second order partial derivatives of u appear. If you recall in the case when d could do the classification and characterization of characteristic curves is actually based on the principal part.

So, similarly, it is going to play a role even in higher dimensions. That is why I have introduced the word formerly, that word is a principal part of a linear partial differential equation. It is that part where the highest order partial derivatives appear. And some people call this one as a lower order term sometimes, because it involves derivatives of u not of second order but of orders less than 2 that is 1 or no derivative.

We have to make some place assumptions on the equations without which we cannot do the theory. So, we are going to assume that a ij, bi, c, d, of course, they are functions of x defined on a domain that is x belongs to some domain R d, we also assume that these are continuous functions for simplicity. We most importantly assume that a ij=a ji because a ij is a coefficient of dou 2 u by dou x i dou x j, a ji is a coefficient of dou 2 u by dou xj dou xi.

And if you are expecting twice continuously differentiable function as a solution to this equation, then dou 2 u by dou x i dou x j is same as dou 2 u by dou xj dou xi. The order in which you take the second order partial derivative does not matter. Therefore, we can always make this to be symmetric a ij equal to a ji. So, this is a condition that we must place, so let us assume this. And this x which is in the boldface actually stands for x1, x2, xd in R d.

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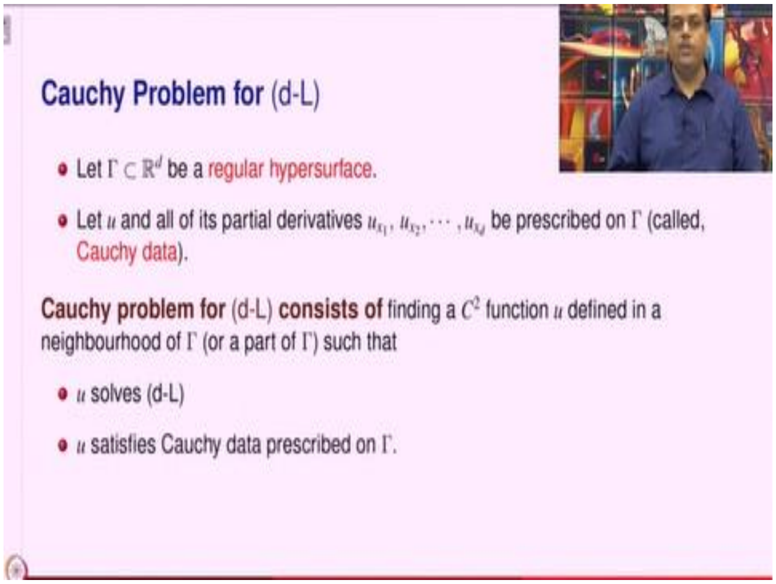

So, what is the Cauchy problem for the equation d-L? That is d-dimensional linear equation, this is a shortcut form of calling d-dimensional linear equation, we just call d-L. So, what is the Cauchy problem? Take gamma subset of R d, which is a regular hypersurface. Let u and on all of its partial derivatives namely u x 1, u x 2 up to u x d be described on gamma. This is often called Cauchy data.

Cauchy problem for d-L consists of finding a C 2 function defined in a neighbourhood of gamma. Because the x where the function u and the partial derivatives are prescribed or even a part of gamma. Now, we are used to this idea that Cauchy problem need not have solution defined around entire gamma; it could be a part of gamma. Such that u solves d-L, it should be a solution of the partial differential equation and the given Cauchy data should be satisfied by u.

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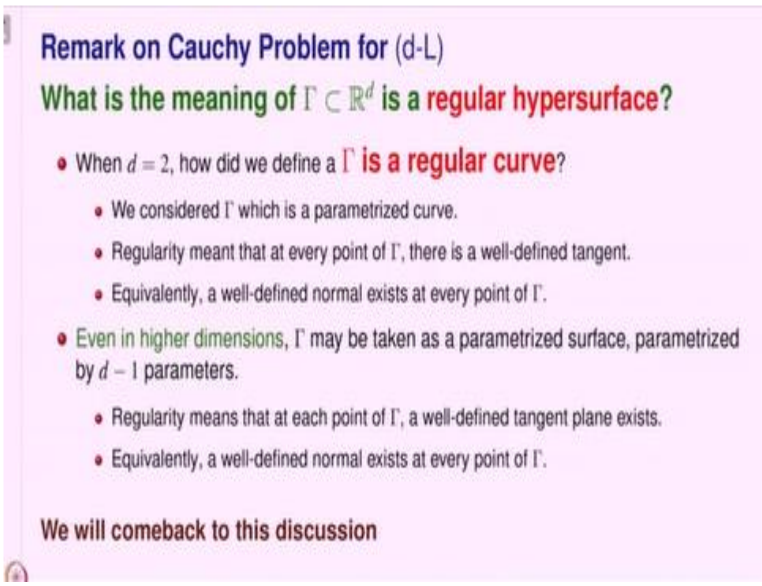

Now, what is the meaning of gamma subset of R d is a regular hypersurface? There are two words regular and hypersurface. Normally, hypersurface means somebody who just one dimensionless in the whole space. For example, in R3 a plane would be a hypersurface, two dimensional planes or a sphere that would be hypersurface. Because a sphere; can be described using two parameters whereas in R3, we need kind of three parameters.

So, that is the meaning of the hypersurface and what is regular? It is something about smoothness, we will see that. So, when d equal 2, how did we define gamma is a regular curve? First of all, we can start gamma which is a parameter is curved to start with and regularity meant that at every point of gamma, there is a well-defined tangent. Equivalently, it means that a well-defined normal exists at every point of gamma.

Even in higher dimensions gamma may be taken as a parameterized surface, parameterized by d -1 parameters. When d = 2, the curve prescribed by one parameter when you are an R d it will be prescribed by d - 1 parameters, so parameterized by d - 1 parameters. Regularity means that at each point of gamma in the case of d equal 2 we said well-defined tangent, now, that becomes well defined tangent plane exists.

Therefore, equivalently a well-defined normal exists at every point of gamma. We will come back to this discussion towards the end of this lecture.

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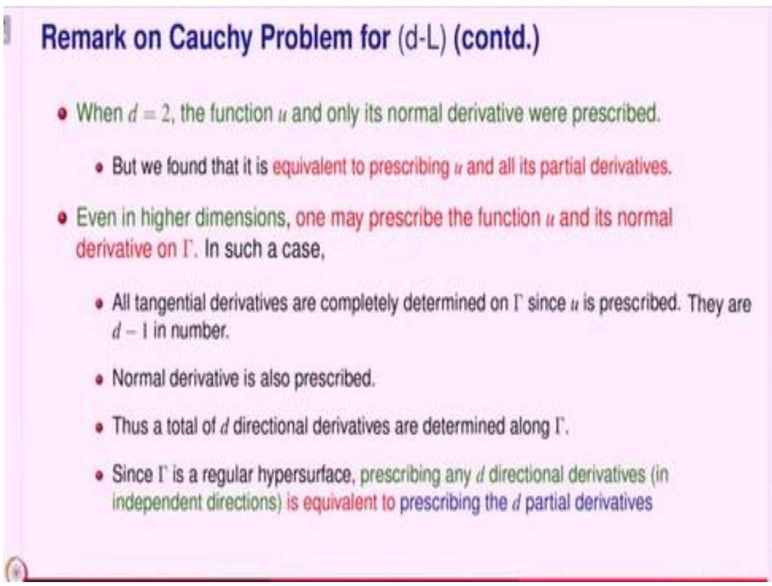

When d = 2 the function u and only as normal derivative were prescribed, like recall lecture 3.1. But we found that it is equivalent, what is the equivalent? Prescribing functions new and only it is normal directive. This is equivalent to prescribing u on all its partial derivatives. Why is it so? Because the; gamma was a regular curve. We can determine all derivatives if we knew the function and the normal derivative.

So, even in higher dimensions one may prescribe the function u and its normal derivative on gamma, it can be done. In such a case what do you have? All tangential derivatives are completely determined on gamma, why is that? Because the function is given on gamma and they are d - 1 in number. Normal derivative is also prescribed. For example, imagine your gamma is let us say in d=3 gamma is xy plane.

So, you are prescribing u on xy plane, it means automatically ux and uy are determine on xy plane. So, these are the tangential derivatives which are obviously d -1 in number that is 2 in number, ux and uy. Normal derivative is nothing but uz. So, if you give uz, you have three derivatives. Now, same thing is true even if gamma is not xy plane, gamma is a regular hypersurface.

So, even in case of regular hypersurface you are given u on gamma, therefore, all tangential derivatives can be determined. Keep in mind the example of the xy plane has gamma and R d is R 3. Normal derivatives also prescribed, thus a total of d directional derivatives are determined along gamma, d -1 is here and one more here therefore, d -1+ 1 that is d. So, you have d directional derivatives and the important thing is directions are independent derivatives.

Since gamma is a regular surface, prescribing any d directional derivatives in independent directions is equivalent to prescribing all the d partial derivatives u x 1, u x 2 up to u x d.

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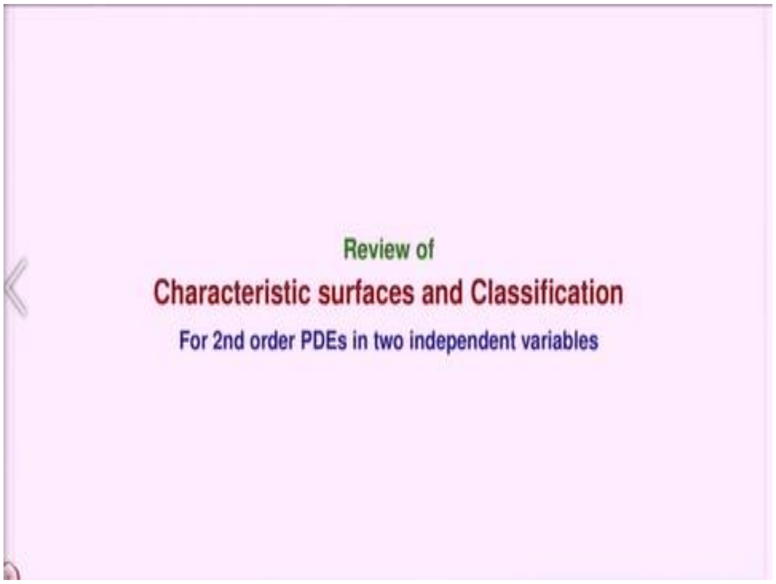

So, let us review characteristic surfaces and classification as was done in the case of two independent variables.

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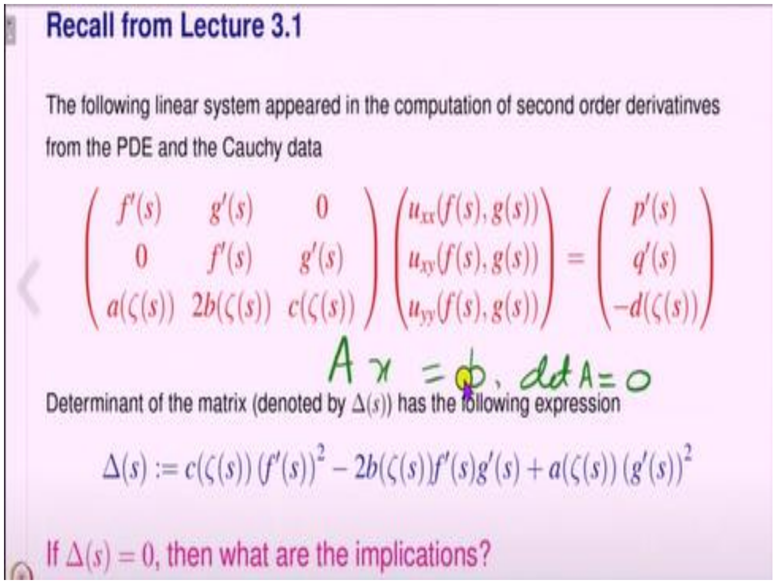

So, let us recall from lecture 3.1. We had a linear system when we are trying to compute the second order derivatives from the PDE and the Cauchy data alone. So, this is the system, f and g was the parameterization of the gamma, gamma is fs, x = fs, y = gs. So, we are trying to determine the second order derivatives, then it satisfies system of linear equations non homogeneous. Determinant of this matrix was what is called delta of s, this is the determinant.

If you expand this determinant, you will get this formula; we did this in lecture 3.1. Now, if delta s = 0 then what are the implications, what will happen? So, in other words you have a system let us call this A x = b, we have a system A x = b and determinant of A = 0. What can you say about the solutions of this system A x = b? What we can say is there will be b for which there is no solution, there will be for which there will be solution.

But the moment you have one solution youhave infinitely many solutions. So, you will never have a unique solution either you have no solution or infinitely many solutions.

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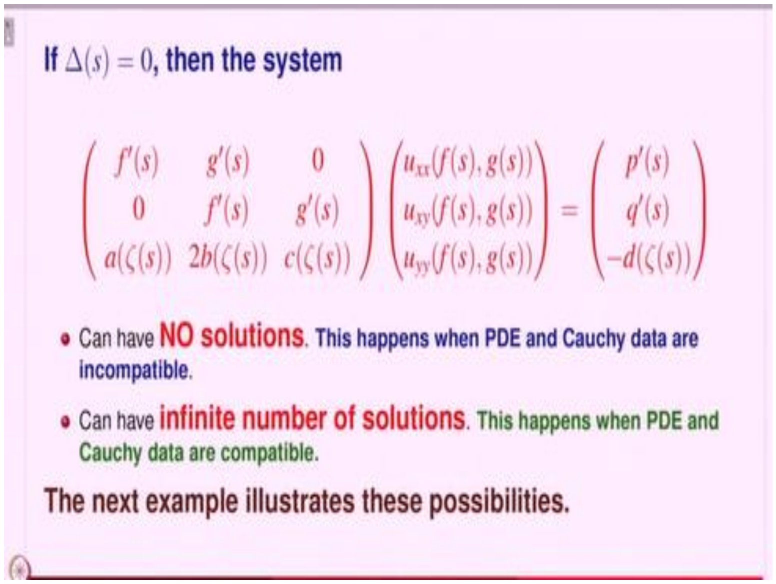

So, if delta s = 0, then this system can have no solutions. When will that happen? When the; data is incompatible with a PDE. It can have infinite number of solutions; this will happen when you have compatibility between the PDE and the Cauchy data. The next example illustrates these possibilities very clearly; it is a very simple partial differential equation.

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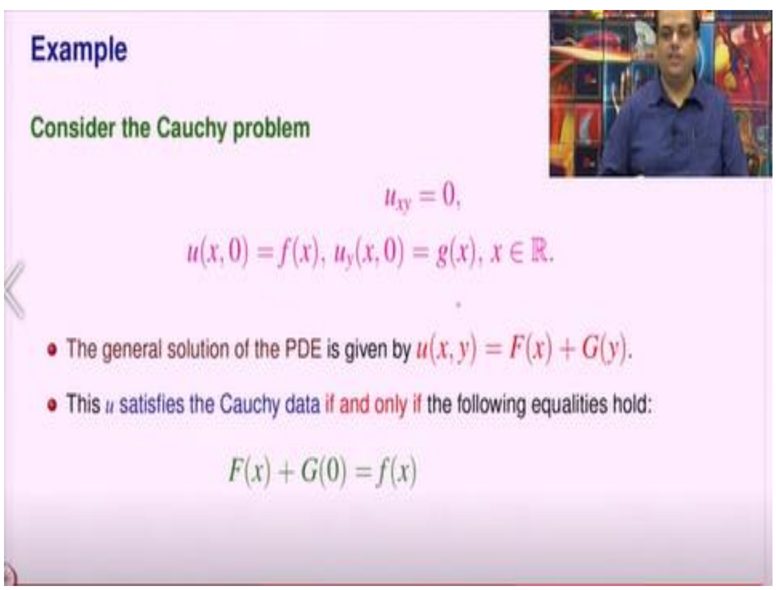

So, the Cauchy problem is u x y = 0 and u of x 0 is given to be fx, u y of x 0 is given to be gx, that is u is prescribed. And the normal derivative is prescribed on x axis, here gamma is x axis. What about this problem? The general solution of the PDE, u xy = 0 is given by a function of x plus function of y. Because you integrate this, u xy = 0, imagine it is d by dy of u x =0, then u x becomes a function of x and you integrate you get this is a function of x plus function of y.

This u satisfies the Cauchy data. What is the Cauchy data? These are the equations. So, when is u x, 0 = f x, substitute here, put y = 0, you get f of x + G of 0 must be equal to small f of x, this is the first condition. What about the second condition? You need to differentiate u with respect to y that will give you a G prime y and when y = 0, then G prime of 0 that will give you g x. So, these are the two conditions that we get.

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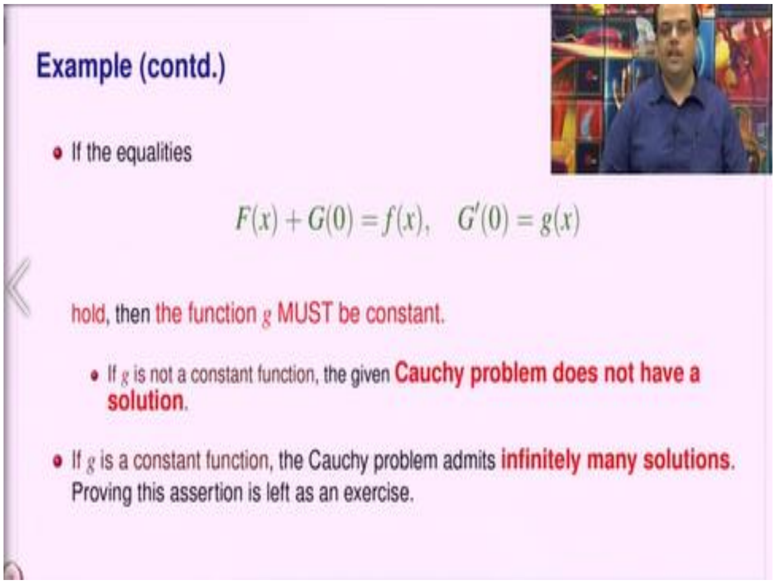

So, if these equations hold, then the function g must be constant, because G prime 0 is a number that means g is a number. If g is not a constant function, then the Cauchy problem does not have a solution that is very clear from this, this is not satisfied. If g is a constant function, then Cauchy problem admits infinitely many solutions. Just convince yourself, it is a very simple to show that you can have infinitely many solutions. Proving this assertion is left as an exercise for you, very simple.

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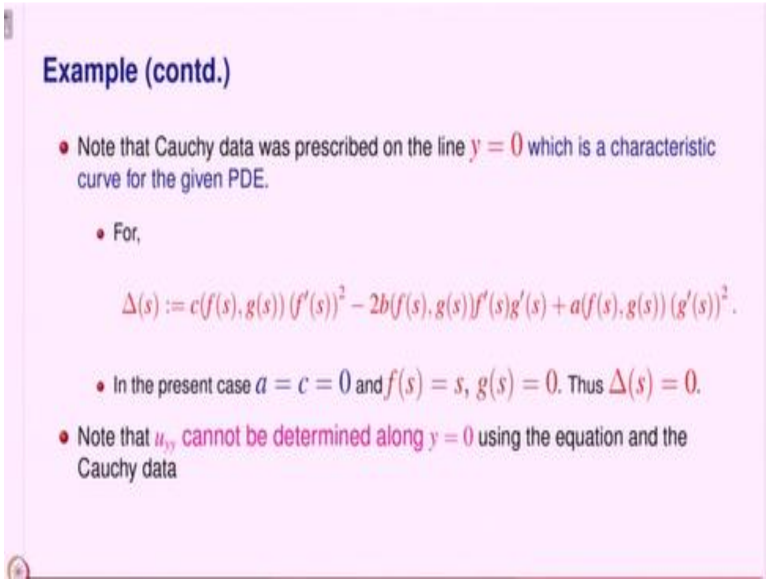

Now, observe that the Cauchy data is prescribed on the line y = 0. That is the x axis which is a characteristic curve for the given PDE. Let us check that. Delta s, this is a formula c f prime square - 2b f prime g prime + a g f prime g prime squared. What is a, b, c in our case? a and c are 0, our equation was u x y = 0. So, a and c are 0 and gamma is x axis, therefore fs is s and gs is 0. When we plug in this into this what we get is delta s is 0.

So, it as a characteristic curve and clearly u yy cannot be determined along the x axis. Using the equation and the Cauchy data because Cauchy data gives you only u ux and u y anyway is given, but u yy you cannot get because it is not there in the equation.

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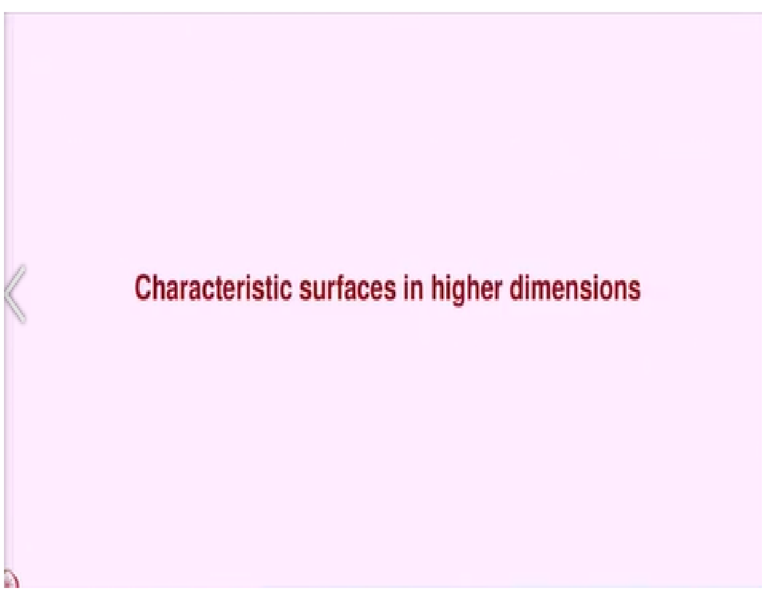

So, now we are ready to define what are characteristic surfaces in higher dimensions.

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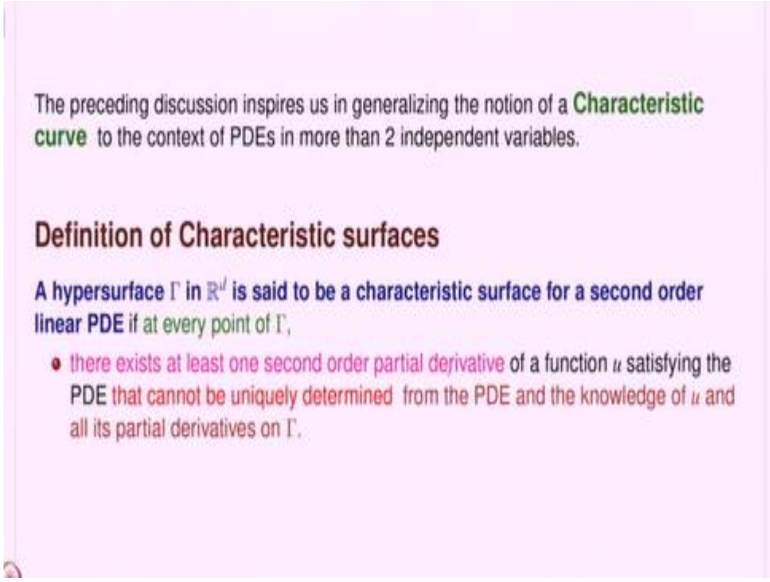

So, the preceding discussion inspires us in generalizing the notion of a characteristic curve to the context of PDFs in more than 2 independent variables. So, let us define the notion of characteristic surfaces, what is that? A hypersurface in R d is said to be a characteristic surface for a second order linear PDE. If at every point of gamma, there exists at least one second order partial derivative of a function u, which satisfies the PDE.

And that cannot be uniquely determined from the PDE on the Cauchy data. That is from the PDE and the values of u and all its partial derivatives alone come. From this knowledge of u and all its partial derivatives on gamma you cannot determine at least one second order partial derivative, uniquely determined, cannot be uniquely determined means either you have too many values or no values.

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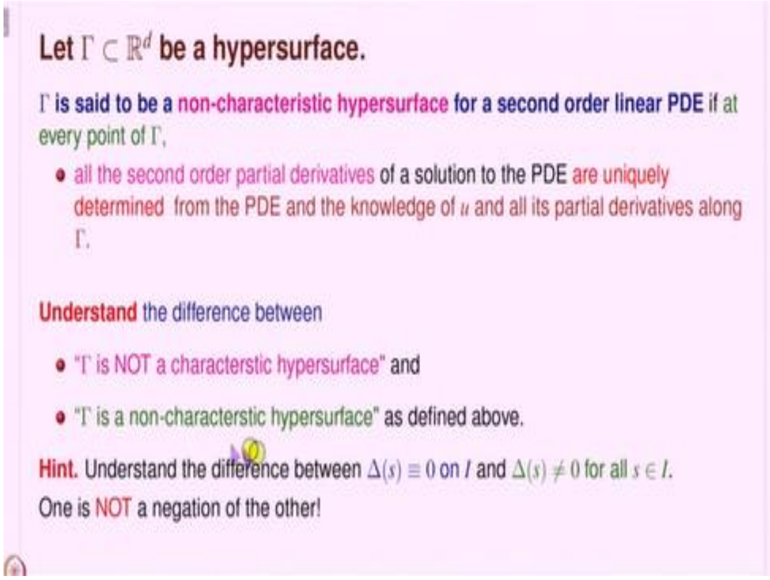

That is no solution or infinitely many solutions. If you recall that is what is a true within two dimensions, when delta s = 0 we have the two possibilities which are seen in the last example. So, let gamma be hypersurface, now, we are going to define a non characteristic hypersurface. Gamma is said to be a non characteristic hypersurface recall what we are defined in the previous slide, what is a characteristic surface?

Now, we are defining what is called non characteristic hypersurface. What is that? At every point of gamma, all the second order partial derivatives of a solution to the PDE can be determined uniquely from the PDE and the knowledge of u along the gamma and partial derivatives of u along the gamma. Now, I would like you to think about the following; understand the differencebetween gamma is not a characteristic hypersurface.

On the previous slide we define what is the meaning of gamma is a characteristic hypersurface. Therefore, gamma is not a characteristic hypersurface and gamma is a non characteristic hypersurface, there is a difference. I just give you a hint understand the difference between delta s identically equal to 0 on I, in fact these we call a characteristic curve if this happens. And delta s is never 0, that means that curve was non characteristic curve that is a difference.

One is not a negation of the other. You may call one is the extreme negation of other, that is fine. But you must understand the difference between not a characteristic hypersurface and non characteristic hypersurface, both are different.

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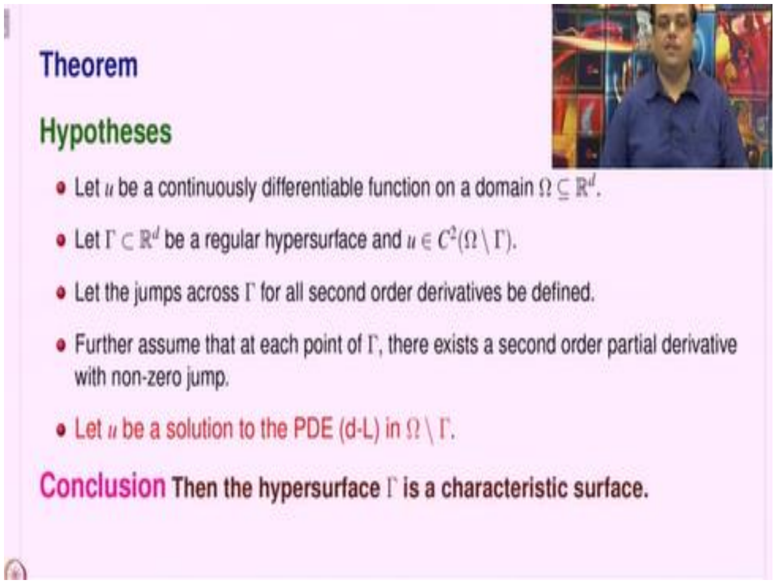

We have a theorem, let u be a C 1 function on a domain omega in R d, let gamma be a regular hypersurface and u E C 2 on omega minus gamma that means, what we are thinking of is a situation like this you have an omega and you have a gamma. Then u E C 2 here at all these points. But u E C 1 throughout and jumps across gamma for all secondary derivatives are defined, that means that at points here at points on gamma, look at any derivative that you think of u x 1 x 2 this is one partial derivative.

This jump is defined, that means u x 1 x 2 coming from this domain let us call it omega 1, call this omega 2, a point here the u x 1 x 2 the second derivative has a meaning. That means it is continuous from this side. So, that means in other words u restricted to omega 1 is actually a continuous function of up to closure. So, that the values of second orderpartial derivatives make sense on gamma. Similarly, u restricted to omega 2 is C of omega 2 bar, not C in fact C 2.

Because we already assumed u E C 1, there is no need of writing C of omega 1 bar, omega 2 bar, what we want actually is about second order derivatives have values at points of gamma coming from the right-hand side or left-hand side. I have written a picture in r 2, that is why I am using the word right hand side of things. So, in other words, what I am assuming is that this hypersurface gamma cuts omegainto two parts.

One part you call it omega 1, another part you call omega 2, u E C 1 on the whole domain omega, but on each part, u is C 2 of omega I bar when you restricted to omega I. So, that jumps these are well defined quantities. So, now I am going to talk about that in the next hypothesis. Assume that at each point of gamma, some second order partial derivative has a non 0 jump and let u be a solution to the PDE in both the parts omega 1 and omega 2 as I have written down.

Then conclusion is about gamma, gamma must be a high characteristic surface, the hypersurface gamma must be a characteristic hypersurface. Sometimes I am calling characteristic surface, sometimes I am calling characters hypersurface does not matter, because anyway, gamma is a hypersurface to start with. So, it is a characteristic surface.

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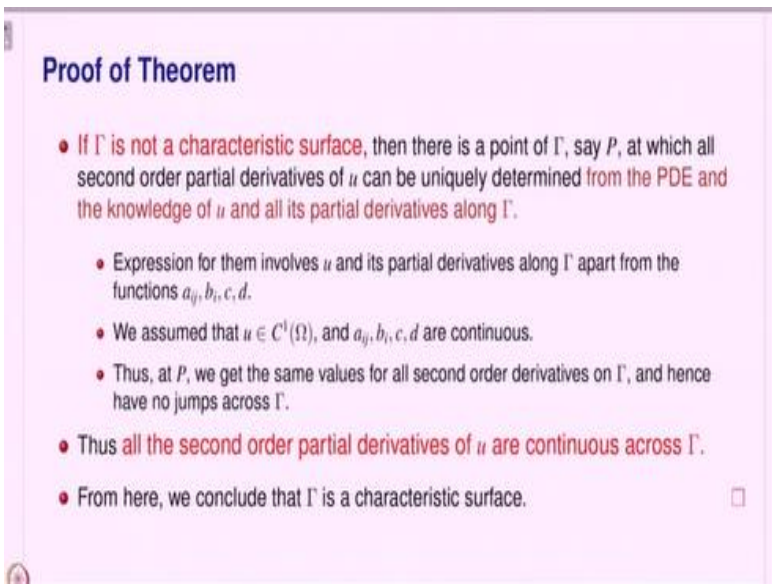

Suppose it is not a characteristic surface, so, the proof is going to be like this, I assume that the conclusion does not hold, and I will show that one of them does not happen. In fact, I am going to show this does not happen, all secondary derivatives are continuous across gamma. I am going to show that, so that we prove this here. So, assume gamma is not a characteristic surface. Then there is a point of gamma at which all second order partial derivatives are determined uniquely called a point P.

There is at least one point so call that point P. All the second order partial derivatives can be uniquely determined from the PDE and the knowledge of u and all its partial derivatives along gamma, this is always there. From this, this is what we call Cauchy data in the Cauchy problem. From there and the PDE, we can determine all second order partial derivatives uniquely at that point. Now, this is where you should compare with what we did in lecture 3.1.

Expression for them involves exponent for what? Exponent for the second order partial derivatives involves u and its partial derivatives along gamma apart from the coefficient functions in the PDE aij, bi, c and d, we assumed u E C 1 of omega and the a ij, bi, c, d are continuous. Therefore, at P we get the same values for all second order partial derivatives on gamma, whether you come from the omega one side or omega two side, you have the same value.

That means no jump across gamma. Thus, all second order partial derivatives are continuous across gamma and as we discussed at the beginning of the proof, this finishes the proof of the theorem.

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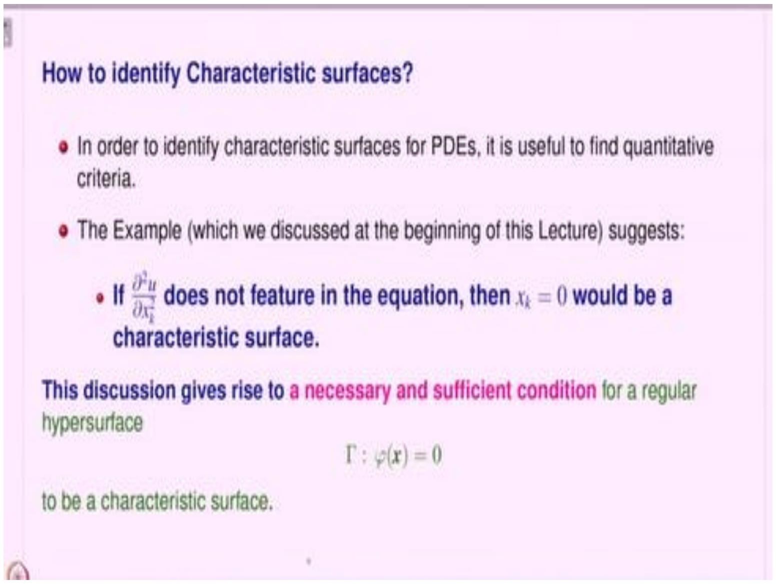

How to identify characteristics surfaces? These are next question because what we have defined characteristic surface is a quality to definition. It just says that some partial second order partial derivative is not uniquely determined at each point of gamma, such a gamma was called characteristic surface, this is a qualitative definition. Now, we would like to have a quantitative version of this. So, that we can go and find some characteristics surfaces.

So, therefore, it is useful to find quantitative criteria. The example which we discussed at the beginning of the lecture suggests the following. If dou 2 u by dou x k squared does not feature in the equation, then x k = 0 would be a characteristic surface. In that example, what was the example? It was u x y = 0 and we have looked at u x,0 is given to the f and u y x, 0 is a normal derivative given to be g.

And we could not get what is u yy using this PDE and the Cauchy data u yy was missing. So, if some derivative is missing, then that variable equal to 0 will be a characteristic surface this idea. So, this discussion gives rise to a necessary and sufficient condition for a regular hypersurface. Now, we are going to look at hyper surfaces of this form phi x =0 to be a characteristic surface. This mean set of all x in R d such that phi x = 0.

So, called level set of the function phi, we are going to derive an excellent sufficient condition for regular hypersurface of this form to be a characteristic surface.

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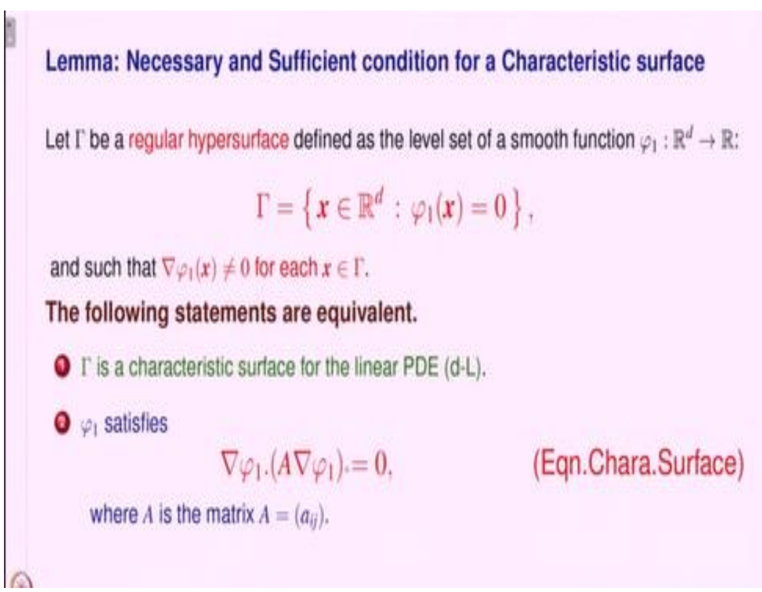

So, let gamma be a regular hypersurface defined as the level set of a smooth function phi 1 from R d to R, what is the definition? Set of all x in R d such that phi 1 x = 0 and gradient of phi one is not 0 for each x in gamma. So, this actually tells you that there is a clearly defined normal at every point of gamma. So, this is the regularity hypothesis for the hypersurface. Then the following statements are equivalent, what are those?

Gamma is a characteristic surface for d-L, that is same as saying that this phi 1 gamma is defined through the function phi 1 that phi 1 satisfies this PDE. What is the PDE? Grad phi 1 dot A of x grad phi 1 = 0. What is A? A is the matrix a ij which is appearing in the principal part of the equation d-L. So, these are the equation for characteristic surface, a level set phi 1 x = 0 is a characteristic hypersurface if and only if this equation is satisfied. Of course, this is a first order PDE nonlinear.

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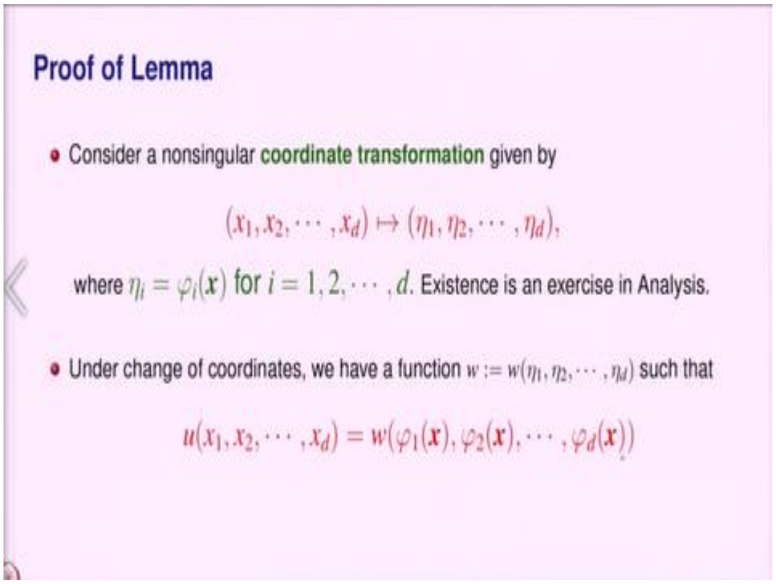

So, consider a nonsingular coordinate transformation given by x 1, x 2, x 3 going to eta 1, eta 2 to eta d and eta equal to phi i of x. We are given only phi 1 of x, therefore eta 1 is phi 1 x, now you find phi 2 of x, phi 3 of x, phi d of x so that we have this coordinate transformation defined which is very easy. So, it can always be done, that is an exercise in analysis. Essentially what we are asking is can you given a function phi 1 with grad phi one non 0.

Can you find phi 2 phi 3 up to phi d is as a certain Jacobin is non 0? That is the question is possible. Now under the change of coordinates, we have a new function w related to u, w of eta 1 eta 2 eta d such that u at the point x is equal to w at phi 1x, phi 2x and phi dx.

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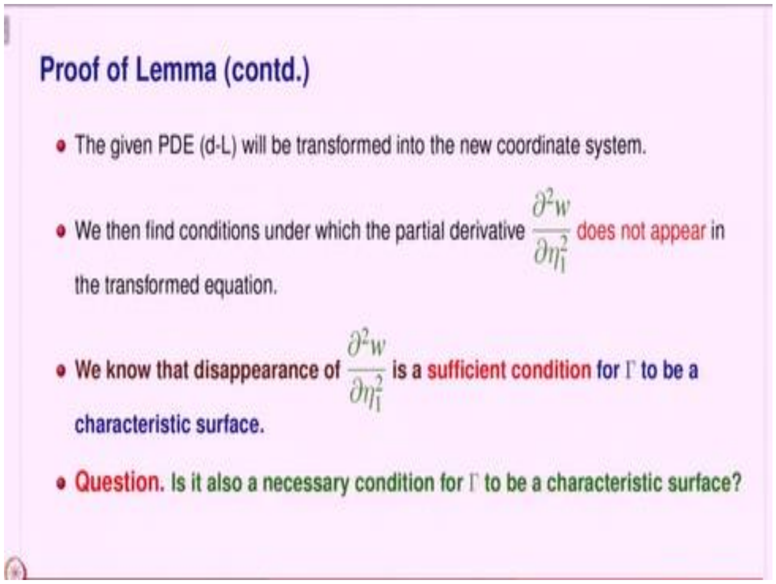

Now, the given PDE will be transformed into the new coordinate system and then we find conditions under which dou 2 w by dou eta 1 squared does not appear. Remember eta 1 equals zero is same as set of all x as at phi 1 x = 0 is a hypersurface and that is precisely our gamma. So, we know that disappearance of this is a sufficient condition for gamma to be a characteristic surface. So, we asked the question, is it also a necessary condition?

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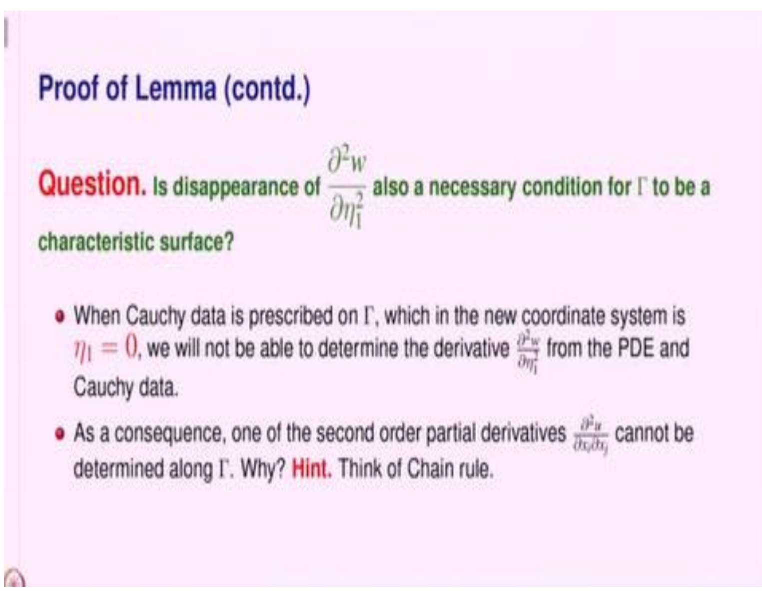

Answer is yes. When Cauchy data is prescribed on gamma, which in the new coordinate system is eta 1 = 0, we will not be able to determine this derivative, dou 2 w by dou eta 1 square for the PDE and the Cauchy data. As a consequence, one of the second order partial derivatives dou 2 to u by dou xi dou xj cannot be determined along gamma uniquely. Why? Because there;

should be some connection between these derivatives this derivative and these derivatives.

Imagine you can find out all of them, and then can I write this as a combination of that? Think about this. Think about Chain rule.

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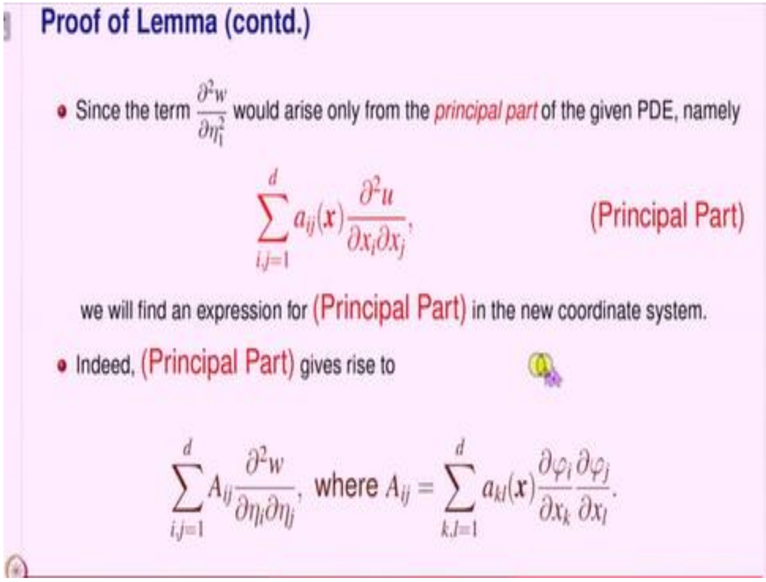

So, since dou 2 w by dou eta 1 square will arise only from the principal part of the given PDE namely this part when you change your d-L equation into the new coordinate system, this one dou 2 w by dou eta 1 squared is a second order derivative it comes only through these terms. Therefore, you will find principal part how this transforms into the new coordinate system. This gives rise to this where each a ij is this expression. So, go back to lecture 3.3 I think where we are done this change of coordinates.

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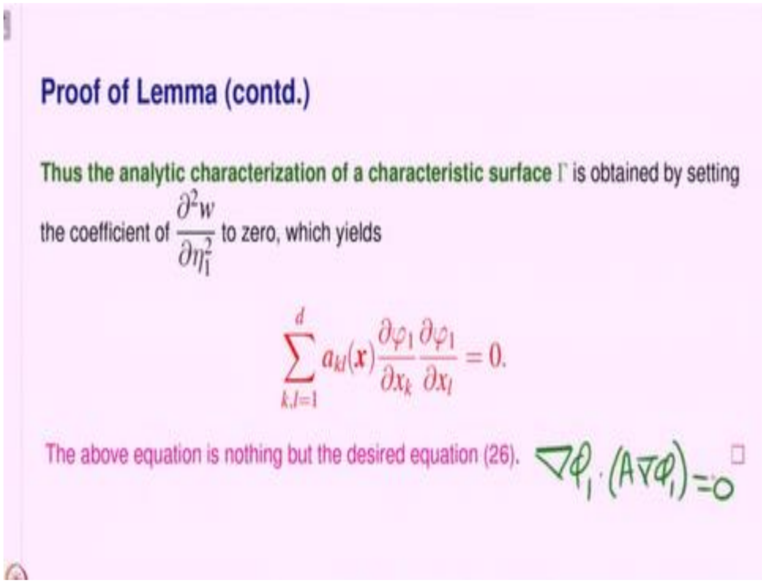

Thus, the analytic characterization of a characteristic surface gamma is obtained by setting the coefficient of dou 2 w by dou eta 1 square to 0, that means this equal to 0. The above equation is nothing but what we want because this is grad phi 1 dot A grad phi 1 = 0. Please ignore this number 26, this refers to the equation grad phi 1. A grad phi 1 = 0.

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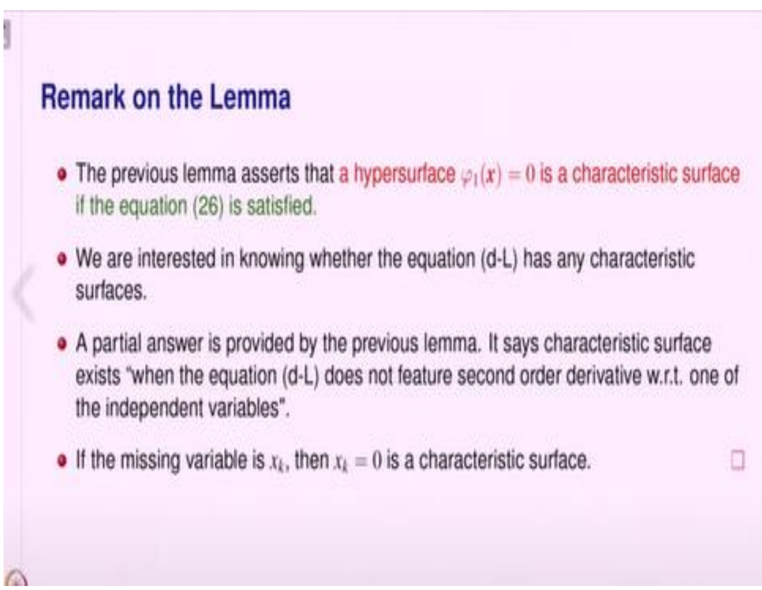

The previous lemma asserts that a hypersurface phi 1 x = 0 is a characteristic surface if the equation grad phi 1. A grad phi 1 is 0 is satisfied. We are interested in knowing whether the equation has any characteristic surfaces. A partial answer is provided by the previous lemma, it also tells us that characteristic surface exists when the equation does not feature the second order derivative with respect to one of the independent variables.

The missing variable is x k, that is dou 2 u by dou x k square is missing, then x k equal to 0 is a characteristic surface.

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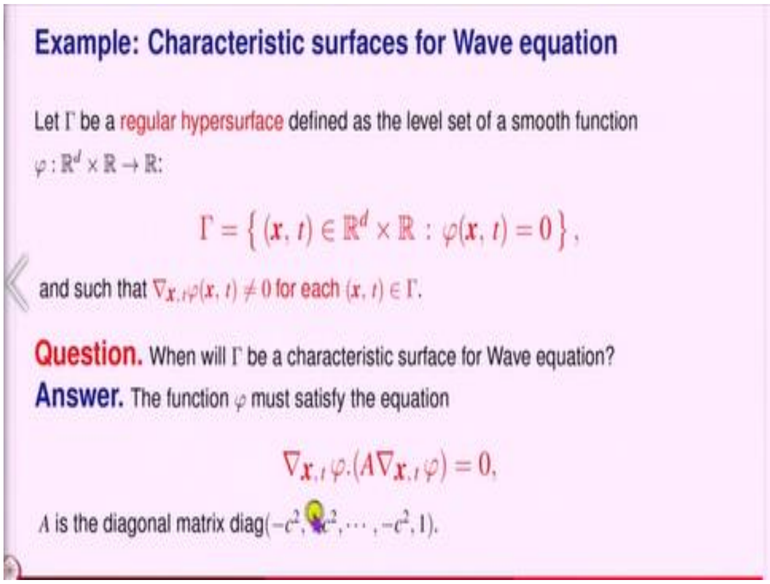

Let us look at an example, which is wave equation. So, we will try to determine characteristic surfaces for wave equation. So, let gamma be a regular hypersurface defined as the level set of a smooth function phi. Now, we have to be careful that wave equation we have x, t. So, therefore, x in R d and t in R, so, we write this. So, gamma is phi of x t = 0 and gradient should be non 0. Here gradient is not with respect to x, x and t just to clear any confusion we write this.

Gradient of x, t of phi of x, t is non 0 for each x, t in gamma. When will this gamma be a characteristic surface for wave equation? Answer; the function phi must satisfy this equation, grad x, t phi. A grad x, t phi = 0 and what is A? A is a diagonal matrix minus c square, minus c square, minus c squared in the end 1. So, this is like u tt equal to c square u x, 1 x, 1 + c square into u x, 2 x, 2 and so on up to c square u x,d x, d. So, these are d number this one, so this is a d + 1 by d + 1 diagonal matrix.

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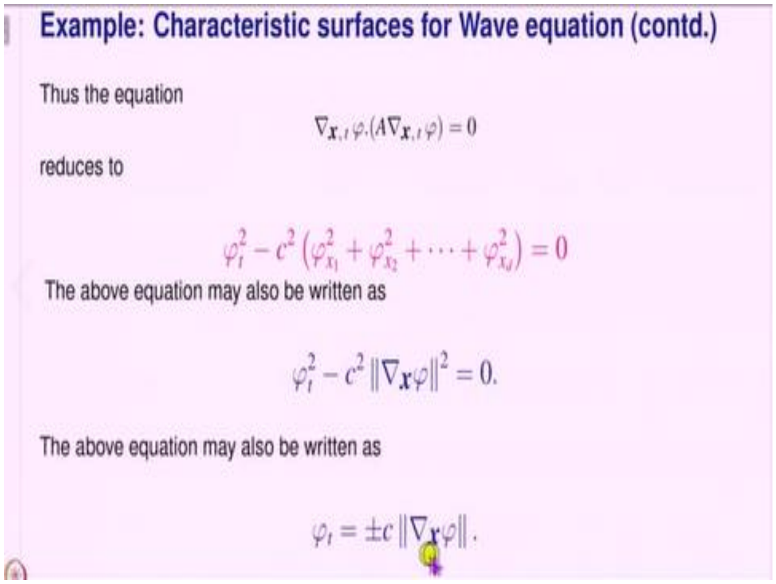

So,this equation grad phi. A grad phi reduces to this equation. This we can write because this is the Euclidean norm is nothing but this whole thing phi x 1 square + phi x 2 square + phi d square is precisely norm grad x phi square Euclidean norm. So, this equation is exactly this. Now, I can get rid of the squares and I get this equation phi t equals plus or minus c norm grad x phi.

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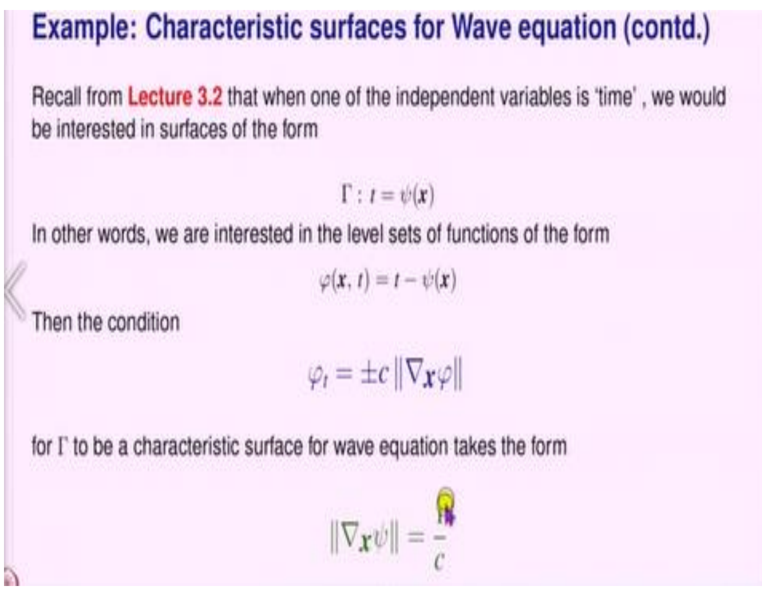

Recall from lecture 3.2 that when one of the independent variables is time which is true in the wave equation. We will be interested in they are the curves of the form t = psi of x. Now, here we will be interested in surfaces t = psi of x. So, we are not interested in arbitrary phi of x = 0 but we are interested in t = psi of x, we explained the reason why so? Because here; it is going to give the location of the discontinuity.

That is what discussed in lecture 3.2, please go back there and understand again. So, in other words, we are interested in the level sets of functions of this time t-psi x. How the equation changes? This equation for a hypersurface to be a characteristic surface will now in terms of psi become this norm grad x psi = 1 by c.

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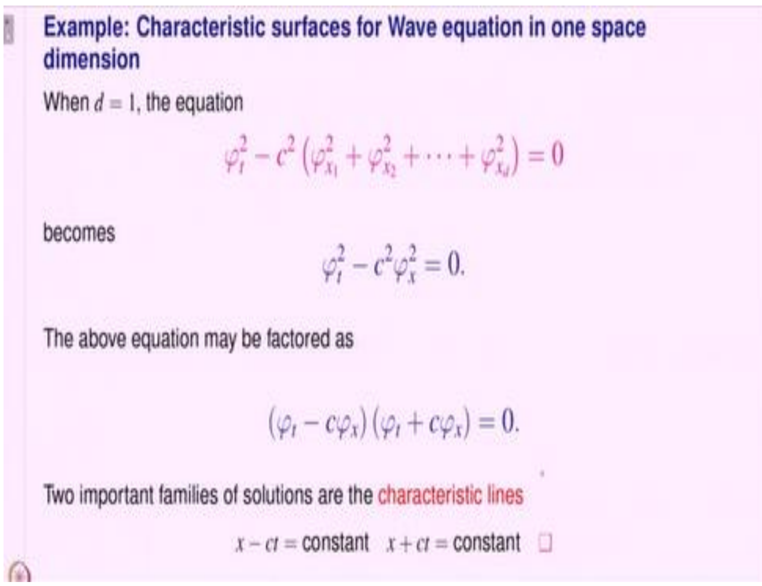

So, when d = 1 this equation is simply this equation. Now, we can factorize them like this phi d square + c square phi x square - c square phi x square is there, then you have phi t c phi x - c phi x phi t which gets cancelled. So, this is exactly the same as this factorization. So, two important families or solutions, the other characteristic lines x - ct = constant and x + ct = constant. They come through solutions of these two equations, phi t - c phi x = 0 and phi t + c phi x = 0.

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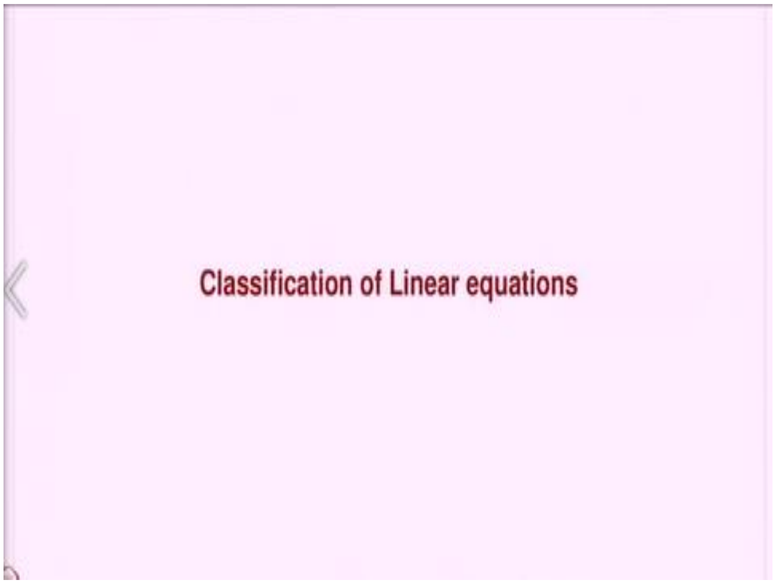

So, now, we are going to classify linear equations in more than two variables.

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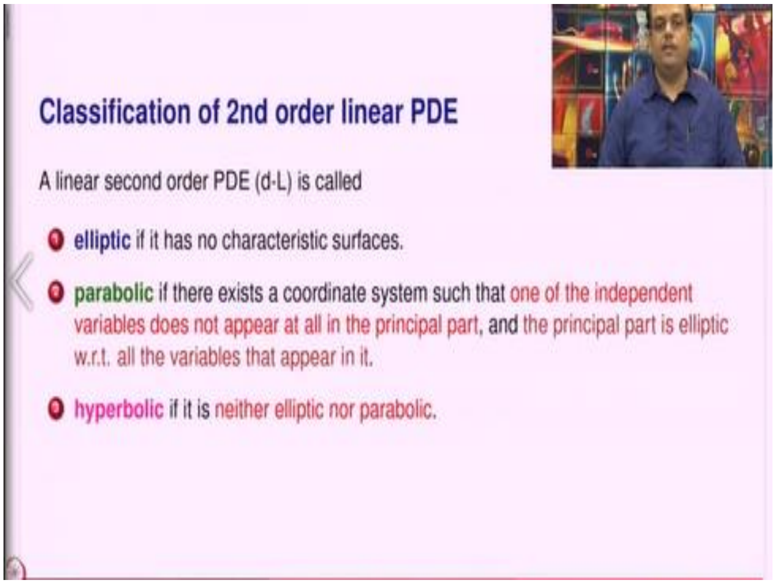

Based on characteristic surfaces. So, linear second order PDE is called elliptic, if it has no characteristic surfaces. Parabolic if there exists a coordinate system such that one of the independent variables does not appear at all in the principal part when the equation is written in that coordinate system and the principal part is elliptic with respect to all the variables that appear in it. Hyperbolic if we did not either elliptic nor parabolic.

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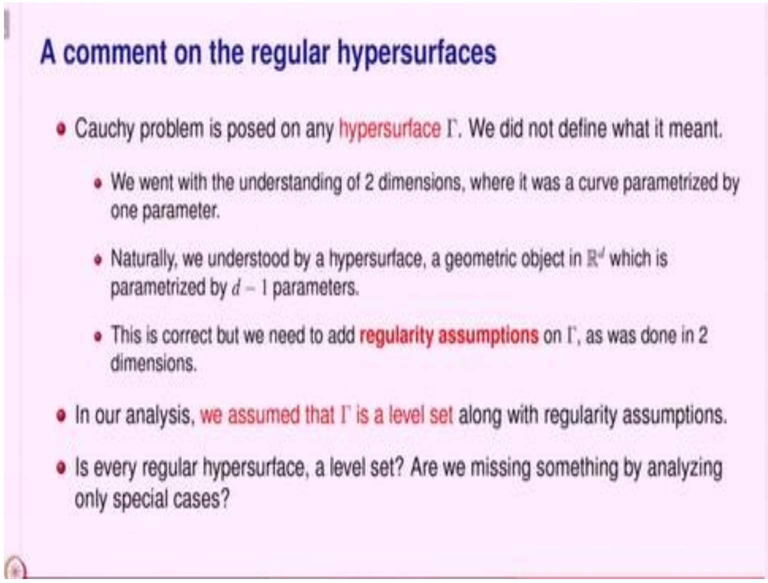

So, now, we finish this lecture with a comment on the regular hyper surfaces that we started discussing at the beginning of the lecture when we introduced the Cauchy problem. So, Cauchy problem is posed on any hypersurface gamma. We did not even define what it is meant, we went with the understanding of what we know in two dimensions or it is a curve, tangent is defined at every point of gamma like that.

So, naturally we understood by hypersurface a geometric object in R d which is parameterize by d -1 parameters is correct, but we need to add regularity assumptions as was done in two dimensions. In our analysis, we assumed that gamma is a level set. So, when we looked at the theorem and the lemma, we assumed a level set of a smooth function with a gradient being non 0. Now, is every regular hypersurface defined like this with the d - 1 parametric hypersurface.

Is it level set? Because if it is not so, we are not proved the theorem for a general hypersurface. So, are we missing something by analysing only maybe possibly a special case?

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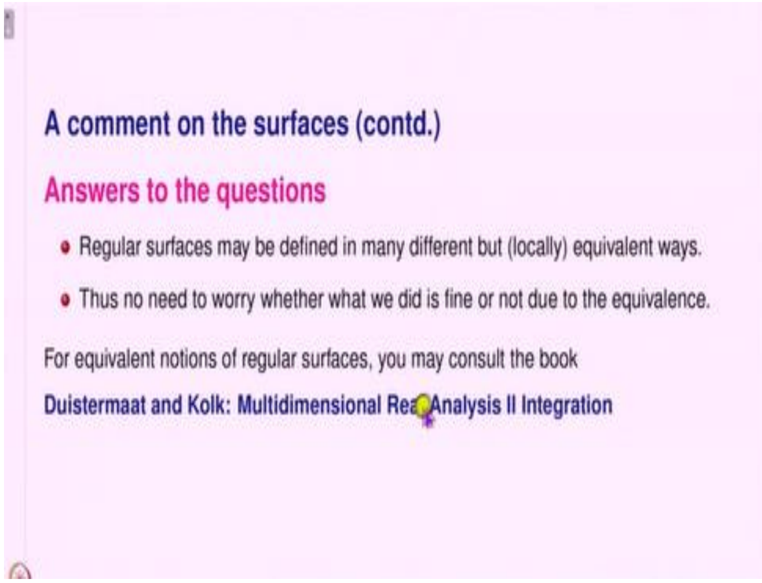

So, regular surface may be defined in many different but locally equivalent ways. There is no need to worry whether what we did is fine or not do not worry due to the equivalence. For equivalent notions of regular surface, there is a wonderful treatment in this book by Duistermaat and Kolk: multidimensional Real Analysis, part 2 integration. They have two books on multidimensional Real Analysis part 1 is differentiation, part 2 is integration. Of course, if you want to study the part 2, they often refer to part 1 also.

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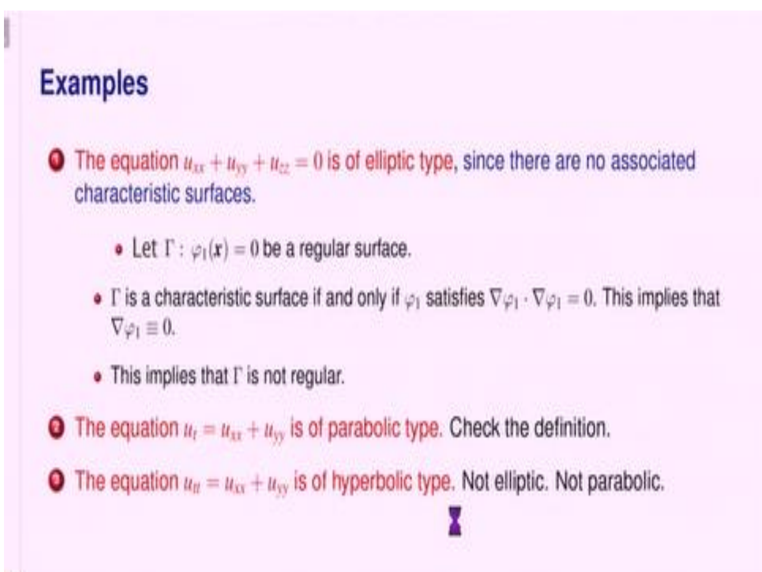

Very beautifully written books. Now, let us look at some examples. This equation u xx + u yy + u zz is an elliptic type. Since there are no associated characteristic surfaces, how do I say it is elliptic? How to ask whether there are character surfaces? If there are none it is elliptic, there are none, why? Let us look at the equation set to be satisfied by hypersurface to be a characteristic surface.

So, assume this gamma given by a level set be a regular surface, gamma is a characteristic surface if and only if phi 1 satisfies this equation grad phi 1. grad phi 1 because the A is identity here. The matrix here you got out of this equation is identity, because only u xx, u yy and u zz are appear or norm exposure level disappear. So, off diagonal terms are 0 in the matrix A and diagonal terms are 1. So, you have this, but what is this? This is mode grad phi 1 square =0.

So, this will be satisfied if and only if grad phi 1 is 0, but we are assuming the set gamma is a regular hypersurface. That means grad phi 1 is never be 0. So, this tells us that there are no characteristic surfaces. Therefore, the equation is elliptic type. Look at this equation u t = u xx + u yy, these are parabolic type. Check the definition. I will not discuss this, please check that by the definition that is not, it is a parabolic equation.

Now, this equation u tt = u xx + u yy, it is a hyperbolic type. It is actually a wave equation in two squares dimensions, it is a hyperbolic type. Check that it is not elliptic is what we are doing it is not elliptic, it is not parabolic. And by definition, it is going to be hyperbolic.

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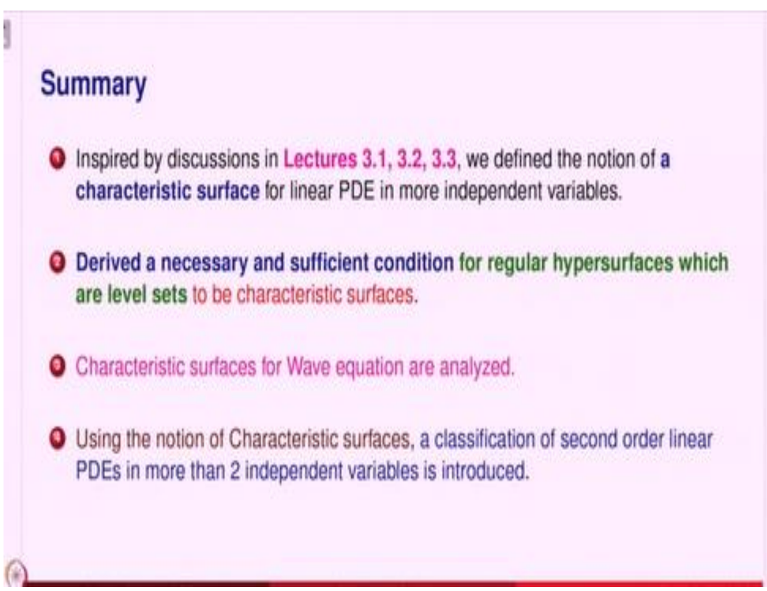

So, let us summarize, inspired by discussions of lectures 3.1, 3.2, 3.3. We define the notion of a characteristic surface for linear PDE in more independent variables, derived a necessary and sufficient condition for regular hyper surfaces which are level sets to be characteristic surfaces. And characteristic surfaces for a wave equation are analysed. That is what we did towards the end.

Using the notion of characteristic surfaces, we have classified linear partial differential equations in more than two independent variables. Thank you.