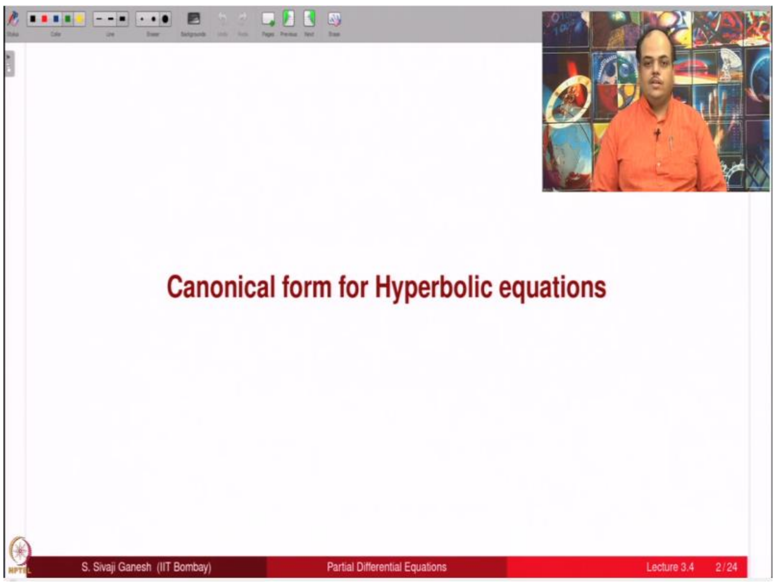
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Science, Bombay

Lecture – 21
Second Order Partial Differential Equations Canonical Form for an Equation of Hyperbolic Type

In this lecture, we are going to find the canonical form for an equation which is a hyperbolic type. In the next two lectures, we will be doing the same, but for equations which are parabolic or elliptic types.

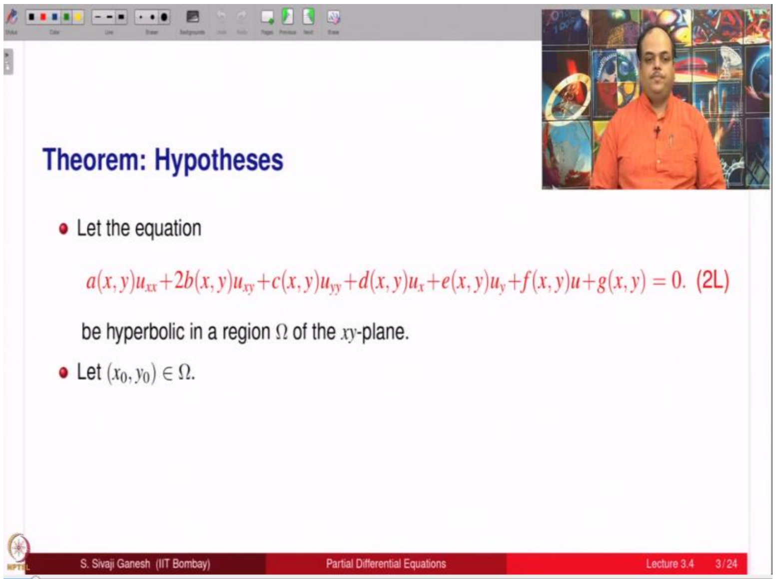
(Refer Slide Time: 00:35)



So, canonical form for hyperbolic equations. What is it? We know what is hyperbolic equation we have defined. What is the canonical form? If somebody asks you give me an example of hyperbolic equation immediately you give an example of the wave equation. So, therefore, finding canonical form for hyperbolic equations consists of the following; you take the equation which is hyperbolic in a domain and then you want to do change of coordinates.

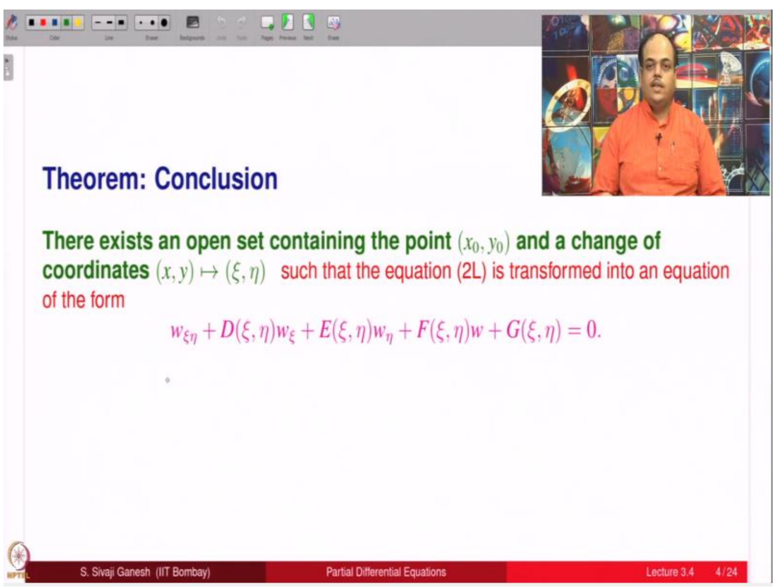
And then convert the PDE into the new coordinate system. But after doing this the part which features the second order derivatives looks like the wave equation, the one in the wave which. We are going to do the canonical forms only for linear equations for quasilinear equations, it is very difficult. So, we will not do that.

(Refer Slide Time: 01:25)



We state what we are going to see as a method as a theorem. So, hypotheses consider the second order linear equation 2L, which is given by this. Assume that this equation is hyperbolic in an open set, region means open and connected set in open set omega of the xy plane. So, throughout the domain omega the equation is a hyperbolic type. Now, take a point x 0, y 0 in omega that means, take a point in omega.

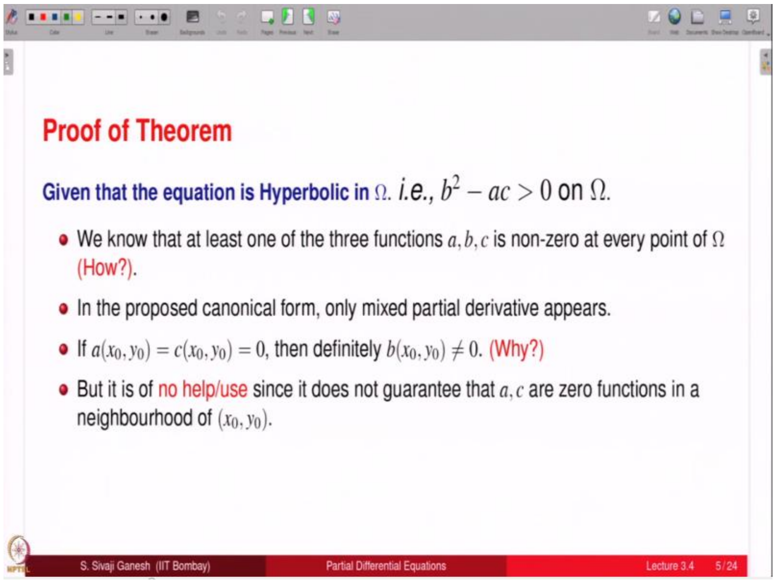
(Refer Slide Time: 01:59)



Conclusion is there is an open set containing the point x 0, y 0 and a change of coordinates defined on that open set such that the equation 2L is transformed into this equation. If you notice here and compare with the equation that we saw in the last lecture, after change the variables we had a, b, c. So, what is missing is a and c? a and c are 0 and b equal to half because 2b is what appeared so, that is equal to half.

So, coefficient of w psi eta is 1, w psi psi and w eta eta do not appear in the equation. Of course, it does not look like the wave equation. That is understood. Wave equation is not like u x u x t equals something wave equation is u tt - u xx. We will come back to that later.

(Refer Slide Time: 02:50)



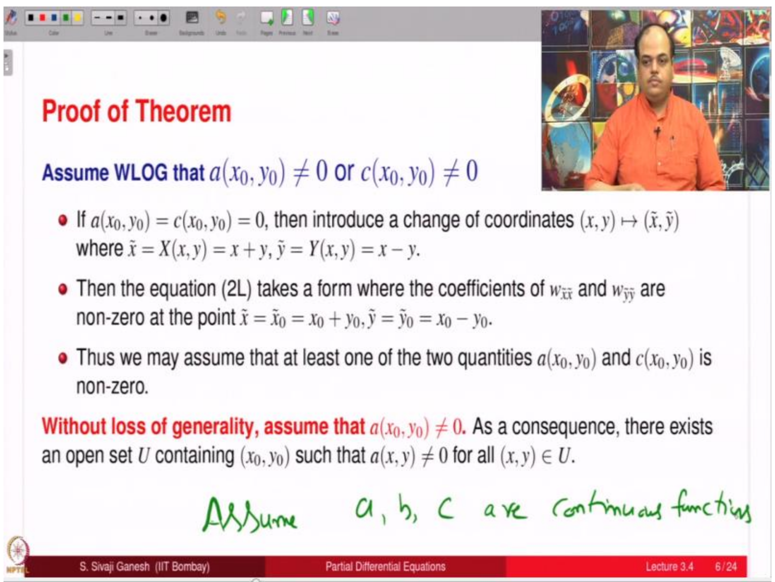
So, given that the equation is hyperbolic, that is b square - ac is positive at every point of omega. We know that at least one of the three functions a, b, c is nonzero at every point, what happens if all of them are 0? PDE is not defined. It is not a second order PDE. So, we have to add this assumption whenever we consider even quasi linear equation or linear equation, we have to assume that none of the three functions a, b, c simultaneously vanish.

That is an important assumption which we forgot to add now we add. In the proposed canonical form, only mixed partial derivative appears we saw that. So, if a and c are 0 at the point x 0, y 0 then definitely b is nonzero, because if both a and c are 0, b square must be positive in particular b is nonzero. But it is of no use or help you may be thinking that a and c are 0. So, in the equation a c is not there what you have is only b of x naught y 0 into u xy.

And it is not b x 0, y 0 is non zero divided by it. Yes, that is fine, you have a equation which looks like canonical form. but remember it is only at the point x 0, y 0 you cannot assert this will happen in open set containing x 0, y 0 because these conditions do not guarantee that if a function is 0 at some point, you cannot say to be 0 in the neighbourhood of that. You can see of course, for this if somebody is nonzero at a point if the function is continuous.

Then definitely you can say it will be nonzero in a neighbourhood. We have to place assumptions on a, b, c. We have to assume that minimum we had assumed that they are continuous functions, we will see as in when we see there is a need to introduce hypotheses on a, b, c we will do that.

(Refer Slide Time: 04:49)



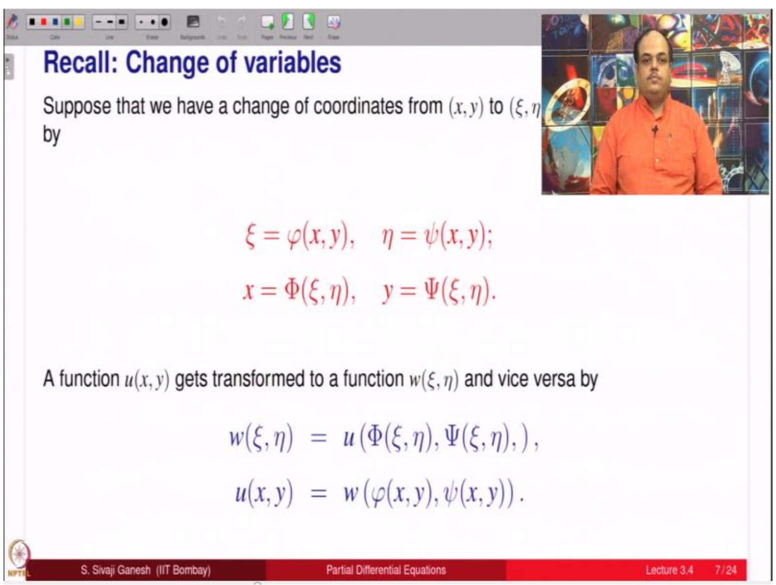
Now, we are seeing a assume without loss of generality this WLOG stands for without loss of generality. That a of x 0, y 0 is not equal 0 or c of x 0, y 0 is not equal to 0, because if they are 0, we can do something, and get at least one of them to be nonzero. So, if they are 0 indeed, then introduce a change of coordinates x y, x tilde y tilde given by x tilde is given by as a function of x, y as x + y and y tilde is x - y.

Now, if you look at the equation that is that will result by this change of variables the equation we have w x tilde, x tilde and w y tilde, y tilde they are nonzero. At the point x 0 tilde which is given by x 0 + y 0 and this point is x 0 - y 0. Because we; are talking about x tilde y tilde that coordinate system. So, x tilde = x 0 y 0 that is a point y tilde = x 0 - y 0 that is the point.

So, x tilde y tilde what is the point precisely maybe this is what you want to write x tilde y tilde this point is x 0 + y 0 is a non point because x naught y naught are known as this point. So, therefore, we may assume that at least one of the two quantities is nonzero. Other if they are 0, we will exactly first implement this change of variables we will get to an equation. In that equation we have this nonzero.

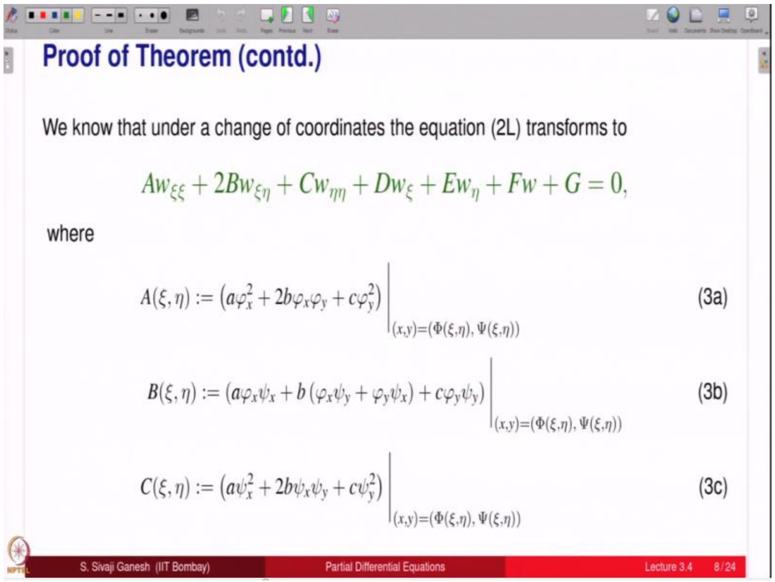
Therefore, we can as well start with assuming that there is no loss of generality in assuming that a is not zero or c is not zero. So, let us assume that a is not 0 as x 0 y 0 is not 0, and we need to now assume that is a is continuous. So, that there will be an open set in which a is not 0, containing that point. So, assume that a is continuous a, b, c are continuous functions.

(Refer Slide Time: 07:23)



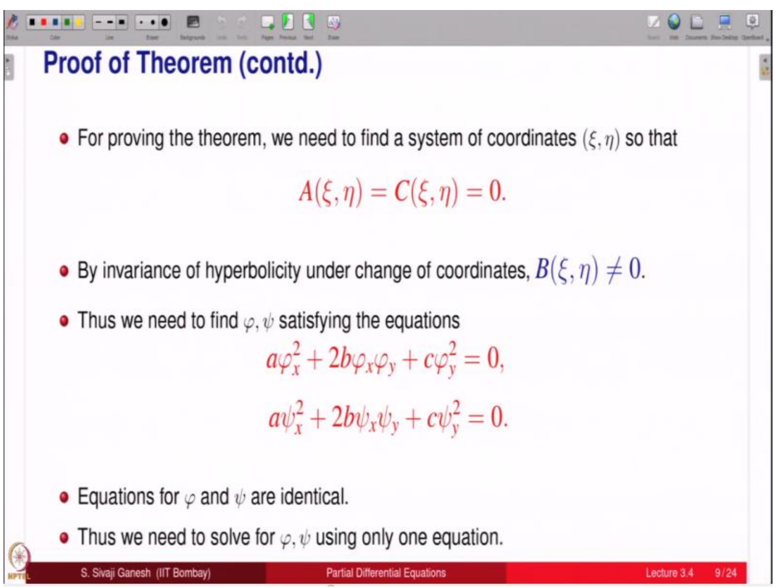
So, recall change your variables are going to look like this and the function u will become w in the new coordinate system and the connection between u and w is given by these equations.

(Refer Slide Time: 07:37)



And the second order linear equation transforms to this, and we are listed on A, B, C in the last lecture, we said the D, E, F, G does not matter but if needed, we can always compute. But our requirement now is that I want to make A and C as 0 and B as 1 by two are nonzero, that is good enough. Once B is nonzero, then I divide everything the entire equation with 2B so that I get one times w psi eta.

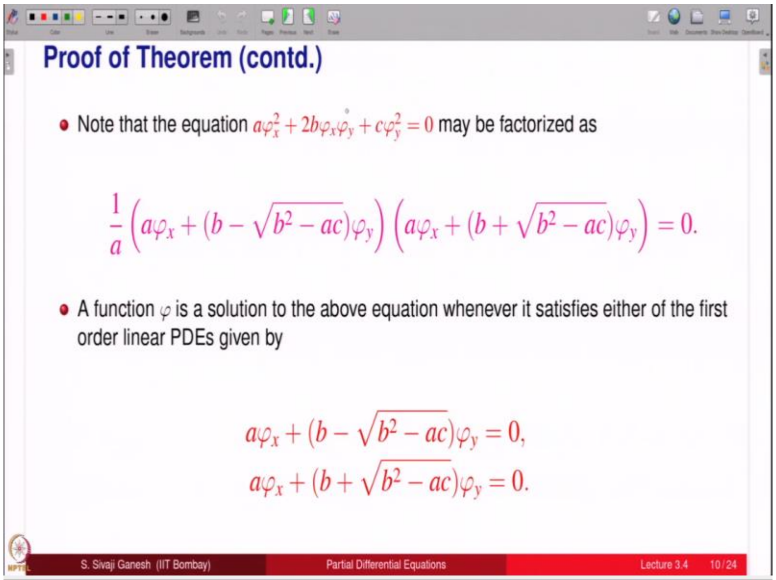
(Refer Slide Time: 08:09)



So, proving the theorem, we need to find a coordinate systems psi eta so, that A and C are 0. And we know that the type classification type of an equation does not change under change of coordinates. Once A and C is 0 B will be nonzero, because b square - ac is positive and, A C is 0. Therefore, b squared is positive that means B is nonzero. So, we need to find phi and psi satisfying a phi x square + 2b phi x phi y + c phi y square = 0 and a psi x square = 2b psi x psi y + c phi y square = 0.

This is c that is expression we get. What are these? These are for starter PDEs nonlinear PDEs. Equations are identical if you see phi solves these are solved it does not matter is the same equation. So, phi and psi solve the same equation. So, we need to solve for phi psi using only one equation.

(Refer Slide Time: 09:14)

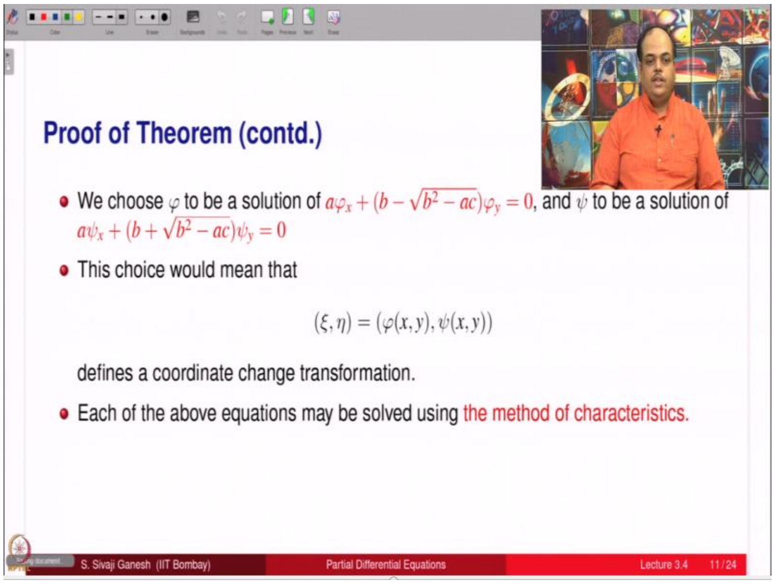


Now, we write this equation like this. If you multiply and divide with a you get back this. So, you can write like this. And we have a license to do this that is why we have shown in the beginning that a is not equal to 0 in that neighbourhood, some neighbourhood of x 0, y 0. Therefore, 1 by a is meaningful. For this we have done that exercise. So, a function phi is a solution to the above equation.

As long as it satisfies either this or this any one of them if it satisfies then the product will be 0. If phi is such that this first part is 0 product is 0. Therefore, this is 0 and these are equations are not similar. I mean, they are similar but not exactly the same, because there is a minus sign here is a plus sign here. That is what helps us in getting phi and psi. So, whenever find a solution to any of these two equations, it is a solution to this equation.

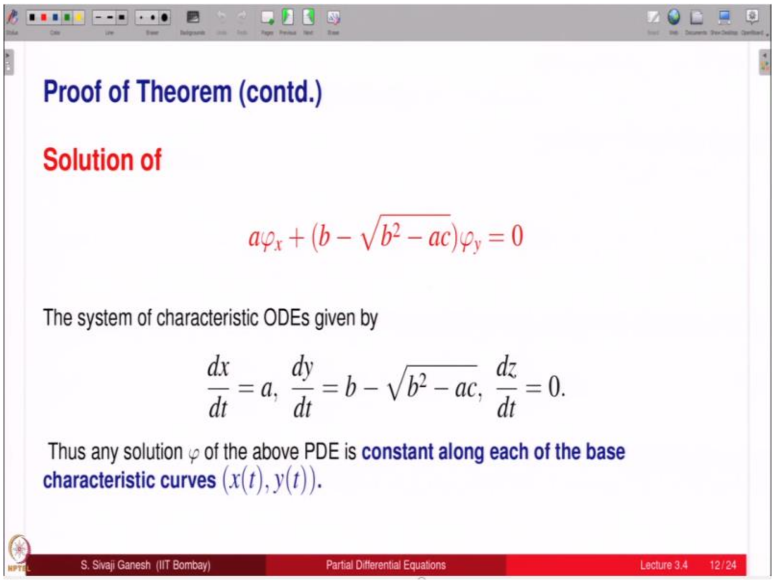
Because it is just a factorization of this, it is the same equation expressed differently. So, we choose phi to be a solution of the first equation from here and we choose psi to be a solution of the second equation.

(Refer Slide Time: 10:36)



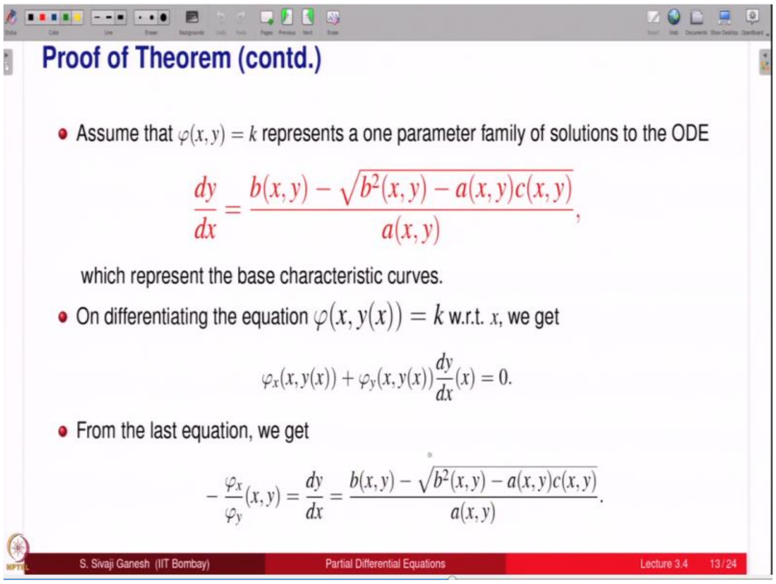
This choice means we have a change of coordinates, why is that? We have to look at the Jacobian and we will see that. So, each of these equations can be solved using method of characteristics, after these are first order linear partial differential equations. Because the a, b, c are known functions of x and y only phi x and phi y. So, these are linear equations.

(Refer Slide Time: 10:59)



Of course, how do you solve by characteristics method you are right equations for characteristics. So, characteristic ODEs are this. So, this equation look at particular third equation, it says dz by dt is 0. It means any solution is constant along the base characteristics any of the each of the base characteristic if you consider the function phi will be a constant on that.

(Refer Slide Time: 11:26)

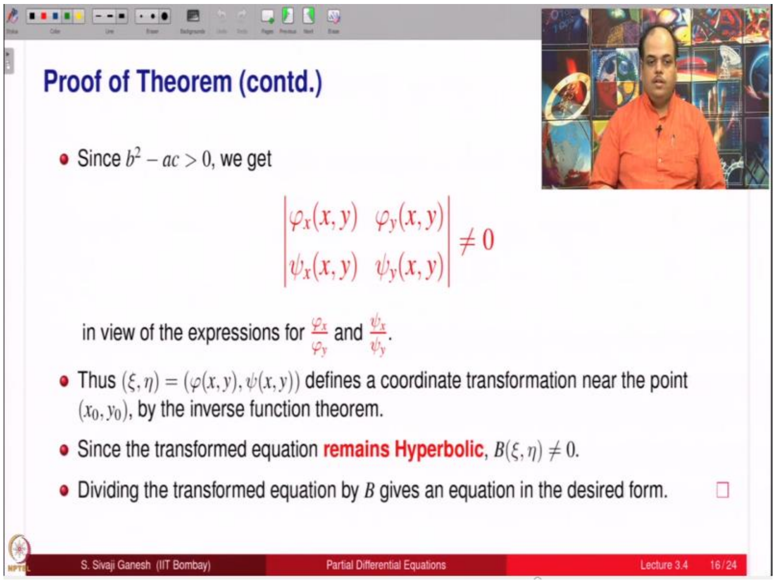


Now, assume that phi of x, y = k represents a one parameter family of solutions to this ODE. This ODEs is what? It is nonparametric expression of this base characteristic curves. On differentiating this equation phi of x, y x = k with respect to x. In other words, we are assuming that there is a solution hidden inside this of this ODE that is part of this assumption. So, we get this by chain rule and from here we can solve minus phi x by phi y has certain expression.

Why are we doing this? Because we want to show that the phi and psi chosen the way that we are described will give a change of coordinates. So, we need to find what is phi x by phi y we have an expression. Now, we decided psi will solve the other second equation. Therefore, the corresponding characteristics ODEs, is given by this. So, any solution as before because of this relation, it is constant along base characteristics. And what is the non parameter expression of base characteristics.

That is given by dy by dx equal to this b + root b square - ac by a. So, these represent base characteristic curves as before differentiating psi of x, y x = k. We end up with an expression for psi x, y x = psi y equal to this.

(Refer Slide Time: 13:04)

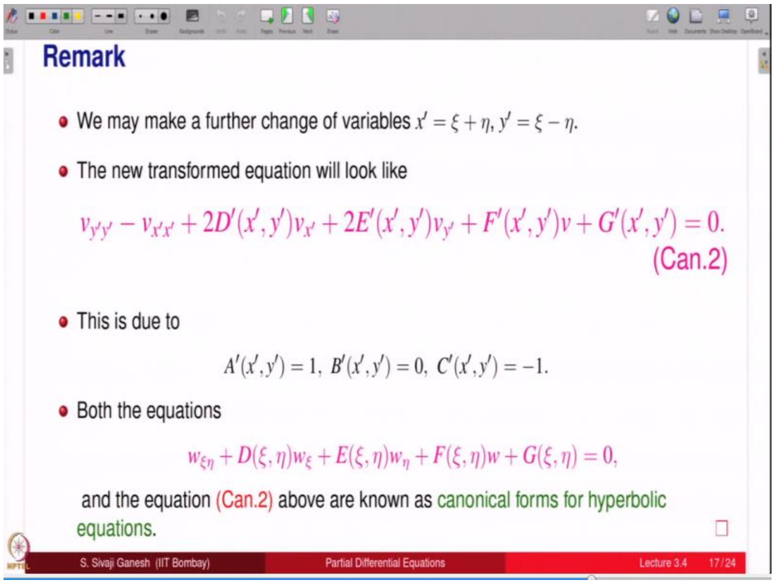


Now, b square - ac is positive, therefore, this determinant is nonzero. So, what is this determinant? If we expand and because these expressions that we are derived for phi x by phi y and psi x by psi y I will just bring back see here b square – ac non zero. And if this is 0, for example, it will just be by a for both the cases, then then the Jacobian will be 0. So, since b square - ac is nonzero, this slope is different from the other one where we had a minus sign here, with a minus sign that was for phi x by phi y.

So, go back and see that slide. That is the reason. Therefore, you have a change of coordinates. Because of the inverse function theorem, I am not writing all the details here after applying what do you get from inverse function theorem and formally conclude I am not doing that, because now we are experienced with writing such conclusions because in first order PDE we dealt with them very clearly.

Now, since the transformed equation remains hyperbolic, we observed if you set A and C are 0, B has to be nonzero. Dividing the transform the equation by B gives an equation that you wanted.

(Refer Slide Time: 14:22)

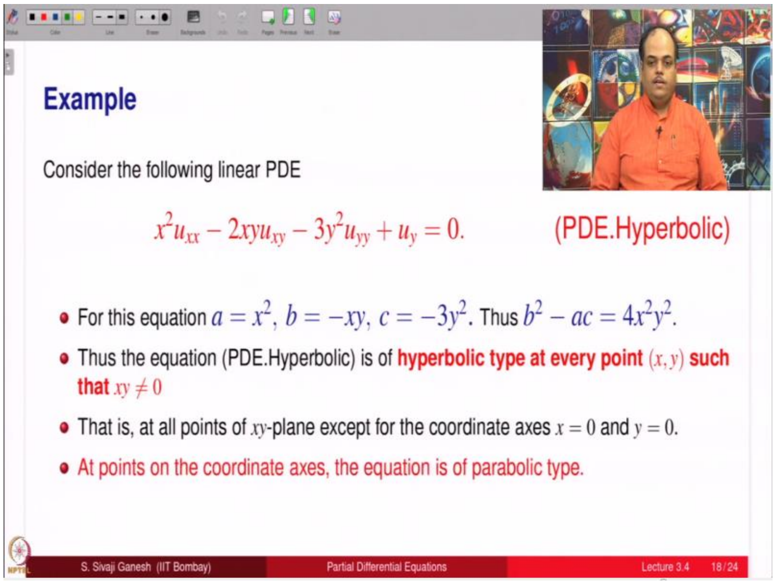


We make one more change of variables. What is the equation that we got now? w psi eta plus something into w psi plus something into w eta plus something into w plus some function of w psi eta equals 0. Now, we are going to do another change of coordinate we write x dash and y dash as a function of psi eta which is psi plus eta and these are functions psi eta which is psi minus eta. When we do this change of variables, we get a new transform the equation for v of x dash y dash.

If you observe v y dash y dash minus v x dash x dash suggests this precisely how the wave equation looked like u tt minus u xx. This rest of the part features only first order derivatives or more derivatives, the second order derivative is involved only in these two terms and that part looks like the one in the wave equation. So, both the equations which we got which features only the mixed partial derivative w psi eta are the equation which we obtained here.

Both are admitted as canonical forms for hyperbolic equation. Of course, actually somebody has defined a canonical form with some other definition, but we need not give value to such definition because after all is just naming. What is admitted as class as canonical form? What is not admirable canonical form? It depends on what you idolized one. What do you model? What do you like? So, we will admit both of them as canonical forms no problem, we do not lose anything.

(Refer Slide Time: 16:04)

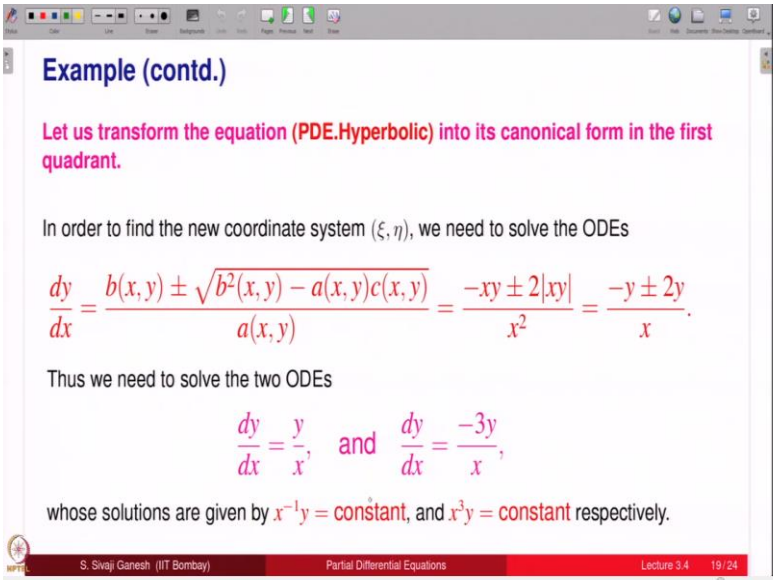


Let us look at an example. I have already called this PDE as PDE dot hyperbolic instead of calling with some number and this equation is nothing very special equation it is just some second order PDE. Therefore, I call this already PDE hyperbolic because it turns out the equation is hyperbolic somewhere. So, these just name; do not give too much value to this name this maybe I could have call just PDE also.

So, for this equation what is a x square? What is b - x y? What is c - 3 y square. So, what is b squared - ac 4x square y squared. It is always positive whenever x y is nonzero because it is 2 xy whole square that is always positive when x y is nonzero. Therefore, except the places where x y is 0, this equation is hyperbolic and xy = 0, if and only if either x is 0 or y 0. That means, either you are on x axis or on y axis.

Outside this, which means in each of the four quadrants which do not include the axis the equation is hyperbolic. When such thing happens? Just note that on the axis it is parabolic because b squared minus ac is 0.

(Refer Slide Time: 17:29)

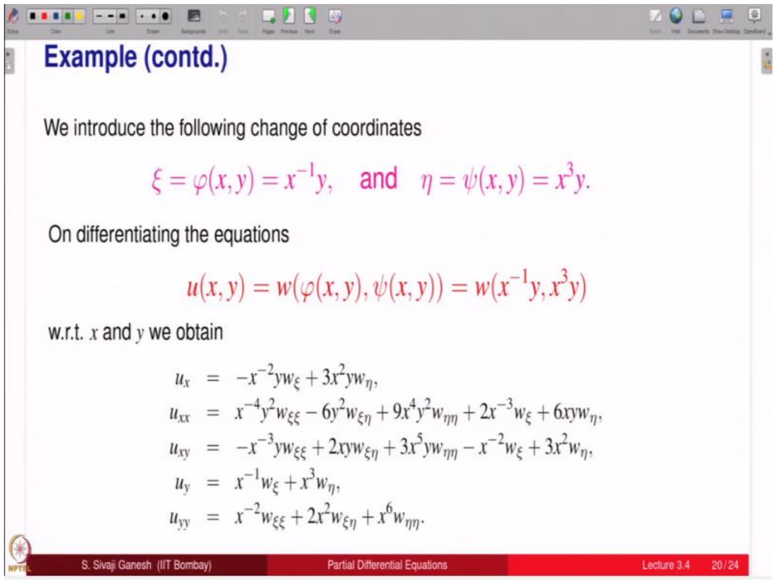


Now, we will transform the equation into canonical form in the first quadrant. We need to do that because in first continent we know what is x and y, x is positive y is positive. Because when we integrate certain ordinary differential equations logarithm will invariably come a logarithm mod x logarithm mod y these kinds of things and which can be uniquely fixed you know if you know where you are working. That is the reason why we determine in first quadrant.

Same in the other remaining quadrants also you can determine the canonical form. So, in order to find a new coordinate system, we need to solve this ODEs dy by dx = b plus or minus root b square - ac by a. We substitute the values of a, b, c from our equation we get this. So, in other words, the two ODEs are this and this. These are very simple ODEs; you can solve them. And solutions of this are given by y by x equal to constant, solutions of the second ODE are given by execute y equal to constant, respectively.

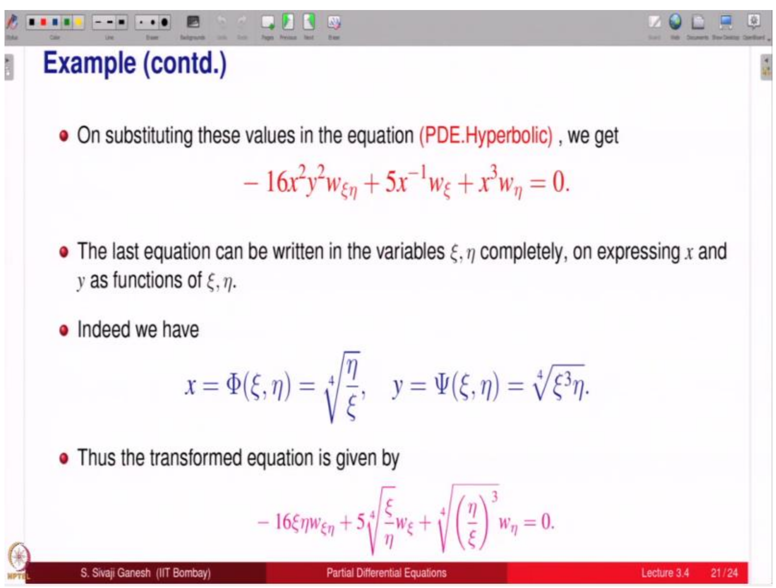
At this point, do not worry that yes, this is not defined when x is 0. Of course, x is never 0 we are in the first quadrant and which constants do not worry which constants.

(Refer Slide Time: 18:47)



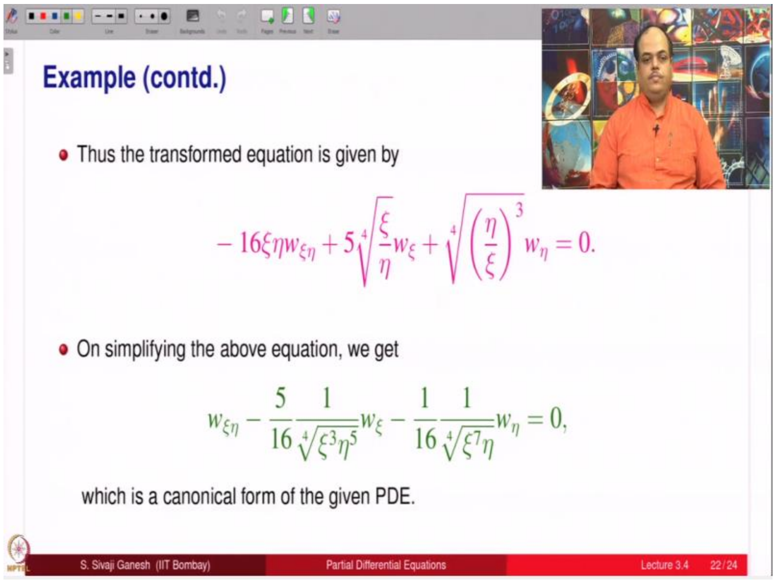
Because finally, we see what we get look at this psi eta which is defined by this is a solution of the first ODE, this is a solution of the second ODE. And differentiate this equation connection between u and w, because we need to plug in the values of u x, u y, u xx, u yy etcetera in the PDE. So, we need to compute various derivatives. Please do the computations on your one pause the video compute the things that derivatives.

(Refer Slide Time: 19:21)



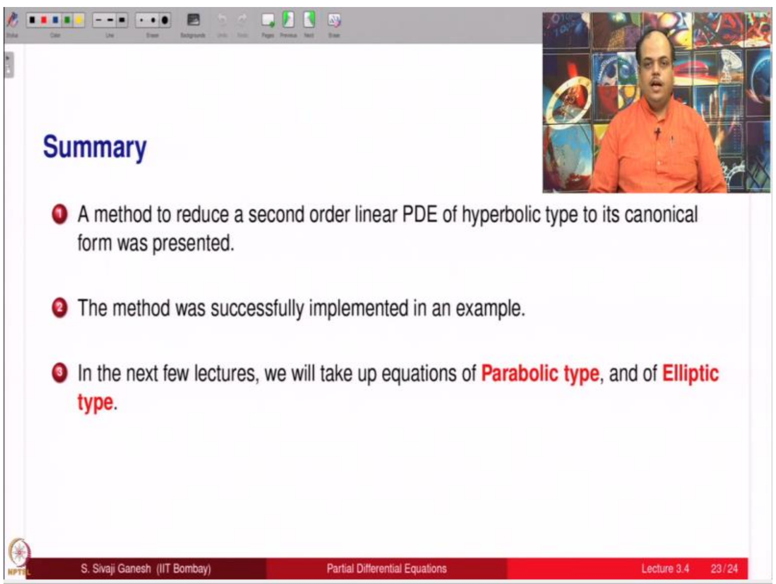
Now go back and substitute in the given PDE you get this. Of course, this is not a good equation because you see that there is a x, y here. We need to eliminate that also express everything in terms of psi eta. And then we have the formula for x and y in terms of psi eta take this go ahead and substitute, we get this.

(Refer Slide Time: 19:48)



So, on simplification we got this equation. Because we wanted w psi eta coefficient should be one side divided everywhere. So, that means we need the psi and eta are never 0. Now we can write down in which domain we have determined this particular canonical form.

(Refer Slide Time: 20:08)



So, a method to reduce a second order linear PDE of hyperbolic type to its canonical form was presented. Of course, it was also successfully implemented in an example. In the next few lectures, I already mentioned we are going to take up the equations of parabolic type and of elliptic type and find canonical forms for them. Thank you.