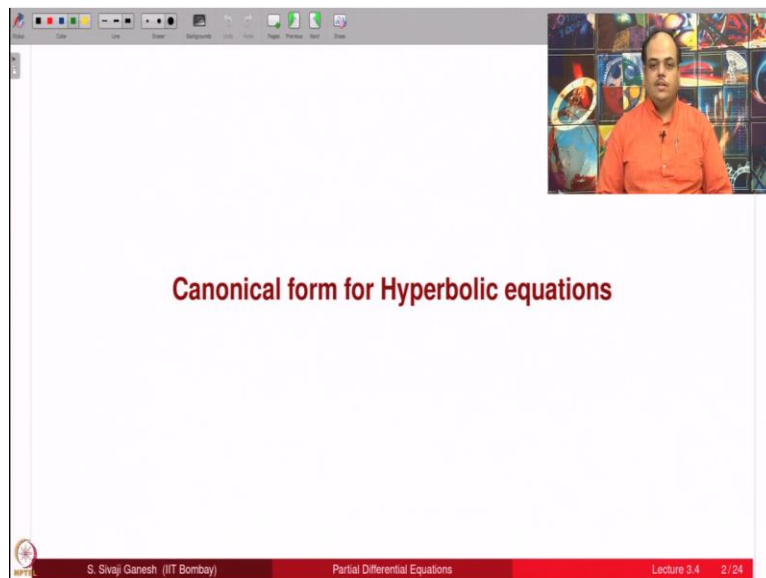


Partial Differential Equations
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Lecture – 21
Second Order Partial Differential Equations Canonical Form for an Equation of Hyperbolic Type

In this lecture, we are going to find the canonical form for an equation which is a hyperbolic type. In the next two lectures, we will be doing the same, but for equations which are parabolic or elliptic types.

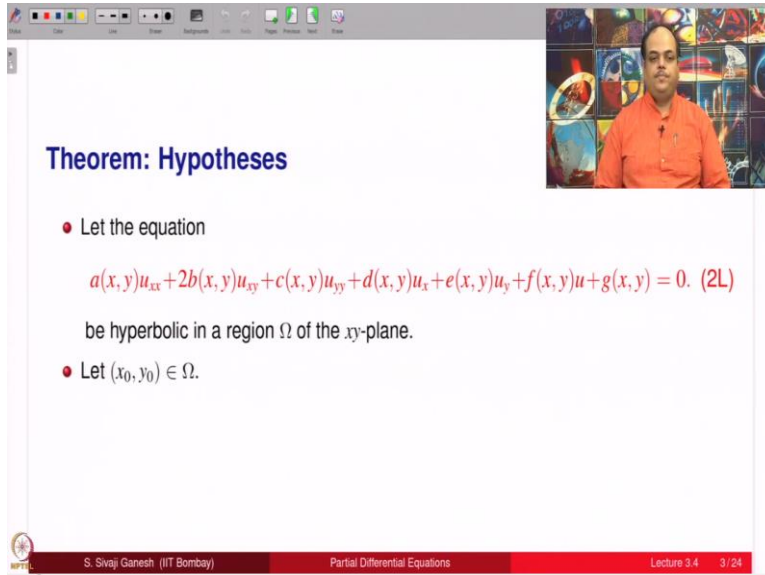
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So, canonical form for hyperbolic equations. What is it? We know what is hyperbolic equation we have defined. What is the canonical form? If somebody asks you give me an example of hyperbolic equation immediately you give an example of the wave equation. So, therefore, finding canonical form for hyperbolic equations consists of the following; you take the equation which is hyperbolic in a domain and then you want to do change of coordinates.

And then convert the PDE into the new coordinate system. But after doing this the part which features the second order derivatives looks like the wave equation, the one in the wave which. We are going to do the canonical forms only for linear equations for quasilinear equations, it is very difficult. So, we will not do that.

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The screenshot shows a presentation slide with a video inset of a man in an orange shirt. The slide title is "Theorem: Hypotheses". It contains a list of hypotheses and a second-order linear equation.

Theorem: Hypotheses

- Let the equation

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u + g(x, y) = 0. \quad (2L)$$

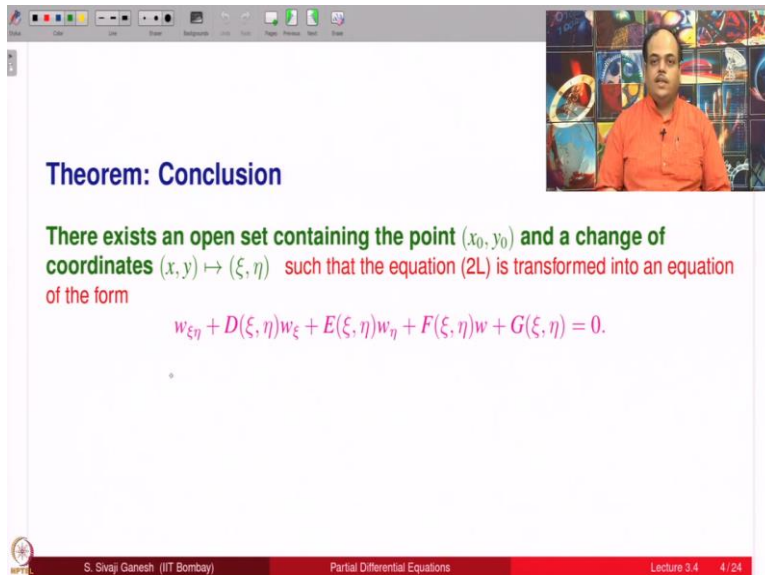
be hyperbolic in a region Ω of the xy -plane.

- Let $(x_0, y_0) \in \Omega$.

At the bottom of the slide, there is a footer with the text: "S. Sivaji Ganesh (IIT Bombay) Partial Differential Equations Lecture 3.4 3 / 24".

We state what we are going to see as a method as a theorem. So, hypotheses consider the second order linear equation 2L, which is given by this. Assume that this equation is hyperbolic in an open set, region means open and connected set in open set Ω of the xy plane. So, throughout the domain Ω the equation is a hyperbolic type. Now, take a point x_0, y_0 in Ω that means, take a point in Ω .

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The screenshot shows a presentation slide with a video inset of the same man in an orange shirt. The slide title is "Theorem: Conclusion". It states the existence of an open set and a change of coordinates that transforms equation (2L) into a specific form.

Theorem: Conclusion

There exists an open set containing the point (x_0, y_0) and a change of coordinates $(x, y) \mapsto (\xi, \eta)$ such that the equation (2L) is transformed into an equation of the form

$$w_{\xi\eta} + D(\xi, \eta)w_{\xi} + E(\xi, \eta)w_{\eta} + F(\xi, \eta)w + G(\xi, \eta) = 0.$$

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Conclusion is there is an open set containing the point x_0, y_0 and a change of coordinates defined on that open set such that the equation 2L is transformed into this equation. If you notice here and compare with the equation that we saw in the last lecture, after change the variables we

had a, b, c . So, what is missing is a and c ? a and c are 0 and b equal to half because $2b$ is what appeared so, that is equal to half.

So, coefficient of $w_{\psi\eta}$ is 1, $w_{\psi\psi}$ and $w_{\eta\eta}$ do not appear in the equation. Of course, it does not look like the wave equation. That is understood. Wave equation is not like $u_{xx} + u_{yy} = u_t$ equals something wave equation is $u_{tt} - u_{xx}$. We will come back to that later.

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Proof of Theorem

Given that the equation is Hyperbolic in Ω . i.e., $b^2 - ac > 0$ on Ω .

- We know that at least one of the three functions a, b, c is non-zero at every point of Ω (How?).
- In the proposed canonical form, only mixed partial derivative appears.
- If $a(x_0, y_0) = c(x_0, y_0) = 0$, then definitely $b(x_0, y_0) \neq 0$. (Why?)
- But it is of **no help/use** since it does not guarantee that a, c are zero functions in a neighbourhood of (x_0, y_0) .

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So, given that the equation is hyperbolic, that is $b^2 - ac$ is positive at every point of Ω . We know that at least one of the three functions a, b, c is nonzero at every point, what happens if all of them are 0? PDE is not defined. It is not a second order PDE. So, we have to add this assumption whenever we consider even quasi linear equation or linear equation, we have to assume that none of the three functions a, b, c simultaneously vanish.

That is an important assumption which we forgot to add now we add. In the proposed canonical form, only mixed partial derivative appears we saw that. So, if a and c are 0 at the point x_0, y_0 then definitely b is nonzero, because if both a and c are 0, b^2 must be positive in particular b is nonzero. But it is of no use or help you may be thinking that a and c are 0. So, in the equation a c is not there what you have is only b of x_0, y_0 into u_{xy} .

And it is not $b \neq 0$, $y \neq 0$ is non zero divided by it. Yes, that is fine, you have a equation which looks like canonical form. but remember it is only at the point x_0, y_0 you cannot assert this will happen in open set containing x_0, y_0 because these conditions do not guarantee that if a function is 0 at some point, you cannot say to be 0 in the neighbourhood of that. You can see of course, for this if somebody is nonzero at a point if the function is continuous.

Then definitely you can say it will be nonzero in a neighbourhood. We have to place assumptions on a, b, c. We have to assume that minimum we had assumed that they are continuous functions, we will see as in when we see there is a need to introduce hypotheses on a, b, c we will do that.

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Proof of Theorem

Assume WLOG that $a(x_0, y_0) \neq 0$ or $c(x_0, y_0) \neq 0$

- If $a(x_0, y_0) = c(x_0, y_0) = 0$, then introduce a change of coordinates $(x, y) \mapsto (\tilde{x}, \tilde{y})$ where $\tilde{x} = X(x, y) = x + y$, $\tilde{y} = Y(x, y) = x - y$.
- Then the equation (2L) takes a form where the coefficients of $w_{\tilde{x}\tilde{x}}$ and $w_{\tilde{y}\tilde{y}}$ are non-zero at the point $\tilde{x} = \tilde{x}_0 = x_0 + y_0, \tilde{y} = \tilde{y}_0 = x_0 - y_0$.
- Thus we may assume that at least one of the two quantities $a(x_0, y_0)$ and $c(x_0, y_0)$ is non-zero.

Without loss of generality, assume that $a(x_0, y_0) \neq 0$. As a consequence, there exists an open set U containing (x_0, y_0) such that $a(x, y) \neq 0$ for all $(x, y) \in U$.

Assume a, b, c are continuous functions

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Now, we are seeing a assume without loss of generality this WLOG stands for without loss of generality. That a of x_0, y_0 is not equal 0 or c of x_0, y_0 is not equal to 0, because if they are 0, we can do something, and get at least one of them to be nonzero. So, if they are 0 indeed, then introduce a change of coordinates x, y , \tilde{x}, \tilde{y} given by \tilde{x} is given by as a function of x, y as $\tilde{x} = x + y$ and \tilde{y} is $\tilde{y} = x - y$.

Now, if you look at the equation that is that will result by this change of variables the equation we have $w_{\tilde{x}\tilde{x}}, \tilde{x}$ and $w_{\tilde{y}\tilde{y}}, \tilde{y}$ they are nonzero. At the point \tilde{x}_0 which is

given by $x_0 + y_0$ and this point is $x_0 - y_0$. Because we are talking about x tilde y tilde that coordinate system. So, x tilde $= x_0$ y tilde $= y_0$ that is a point y tilde $= x_0 - y_0$ that is the point.

So, x tilde y tilde what is the point precisely maybe this is what you want to write x tilde y tilde this point is $x_0 + y_0$ is a non point because x naught y naught are known as this point. So, therefore, we may assume that at least one of the two quantities is nonzero. Other if they are 0, we will exactly first implement this change of variables we will get to an equation. In that equation we have this nonzero.

Therefore, we can as well start with assuming that there is no loss of generality in assuming that a is not zero or c is not zero. So, let us assume that a is not 0 as $x_0 y_0$ is not 0, and we need to now assume that a is continuous. So, that there will be an open set in which a is not 0, containing that point. So, assume that a is continuous a, b, c are continuous functions.

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Recall: Change of variables

Suppose that we have a change of coordinates from (x, y) to (ξ, η) by

$$\xi = \varphi(x, y), \quad \eta = \psi(x, y);$$

$$x = \Phi(\xi, \eta), \quad y = \Psi(\xi, \eta).$$

A function $u(x, y)$ gets transformed to a function $w(\xi, \eta)$ and vice versa by

$$w(\xi, \eta) = u(\Phi(\xi, \eta), \Psi(\xi, \eta)),$$

$$u(x, y) = w(\varphi(x, y), \psi(x, y)).$$

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So, recall change your variables are going to look like this and the function u will become w in the new coordinate system and the connection between u and w is given by these equations.

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Proof of Theorem (contd.)

We know that under a change of coordinates the equation (2L) transforms to

$$Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_{\xi} + Ew_{\eta} + Fw + G = 0,$$

where

$$A(\xi, \eta) := (a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2) \Big|_{(x,y)=(\Phi(\xi,\eta), \Psi(\xi,\eta))} \quad (3a)$$

$$B(\xi, \eta) := (a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y) \Big|_{(x,y)=(\Phi(\xi,\eta), \Psi(\xi,\eta))} \quad (3b)$$

$$C(\xi, \eta) := (a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2) \Big|_{(x,y)=(\Phi(\xi,\eta), \Psi(\xi,\eta))} \quad (3c)$$

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And the second order linear equation transforms to this, and we are listed on A, B, C in the last lecture, we said the D, E, F, G does not matter but if needed, we can always compute. But our requirement now is that I want to make A and C as 0 and B as 1 by two are nonzero, that is good enough. Once B is nonzero, then I divide everything the entire equation with 2B so that I get one times w psi eta.

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Proof of Theorem (contd.)

- For proving the theorem, we need to find a system of coordinates (ξ, η) so that

$$A(\xi, \eta) = C(\xi, \eta) = 0.$$
- By invariance of hyperbolicity under change of coordinates, $B(\xi, \eta) \neq 0$.
- Thus we need to find φ, ψ satisfying the equations

$$a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 = 0,$$

$$a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = 0.$$
- Equations for φ and ψ are identical.
- Thus we need to solve for φ, ψ using only one equation.

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So, proving the theorem, we need to find a coordinate systems psi eta so, that A and C are 0. And we know that the type classification type of an equation does not change under change of coordinates. Once A and C is 0 B will be nonzero, because b square - ac is positive and, A C is 0. Therefore, b squared is positive that means B is nonzero. So, we need to find phi and psi

satisfying $a\phi^2 + 2b\phi\psi + c\psi^2 = 0$ and $a\psi^2 + 2b\psi\phi + c\phi^2 = 0$.

This is the expression we get. What are these? These are for starter PDEs nonlinear PDEs. Equations are identical if you see ϕ solves these are solved it does not matter is the same equation. So, ϕ and ψ solve the same equation. So, we need to solve for ϕ ψ using only one equation.

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Proof of Theorem (contd.)

- Note that the equation $a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0$ may be factorized as

$$\frac{1}{a} \left(a\phi_x + (b - \sqrt{b^2 - ac})\phi_y \right) \left(a\phi_x + (b + \sqrt{b^2 - ac})\phi_y \right) = 0.$$

- A function ϕ is a solution to the above equation whenever it satisfies either of the first order linear PDEs given by

$$a\phi_x + (b - \sqrt{b^2 - ac})\phi_y = 0,$$

$$a\phi_x + (b + \sqrt{b^2 - ac})\phi_y = 0.$$

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Now, we write this equation like this. If you multiply and divide with a you get back this. So, you can write like this. And we have a license to do this that is why we have shown in the beginning that a is not equal to 0 in that neighbourhood, some neighbourhood of $x = 0, y = 0$. Therefore, $1/a$ is meaningful. For this we have done that exercise. So, a function ϕ is a solution to the above equation.

As long as it satisfies either this or this any one of them if it satisfies then the product will be 0. If ϕ is such that this first part is 0 product is 0. Therefore, this is 0 and these are equations are not similar. I mean, they are similar but not exactly the same, because there is a minus sign here is a plus sign here. That is what helps us in getting ϕ and ψ . So, whenever find a solution to any of these two equations, it is a solution to this equation.

Because it is just a factorization of this, it is the same equation expressed differently. So, we choose ϕ to be a solution of the first equation from here and we choose ψ to be a solution of the second equation.

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Proof of Theorem (contd.)

- We choose φ to be a solution of $a\varphi_x + (b - \sqrt{b^2 - ac})\varphi_y = 0$, and ψ to be a solution of $a\psi_x + (b + \sqrt{b^2 - ac})\psi_y = 0$
- This choice would mean that

$$(\xi, \eta) = (\varphi(x, y), \psi(x, y))$$
 defines a coordinate change transformation.
- Each of the above equations may be solved using **the method of characteristics**.

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This choice means we have a change of coordinates, why is that? We have to look at the Jacobian and we will see that. So, each of these equations can be solved using method of characteristics, after these are first order linear partial differential equations. Because the a, b, c are known functions of x and y only ϕ_x and ϕ_y . So, these are linear equations.

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Proof of Theorem (contd.)

Solution of

$$a\varphi_x + (b - \sqrt{b^2 - ac})\varphi_y = 0$$

The system of characteristic ODEs given by

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b - \sqrt{b^2 - ac}, \quad \frac{dz}{dt} = 0.$$

Thus any solution φ of the above PDE is **constant along each of the base characteristic curves** $(x(t), y(t))$.

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Of course, how do you solve by characteristics method you are right equations for characteristics. So, characteristic ODEs are this. So, this equation look at particular third equation, it says dz by dt is 0. It means any solution is constant along the base characteristics any of the each of the base characteristic if you consider the function ϕ will be a constant on that.

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Proof of Theorem (contd.)

- Assume that $\phi(x, y) = k$ represents a one parameter family of solutions to the ODE

$$\frac{dy}{dx} = \frac{b(x, y) - \sqrt{b^2(x, y) - a(x, y)c(x, y)}}{a(x, y)},$$

which represent the base characteristic curves.

- On differentiating the equation $\phi(x, y(x)) = k$ w.r.t. x , we get

$$\phi_x(x, y(x)) + \phi_y(x, y(x)) \frac{dy}{dx}(x) = 0.$$

- From the last equation, we get

$$-\frac{\phi_x}{\phi_y}(x, y) = \frac{dy}{dx} = \frac{b(x, y) - \sqrt{b^2(x, y) - a(x, y)c(x, y)}}{a(x, y)}.$$

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Now, assume that ϕ of $x, y = k$ represents a one parameter family of solutions to this ODE. This ODEs is what? It is nonparametric expression of this base characteristic curves. On differentiating this equation ϕ of $x, y = k$ with respect to x . In other words, we are assuming that there is a solution hidden inside this of this ODE that is part of this assumption. So, we get this by chain rule and from here we can solve minus ϕ_x by ϕ_y has certain expression.

Why are we doing this? Because we want to show that the ϕ and ψ chosen the way that we are described will give a change of coordinates. So, we need to find what is ϕ_x by ϕ_y we have an expression. Now, we decided ψ will solve the other second equation. Therefore, the corresponding characteristics ODEs, is given by this. So, any solution as before because of this relation, it is constant along base characteristics. And what is the non parameter expression of base characteristics.

That is given by dy by dx equal to this $b + \sqrt{b^2 - ac}$ by a . So, these represent base characteristic curves as before differentiating ψ of x, y $x = k$. We end up with an expression for ψ x, y $x = \psi$ y equal to this.

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Proof of Theorem (contd.)

- Since $b^2 - ac > 0$, we get

$$\begin{vmatrix} \varphi_x(x,y) & \varphi_y(x,y) \\ \psi_x(x,y) & \psi_y(x,y) \end{vmatrix} \neq 0$$

in view of the expressions for $\frac{\varphi_x}{\varphi_y}$ and $\frac{\psi_x}{\psi_y}$.

- Thus $(\xi, \eta) = (\varphi(x,y), \psi(x,y))$ defines a coordinate transformation near the point (x_0, y_0) , by the inverse function theorem.
- Since the transformed equation **remains Hyperbolic**, $B(\xi, \eta) \neq 0$.
- Dividing the transformed equation by B gives an equation in the desired form. \square

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Now, $b^2 - ac$ is positive, therefore, this determinant is nonzero. So, what is this determinant? If we expand and because these expressions that we are derived for φ_x by φ_y and ψ_x by ψ_y I will just bring back see here $b^2 - ac \neq 0$. And if this is 0, for example, it will just be by a for both the cases, then then the Jacobian will be 0. So, since $b^2 - ac$ is nonzero, this slope is different from the other one where we had a minus sign here, with a minus sign that was for φ_x by φ_y .

So, go back and see that slide. That is the reason. Therefore, you have a change of coordinates. Because of the inverse function theorem, I am not writing all the details here after applying what do you get from inverse function theorem and formally conclude I am not doing that, because now we are experienced with writing such conclusions because in first order PDE we dealt with them very clearly.

Now, since the transformed equation remains hyperbolic, we observed if you set A and C are 0, B has to be nonzero. Dividing the transform the equation by B gives an equation that you wanted.

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Remark

- We may make a further change of variables $x' = \xi + \eta, y' = \xi - \eta$.
- The new transformed equation will look like

$$v_{y'y'} - v_{x'x'} + 2D'(x', y')v_{x'y'} + 2E'(x', y')v_{y'y'} + F'(x', y')v + G'(x', y') = 0. \quad (\text{Can.2})$$
- This is due to

$$A'(x', y') = 1, B'(x', y') = 0, C'(x', y') = -1.$$
- Both the equations

$$w_{\xi\eta} + D(\xi, \eta)w_{\xi} + E(\xi, \eta)w_{\eta} + F(\xi, \eta)w + G(\xi, \eta) = 0,$$
 and the equation (Can.2) above are known as **canonical forms for hyperbolic equations**.

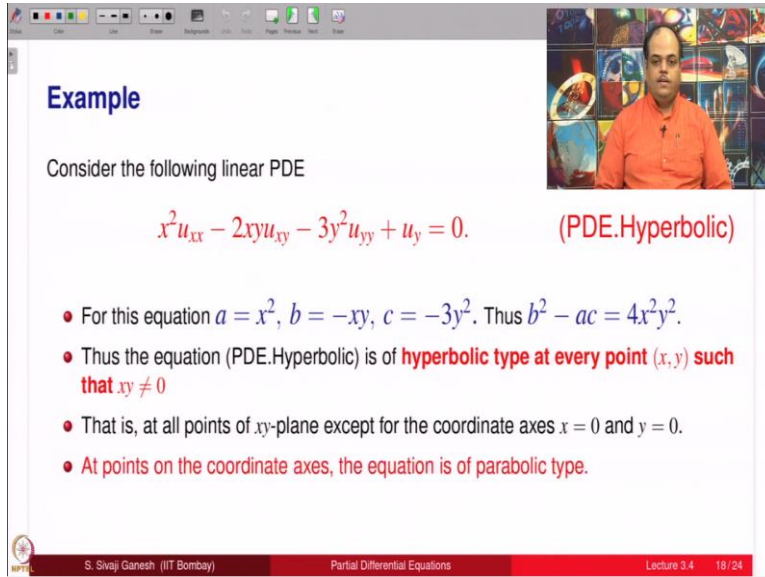
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We make one more change of variables. What is the equation that we got now? $w_{\xi\eta}$ plus something into w_{ξ} plus something into w_{η} plus something into w plus some function of $w_{\xi\eta}$ equals 0. Now, we are going to do another change of coordinate we write x' and y' as a function of ξ and η which is $\xi + \eta$ and these are functions ξ and η which is $\xi - \eta$. When we do this change of variables, we get a new transform the equation for v of x' and y' .

If you observe $v_{y'y'} - v_{x'x'} + 2D'(x', y')v_{x'y'} + 2E'(x', y')v_{y'y'} + F'(x', y')v + G'(x', y')$ suggests this precisely how the wave equation looked like $u_{tt} - u_{xx}$. This rest of the part features only first order derivatives or more derivatives, the second order derivative is involved only in these two terms and that part looks like the one in the wave equation. So, both the equations which we got which features only the mixed partial derivative $w_{\xi\eta}$ are the equation which we obtained here.

Both are admitted as canonical forms for hyperbolic equation. Of course, actually somebody has defined a canonical form with some other definition, but we need not give value to such definition because after all is just naming. What is admitted as class as canonical form? What is not admirable canonical form? It depends on what you idolized one. What do you model? What do you like? So, we will admit both of them as canonical forms no problem, we do not lose anything.

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Example

Consider the following linear PDE

$$x^2 u_{xx} - 2xy u_{xy} - 3y^2 u_{yy} + u_y = 0. \quad (\text{PDE.Hyperbolic})$$

- For this equation $a = x^2$, $b = -xy$, $c = -3y^2$. Thus $b^2 - ac = 4x^2y^2$.
- Thus the equation (PDE.Hyperbolic) is of **hyperbolic type at every point (x,y) such that $xy \neq 0$**
- That is, at all points of xy -plane except for the coordinate axes $x = 0$ and $y = 0$.
- **At points on the coordinate axes, the equation is of parabolic type.**

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Let us look at an example. I have already called this PDE as PDE dot hyperbolic instead of calling with some number and this equation is nothing very special equation it is just some second order PDE. Therefore, I call this already PDE hyperbolic because it turns out the equation is hyperbolic somewhere. So, these just name; do not give too much value to this name this maybe I could have call just PDE also.

So, for this equation what is a x square? What is $b - x y$? What is $c - 3 y$ square. So, what is b squared - ac $4x$ square y squared. It is always positive whenever $x y$ is nonzero because it is $2 xy$ whole square that is always positive when $x y$ is nonzero. Therefore, except the places where $x y$ is 0, this equation is hyperbolic and $xy = 0$, if and only if either x is 0 or y 0. That means, either you are on x axis or on y axis.

Outside this, which means in each of the four quadrants which do not include the axis the equation is hyperbolic. When such thing happens? Just note that on the axis it is parabolic because b squared minus ac is 0.

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Example (contd.)

Let us transform the equation (PDE.Hyperbolic) into its canonical form in the first quadrant.

In order to find the new coordinate system (ξ, η) , we need to solve the ODEs

$$\frac{dy}{dx} = \frac{b(x, y) \pm \sqrt{b^2(x, y) - a(x, y)c(x, y)}}{a(x, y)} = \frac{-xy \pm 2|xy|}{x^2} = \frac{-y \pm 2y}{x}$$

Thus we need to solve the two ODEs

$$\frac{dy}{dx} = \frac{y}{x}, \quad \text{and} \quad \frac{dy}{dx} = \frac{-3y}{x},$$

whose solutions are given by $x^{-1}y = \text{constant}$, and $x^3y = \text{constant}$ respectively.

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Now, we will transform the equation into canonical form in the first quadrant. We need to do that because in first quadrant we know what is x and y, x is positive y is positive. Because when we integrate certain ordinary differential equations logarithm will invariably come a logarithm mod x logarithm mod y these kinds of things and which can be uniquely fixed you know if you know where you are working. That is the reason why we determine in first quadrant.

Same in the other remaining quadrants also you can determine the canonical form. So, in order to find a new coordinate system, we need to solve this ODEs dy by $dx = b$ plus or minus root b square - ac by a . We substitute the values of a , b , c from our equation we get this. So, in other words, the two ODEs are this and this. These are very simple ODEs; you can solve them. And solutions of this are given by y by x equal to constant, solutions of the second ODE are given by y equal to constant, respectively.

At this point, do not worry that yes, this is not defined when x is 0. Of course, x is never 0 we are in the first quadrant and which constants do not worry which constants.

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Example (contd.)

We introduce the following change of coordinates

$$\xi = \varphi(x, y) = x^{-1}y, \quad \text{and} \quad \eta = \psi(x, y) = x^3y.$$

On differentiating the equations

$$u(x, y) = w(\varphi(x, y), \psi(x, y)) = w(x^{-1}y, x^3y)$$

w.r.t. x and y we obtain

$$\begin{aligned} u_x &= -x^{-2}yw_\xi + 3x^2yw_\eta, \\ u_{xx} &= x^{-4}y^2w_{\xi\xi} - 6y^2w_{\xi\eta} + 9x^4y^2w_{\eta\eta} + 2x^{-3}w_\xi + 6xyw_\eta, \\ u_{xy} &= -x^{-3}yw_{\xi\xi} + 2xyw_{\xi\eta} + 3x^5yw_{\eta\eta} - x^{-2}w_\xi + 3x^2w_\eta, \\ u_y &= x^{-1}w_\xi + x^3w_\eta, \\ u_{yy} &= x^{-2}w_{\xi\xi} + 2x^2w_{\xi\eta} + x^6w_{\eta\eta}. \end{aligned}$$

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Because finally, we see what we get look at this psi eta which is defined by this is a solution of the first ODE, this is a solution of the second ODE. And differentiate this equation connection between u and w , because we need to plug in the values of u_x , u_y , u_{xx} , u_{yy} etcetera in the PDE. So, we need to compute various derivatives. Please do the computations on your one pause the video compute the things that derivatives.

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Example (contd.)

- On substituting these values in the equation (PDE.Hyperbolic), we get

$$-16x^2y^2w_{\xi\xi} + 5x^{-1}w_\xi + x^3w_\eta = 0.$$

- The last equation can be written in the variables ξ, η completely, on expressing x and y as functions of ξ, η .
- Indeed we have

$$x = \Phi(\xi, \eta) = \sqrt[4]{\frac{\eta}{\xi}}, \quad y = \Psi(\xi, \eta) = \sqrt[4]{\xi^3\eta}.$$

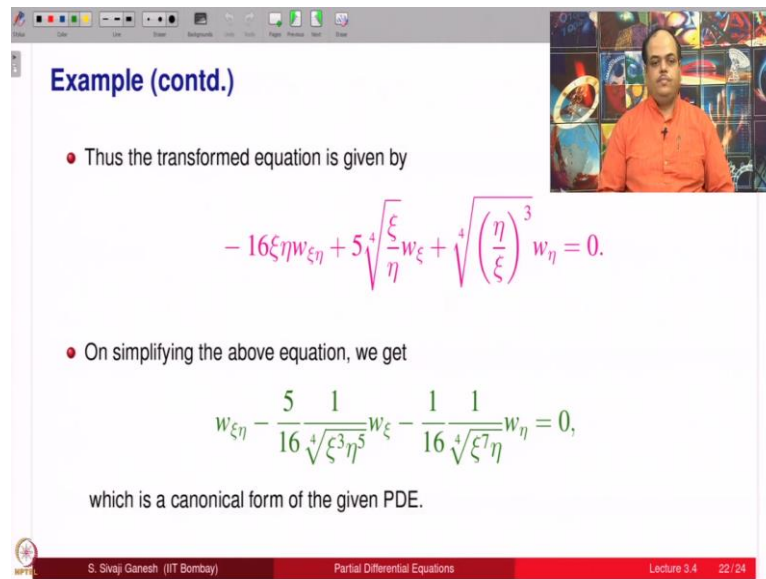
- Thus the transformed equation is given by

$$-16\xi\eta w_{\xi\xi} + 5\sqrt[4]{\frac{\xi}{\eta}}w_\xi + \sqrt[4]{\left(\frac{\eta}{\xi}\right)^3}w_\eta = 0.$$

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Now go back and substitute in the given PDE you get this. Of course, this is not a good equation because you see that there is a x, y here. We need to eliminate that also express everything in terms of psi eta. And then we have the formula for x and y in terms of psi eta take this go ahead and substitute, we get this.

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Example (contd.)

- Thus the transformed equation is given by

$$-16\xi\eta w_{\xi\eta} + 5\sqrt[4]{\frac{\xi}{\eta}} w_{\xi} + \sqrt[4]{\left(\frac{\eta}{\xi}\right)^3} w_{\eta} = 0.$$

- On simplifying the above equation, we get

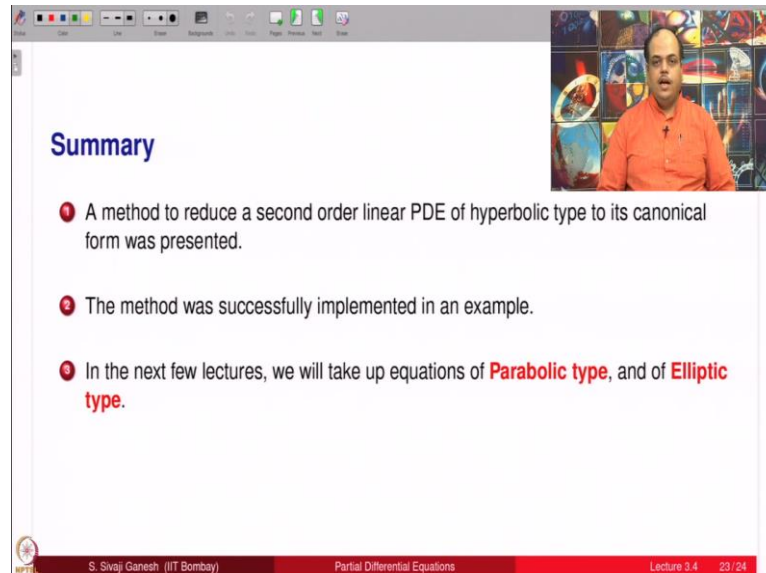
$$w_{\xi\eta} - \frac{5}{16} \frac{1}{\sqrt[4]{\xi^3\eta^5}} w_{\xi} - \frac{1}{16} \frac{1}{\sqrt[4]{\xi^7\eta}} w_{\eta} = 0,$$

which is a canonical form of the given PDE.

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So, on simplification we got this equation. Because we wanted $w_{\xi\eta}$ coefficient should be one side divided everywhere. So, that means we need the ξ and η are never 0. Now we can write down in which domain we have determined this particular canonical form.

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Summary

- 1 A method to reduce a second order linear PDE of hyperbolic type to its canonical form was presented.
- 2 The method was successfully implemented in an example.
- 3 In the next few lectures, we will take up equations of **Parabolic type**, and of **Elliptic type**.

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So, a method to reduce a second order linear PDE of hyperbolic type to its canonical form was presented. Of course, it was also successfully implemented in an example. In the next few lectures, I already mentioned we are going to take up the equations of parabolic type and of elliptic type and find canonical forms for them. Thank you.