Partial Differential Equations
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Lecture – 20
Second Order Partial Differential Equations Classification

In this lecture, we are going to introduce classification of second order partial differential equations. It is also known as classification by characteristics.

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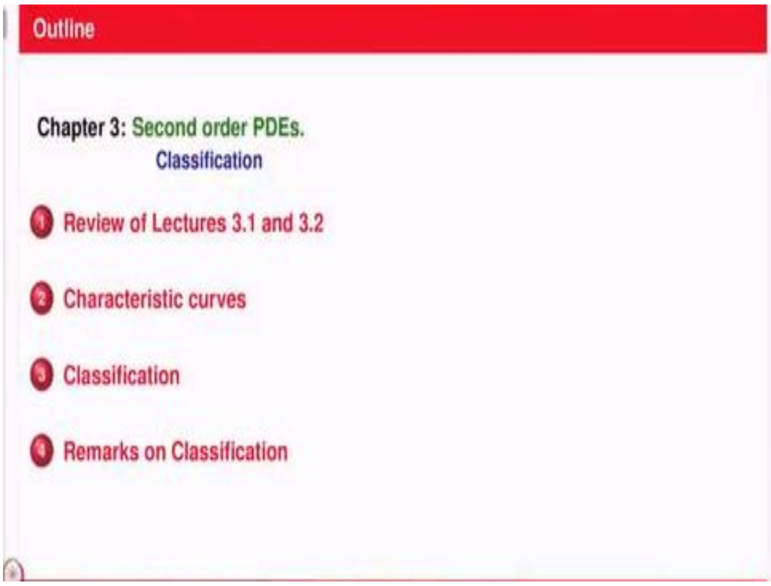

Outline of today's lecture is we start with a review of lectures 3.1 and 3.2, then we introduce define what is called characteristic curves. Using which you will define what is meant by classification of second order partial differential equations. In fact, we are going to classify second order quasi linear partial differential equations and then a few remarks on classification.

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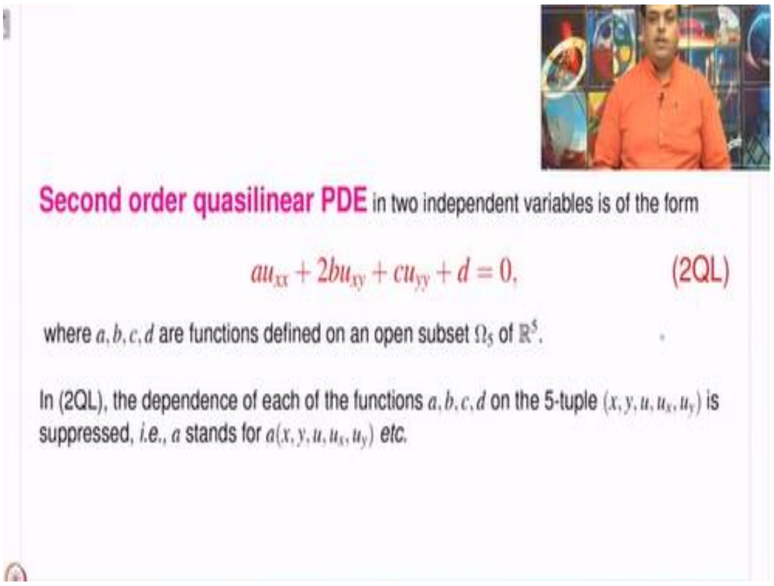

Recall the second order quasi linear equation we do not by to 2QL, it stands for this equation au xx + 2bu xy + cu yy + d = 0 coefficient a, b, c of the second order derivatives and d R functions of x, y, u, u x, u y that is why it is quasi linear. When the coefficient depend only on x, y particularly a, b, c depend only on x, y and d may depend on all the five variables here x, y, u, u x, u y. In that case 2QL is actually called a semi linear equation.

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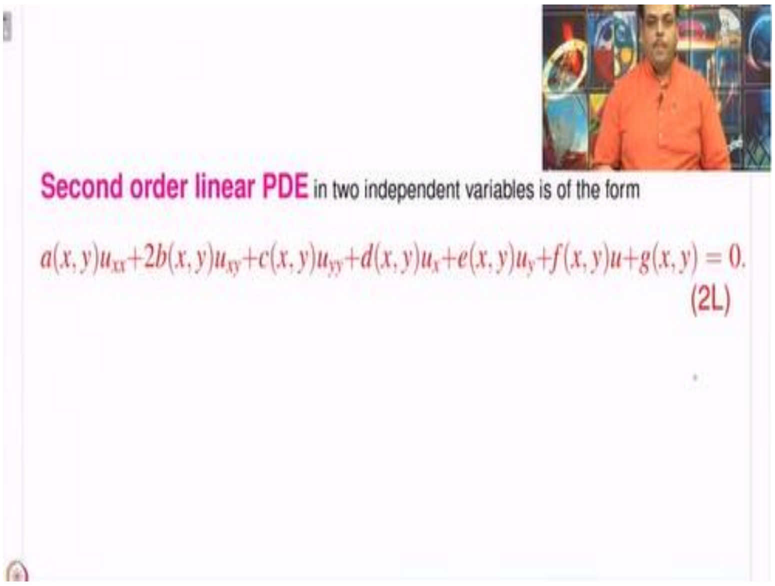

So, this is a linear equation which we will also be looking at. So, in this a, b, c naught depend only on x, y even d its dependence on u, u x, u y is linear u here, u x here, u y are here is a function of x and y equal to 0. So, this is second order linear PDE.

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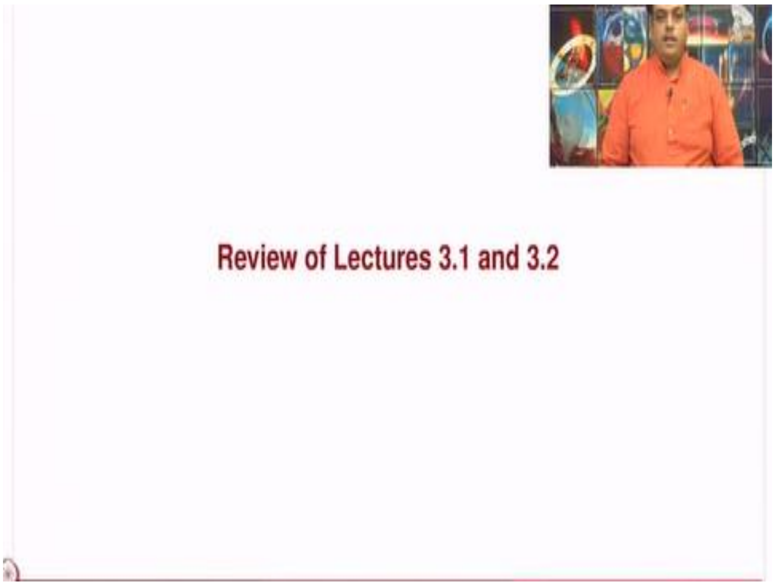

So, let us start with a review of the lecture 3.1, 3.2 what we did there?

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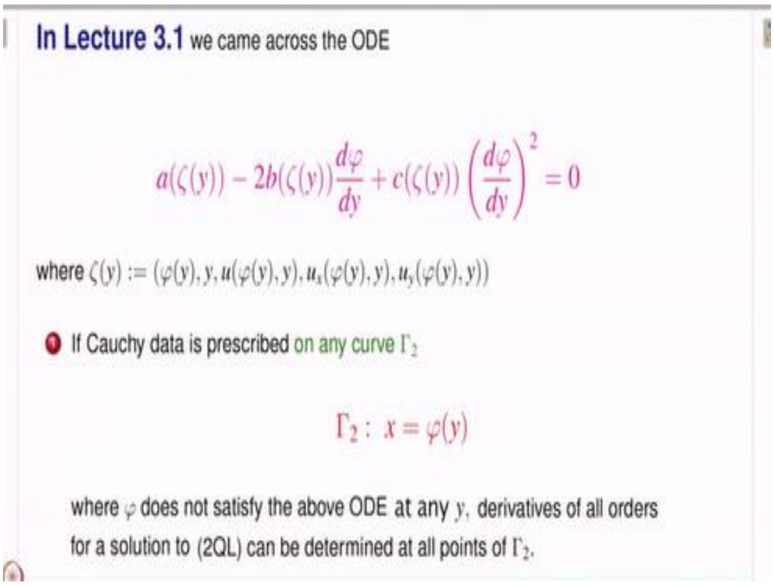

In lecture 3.1, we came across this ODE, a of zeta y - 2b zita y d phi dy + C of zita y d phi dy whole square equals 0. Where zeta y is phi tripple, phi y y, u of phi y, y u x at the point phi y, y, u y phi y, y. So, the Cauchy data is prescribed on any curve gamma 2 given by x = phi y, yes in lecture 3.1 to start with we considered a parameterized curve regular parameters curve gamma 2 given by x equal to fs and y equal to gs.

And later if the curve is given by any equation of this form, in other words it is a graph of function of y, x equal to phi of y then the equation that we got there then namely delta s = 0 reduces to this equation this was already observed. So, if the Cauchy data is prescribed on any curve gamma 2 in the plane given by x = phi y. The equation where phi does not satisfy the above ODE, derivatives of all orders for a solution to 2QL can be determined at all points of gamma 2.

Note here the zeta y of course involves the given solution u of the 2QL the quasi linear equation, second order quasi linear equation. So, given such a curve gamma to such that the phi does not satisfy the solution where u is we have to input the given function u. If it does not satisfy this equation, then we can determine all derivatives of all orders for the solution u can be determined at all points of gamma 2.

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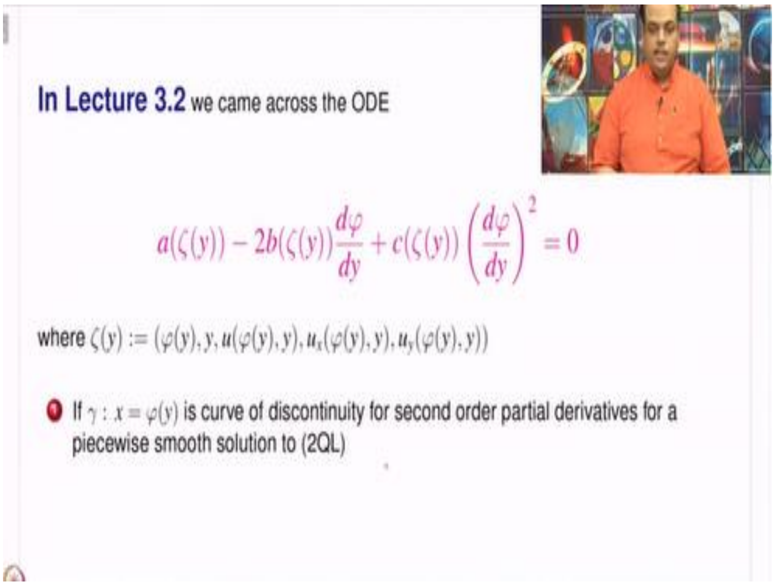

In lecture 3.2 also we came across the same ODE, I will not repeat the exactly same ODE. How did we get that? We assumed that a curve gamma given by x = phi y is a curve of discontinuity for second order partial derivatives for a piecewise smooth solution to the second order quasi linear equation. Then, phi solves the above ODE.

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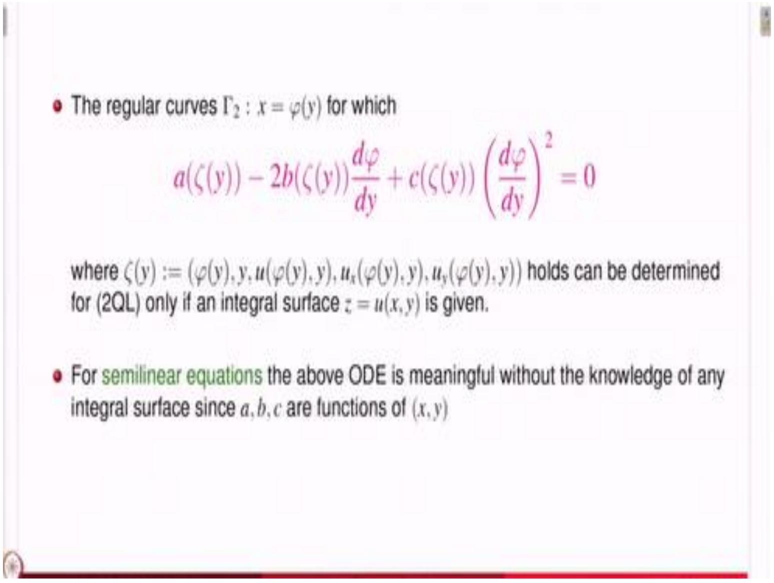

The regular curves for which this equation is satisfied they can be determined for a second order quasi linear equations, only if we are given a solution new. In other words, an integral surface z equals to u x, y is given is another way of saying that a solution u is given. Because only then the coefficients are determined and once coefficient are determined we can try to solve for phi satisfying this ODE.

For semilinear equations as we observe A does not depend on all the five quantities namely x, y u, u x, u y, it just depends on x y same thing is with the b and c. So, all the a, b, c functions that depend only on x and y. Which means what? This u knowledge of u is not required. Therefore, this ODE is meaningful without the knowledge of any integral surface. Since a, b, c are functions of x, y only.

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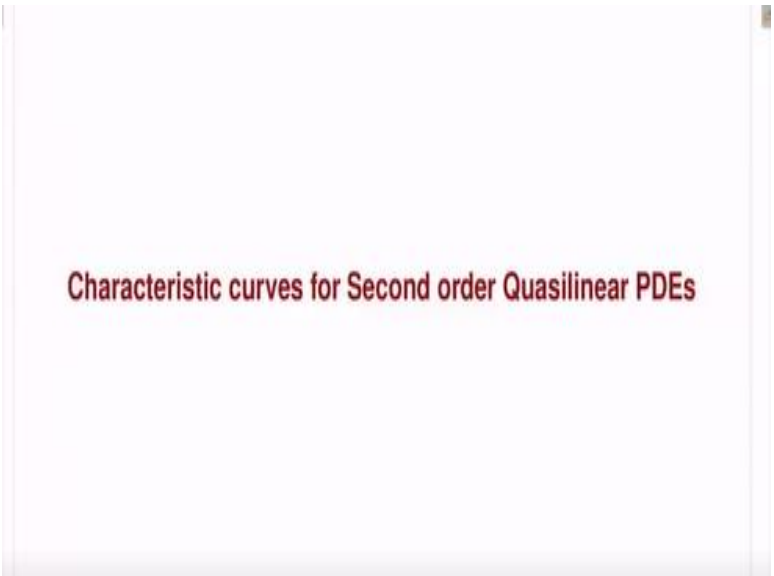

Now, let us introduce characteristic curves for second order quasilinear PDEs.

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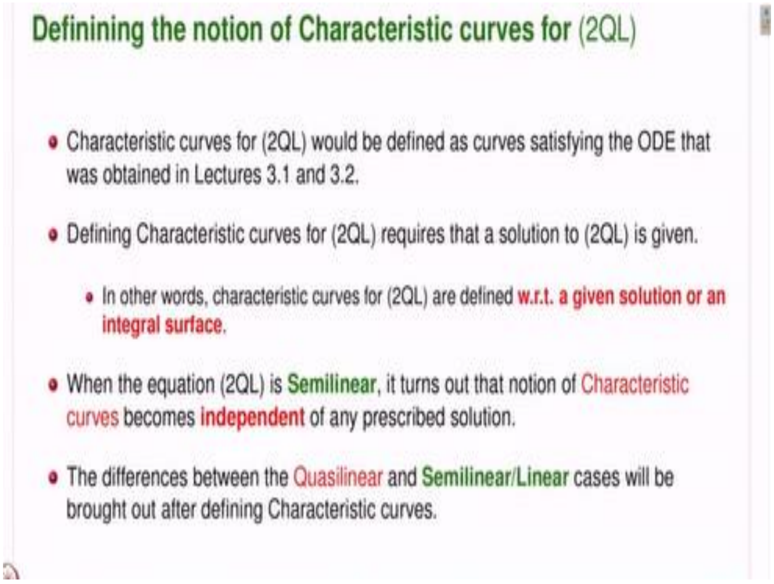

So, defining the notion of characteristic curves for 2QL, they will be defined as curves satisfying the ODE that was obtained in lecture 3.1, 3.2 the equation that we saw just now. Defining characteristics curve for 2QL requires that a solution of 2QL is given. In other words, characteristic curves for 2QL are defined with respect to a given solution or an integral surface. So, when the equation 2QL is actually semi linear that happens in that case a, b, c will be functions of x, y only.

It turns out that the notion of characteristic curves becomes independent of any prescribed solution, we do not require any solution to be prescribed. The ODE itself determines the characteristic curves and the differences between the quasi linear and semi linear cases will be brought out once we finish defining the characteristic curves.

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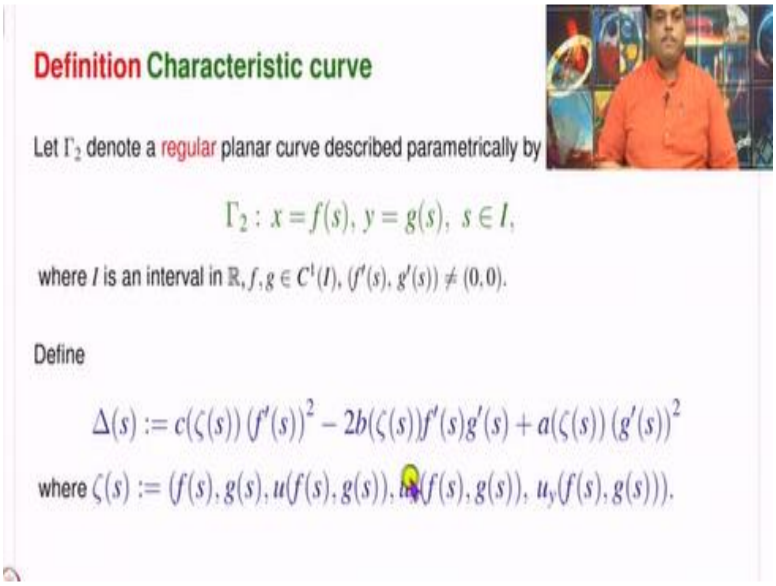

So, definition of a characteristic curve. We consider a regular planar curve described parametrically by x equal to fs y equal to gs and s belongs to some interval I and f and g are C 1 functions and I want to capture the word regular. What does regular means? Both f dash and g dash cannot vanish simultaneously. So, we consider a regular parametrized planar curve. Define delta of s by this expression.

We have zeta s equal to this note you need to know u, u x and u y. So, given a u we can define zeta s. In the definition of characteristic curve, we will see that u has to be defined as a solution of the 2QL equation then this is well defined. So, delta of s is denoted by this.

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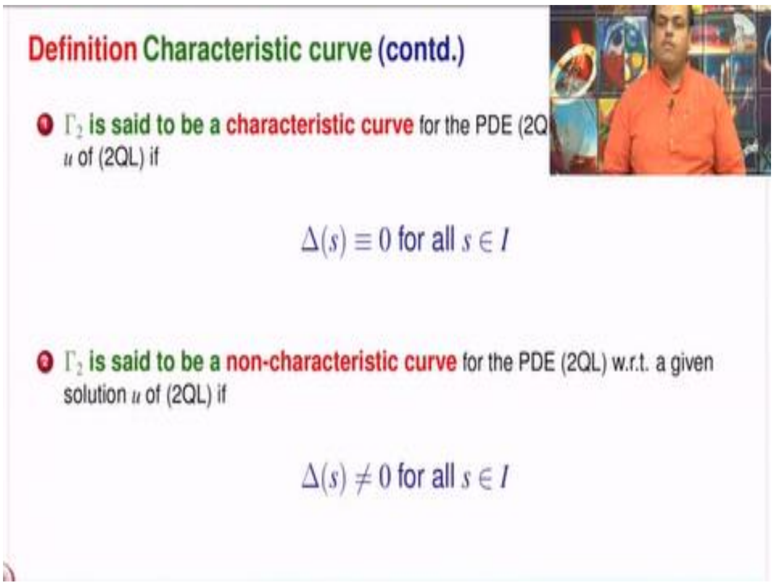

Now, the definition gamma 2 is said to be a characteristic curve for the PDE second order quasi linear equation 2QL with respect to a given solution of 2QL. If delta s is identically equal to 0 for all s and I. At every point on the curve delta s is 0. It is said to be a non-characteristic curve, it is exactly opposite of what is a characteristic curve. It is a non-characteristic curve means at no point delta s equals 0 is satisfied.

Of course, because we are dealing with 2QL, we have to write non characteristic curve for the PDE with respect to a given solution. If delta s is not equal to 0 for every s E I.

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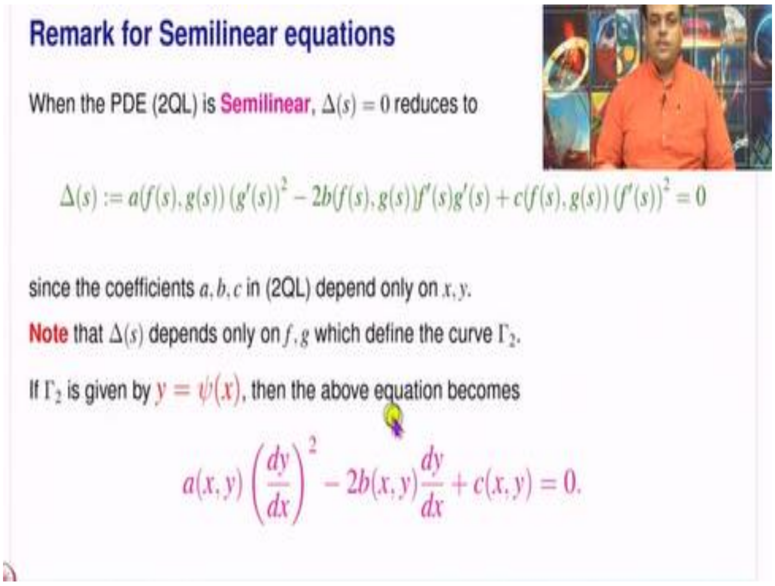

So, when the PDE is actually semi linear, then delta s equal to 0 reduces to this. The zeta s has gone, zeta s is actually fs and gs in this case. Because the a, b, c are functions of x, y only. Therefore, delta s depends only on f and g. What is f and g? They are the ones who are describing the curve gamma 2. The curve gamma 2 is parameterized using f and g. So, if gamma 2 is given by y equal to psi x, what will happened to this equation?

It will be this, psi will be a solution to this ordinary differential equation it is a quadratic in dy by dx. So, it is a first order ordinary differential equation but degree 2.

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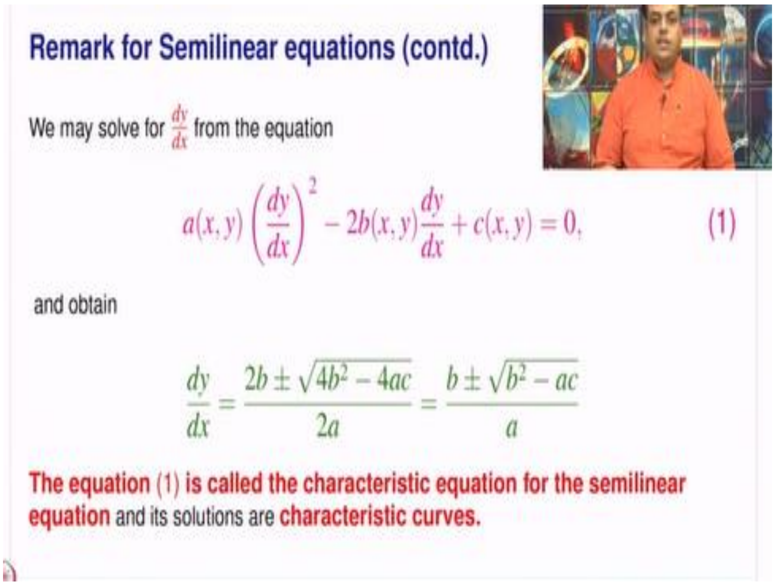

So, we may solve for dy by dx from this equation and get this expression dy by dx equal to b plus or minus root b square minus ac by a. The moment we see a square root we are to become alert. Therefore, if this is negative, then there are no real solutions for this equation for dy by dx. If it is positive b square - ac is positive, then we have two different ODEs given by this one corresponding to plus one corresponding to minus.

And this is equal to 0 we only have one ODE. So, the equation one which is given above is called a characteristic equation for the semi linear equation, and it solutions are characteristic curves.

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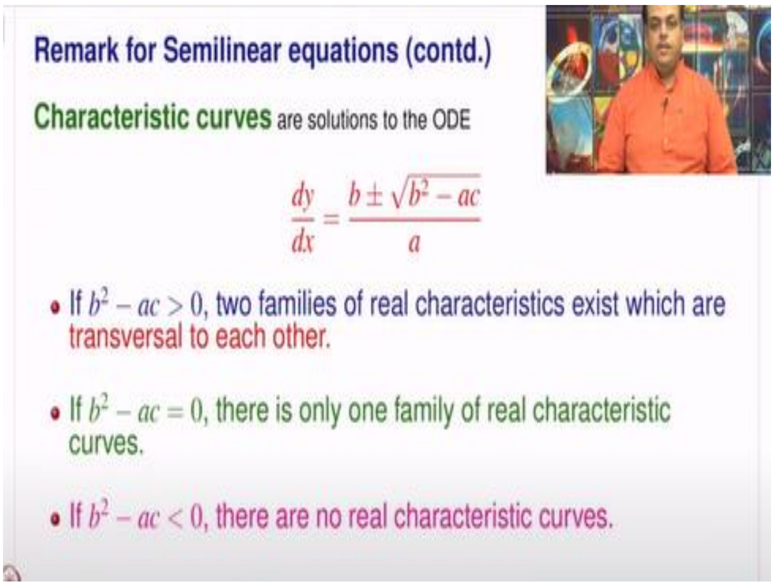

So, characteristic curves are solutions to this ODE, infact two ODE if b square - ac is positive, one ODE if b square - ac is 0 and there is no ODE real value if b square - ac is negative. So, in which case there will be no characteristic curves. So, if b square - ac is positive, two families of real characteristics exists which are transversal to each other, because the slope at any point will any point of intersection will be one will have slope b + route b square - ac by a.

Other one we have b - root b square - ac by a both are different. They are one and the same, if and only b square - ac 0. If b square - ac is 0 there is only one family of real characteristic curves because there is only one ODE here. If b square - ac is negative, there are no real characteristic curves.

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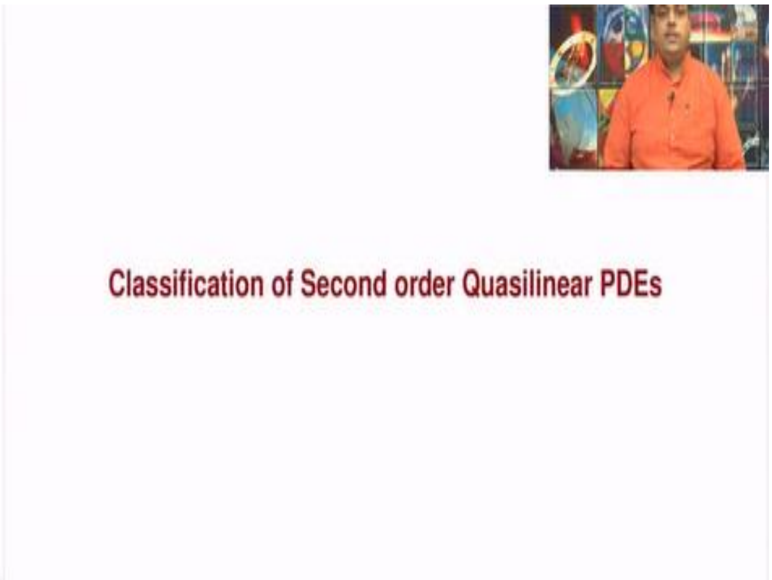

So, this is what leads to classification of second order equations.

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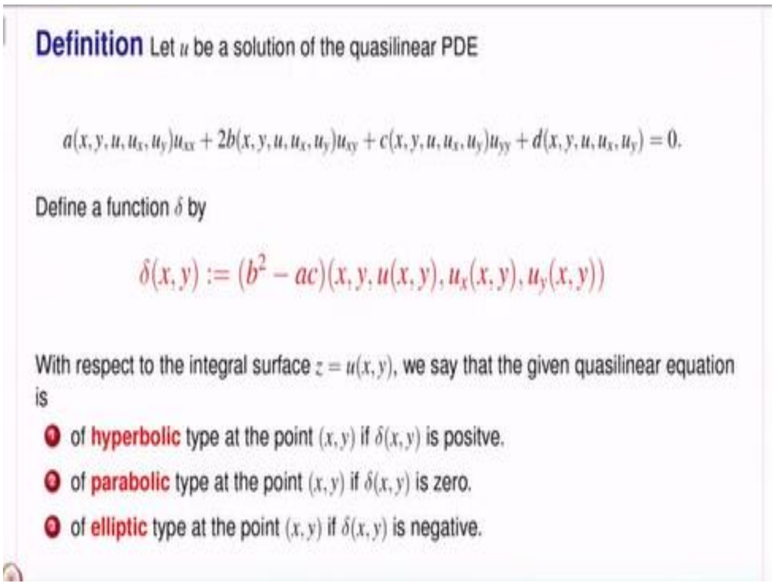

So, let u be a solution of the quasi linear PDE here, define a function delta actually it is b square minus ac at this point x, y, u x, y, u x of x, y, u y of x, y that is delta at the point x, y. So, with respect to the given integral surface that equal to x, y, this delta makes sense now. It is a function of x, y because you have already given to us. Therefore, all the members of this five triple are known at any point x, y.

We say that the given equation is a hyperbolic type at the point x, y if delta x, y is positive. It is a parabolic type if delta x, y = 0. It is an elliptic type if the delta of x, y is negative at this point remember hyperbolic type at the point x, y, parabolic type at the point x y, elliptic type at the point x, y is very important. Equation can change types from point to point we will see such examples also.

So, if you want to define classification only for semi linear equations, we need not start with a solution of the quasilinear PDE, we say consider the semi linear PDE a of x, y u x x, 2b of x, y u x y + c of x, y, u y y + d of the same thing x, y, u x y = 0. Define delta f x, y = b square - ac at the point x, y and exactly the same definition. So, normally people want to avoid this kind of unpleasant thing that you have to say all the time.

That you have to fix a solution to the quasilinear PDE then only classification is defined people simply start with semilinear equations, where you need not mentioned this. But it is of relevance even for quasilinear equations. Therefore, we are presenting it here.

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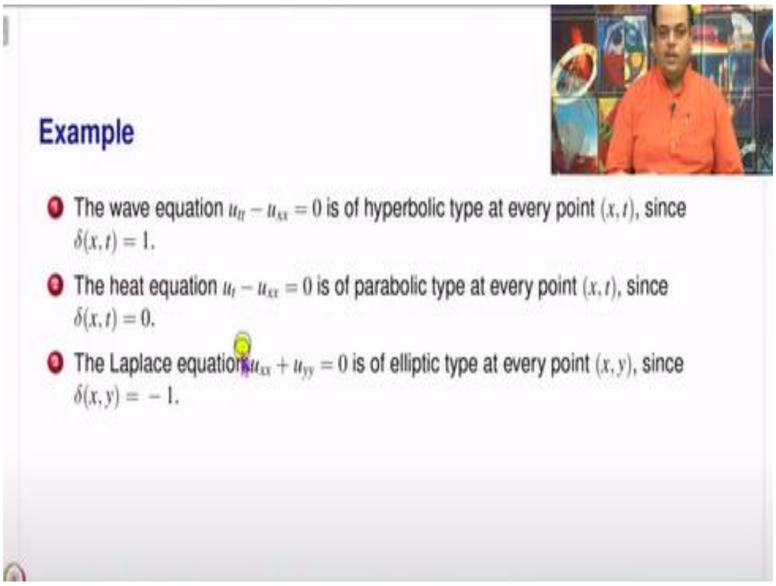

Let us look at some examples, look at this equation u tt - u xx = 0 is called a wave equation. It is a hyperbolic type at every point x, t. See here notice the coefficients here are constants. Therefore, if you go back to the definition, if a, b, c are constants b square - ac will be constant. Therefore, given any equation it will be exactly one type at all the points. So, here what is delta of x t? b square - ac b is 0 here a is minus one c is one.

Therefore, delta is 1, this equation u t - u xx = 0 is called heat equation, it is a parabolic type at every point. Since delta x, t here b and c are 0 only a is there, a is minus 1. Therefore, delta is 0, b square - ac is 0. The Laplace equation u xx + u yy = 0 is called Laplace equation. It is an elliptic type at every point because delta of x, y here b is 0 a and c are 1. So, it is minus 1, delta is minus 1.

In fact these three equations are the ideal equations, they are the simplest examples of each of these types of equations wave equation is a canonical candidate for hyperbolic equation. Whenever you think of hyperbolic equation second order you think of a equation. Similarly, for a parabolic equation if you want to think of it is a heat equation, elliptic equation means Laplace equation.

Therefore, given any other second order partial differential equation where a, b, c are not constants. We will ask the following question is it possible to do a change of coordinates? So, that after changing the coordinates at least the part which involves second order derivatives, we have constant cooffecients and looks like this for a hyperbolic equation, looks like this for a parabolic equation, looks like this for a elliptic equation. These are questions that we will ask later on.

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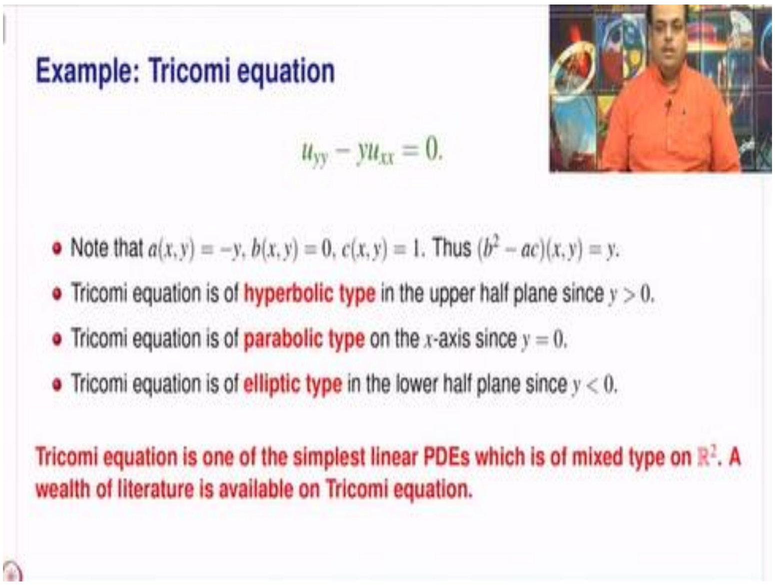

These are very important example Tricomi equation, this is also very simple equation u yy- yu xx = 0. But these are not with constant coefficients, here a of x, y is minus y, b of x, y is 0 because u x, y is absent, c of x, y is 1. So, b square - ac is y and b square - ac can become positive can become negative can become 0. So, that is why this equation will have all the three properties of course at different points.

So, Tricomi equation is of hyperbolic type in the upper half plane, because that is where y is positive, parabolic type on the x axis because y = 0 is the equation of the x axis. It is of elliptic type in the lower half plane, because in the lower half plane, the y coordinate is less than 0 negative. So, Tricomi equation is one of the simplest linear partial differential equations, which is of mixed type on R 2.

In fact, not only in R 2 take any domain, any domain which contains x axis, and therefore, it contains some part of upper half plane as well as some part of the lower half plane of course, some part of the x axis. So, it will have all the three types the equation will be of all the three types. Therefore, there will be natural questions, what kind of problems I should be able to solve for these kinds of equations.

We will not be discussing Tricomi equation as such in this course, in fact, no mixer type equation will be discussed a wealth of literature is available on Tricomi equation.

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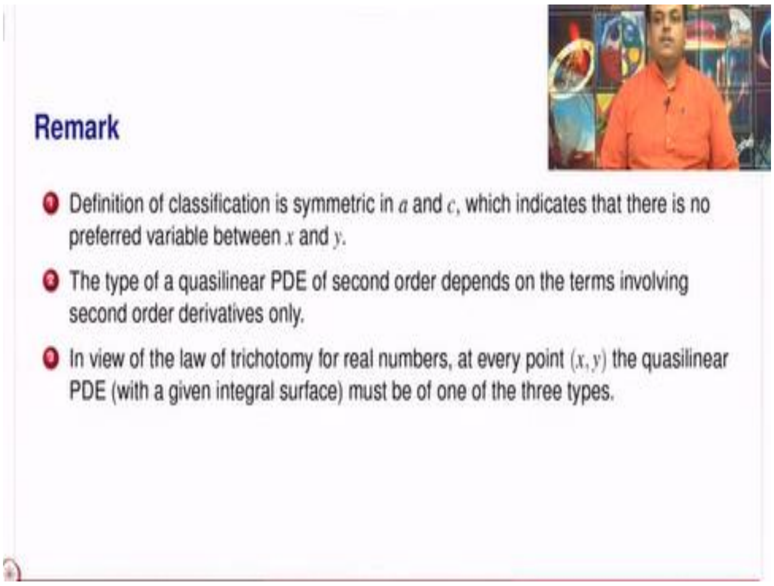

Now, definition of classification is symmetric in a and c. Why? The definition is based on b square – ac, a is a coefficient of u xx, c is the coefficient of u yy, but b square - ac will remain the same even if a and c are interchanged. So, it just says that there is no preferred variable between x and y. The type of a quasi linear PDE of second order depends on the terms involving second order derivatives only, because it is involving b square - ac, namely the function a b c.

What are a, b, c? These are the coefficients of the second order derivatives in the partial differential equation. Of course, in any given the real number has to be positive, negative or 0, that is what is called law of trichotomy. Therefore, a quasilinear PDE always be able to say with a given integral surface, it will be one of the three types.

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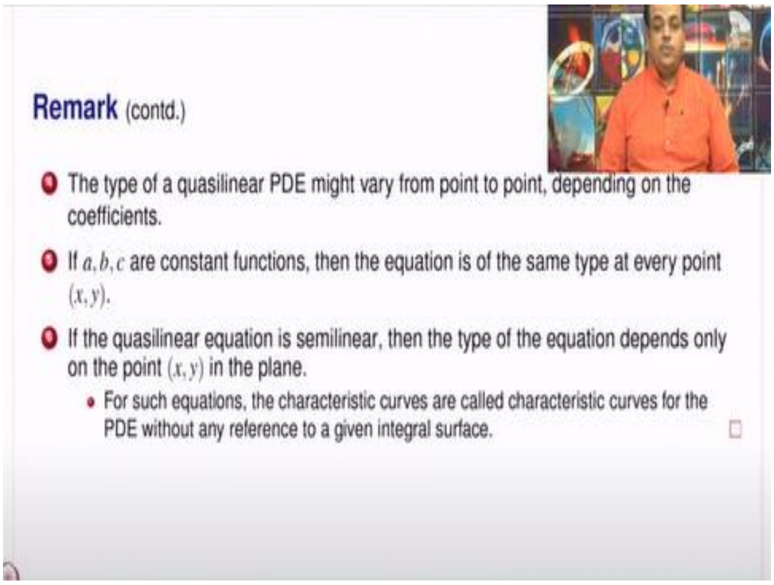

The type of a quasilinear PDE might vary from point to point we already saw that in the trichome equation. If a, b, c or constant function the type does not change, because b square-ac is a constant function. If the quasi linear equation is semi linear, then the type of the equation depends only on the point x, y in the plane. For such equations, the characteristic curves are called characteristic curves for the PDE without any reference to a given integral surface.

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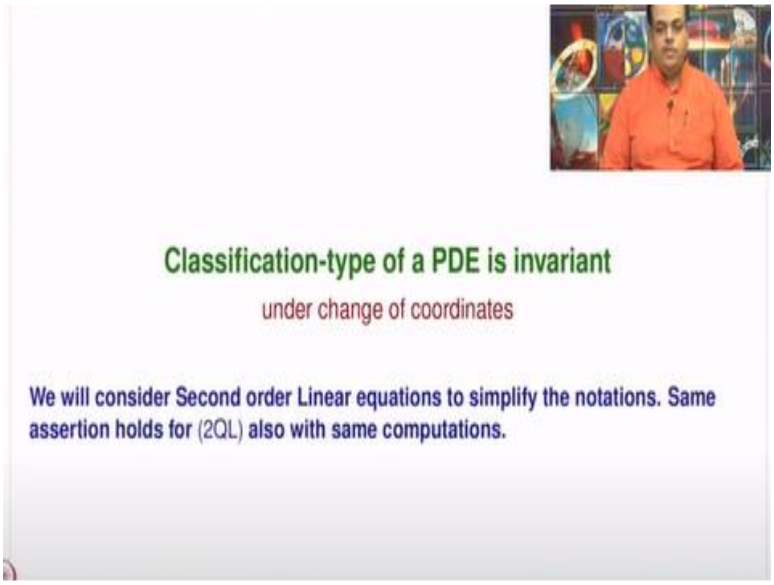

Now, we asked the following question whether the classification type of an equation changes if you change coordinates what we are asserting here is it is invariant under change of coordinates. We will consider second order linear equations only for this purpose just for the simplicity in the notations. Same assertions also holds for two 2QL with same computations.

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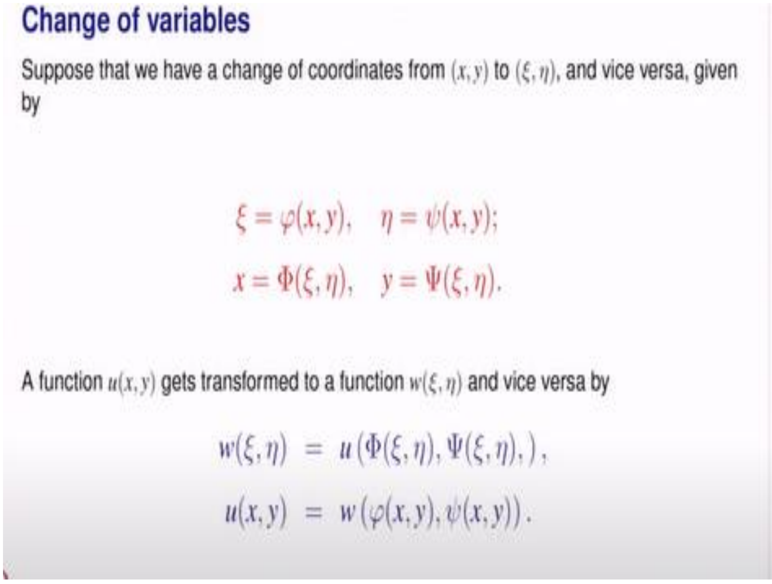

So, change of variables. Suppose that we have a change of coordinates from x, y to psi eta and vice versa given by these equations psi = phi x, y eta = phi x, y and x and y are given as function of psi and zeta. A function of u x,y gets transformed to a function of psi eta and vice versa by this relation.

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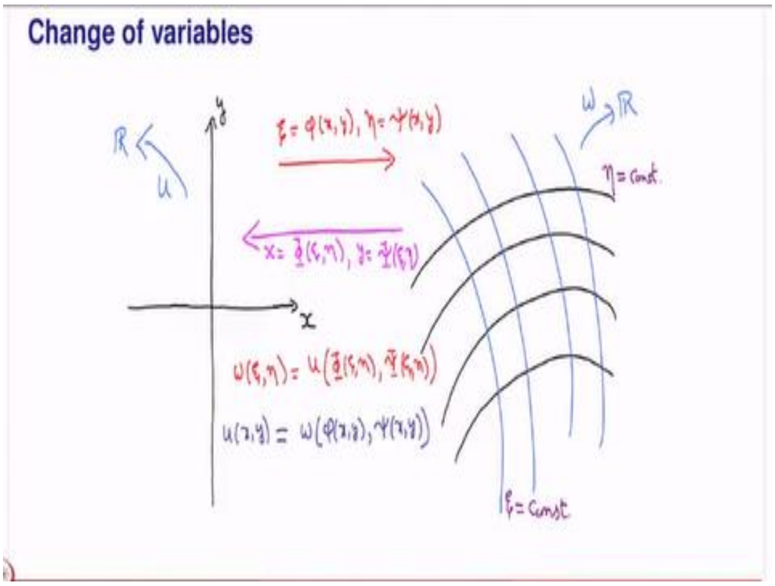

This is a picture we have already come across this in the first order equations discussion, where we employed change of variables to solve some problems for linear and semi linear equations to find general solutions of linear and semi linear equations in the first order case.

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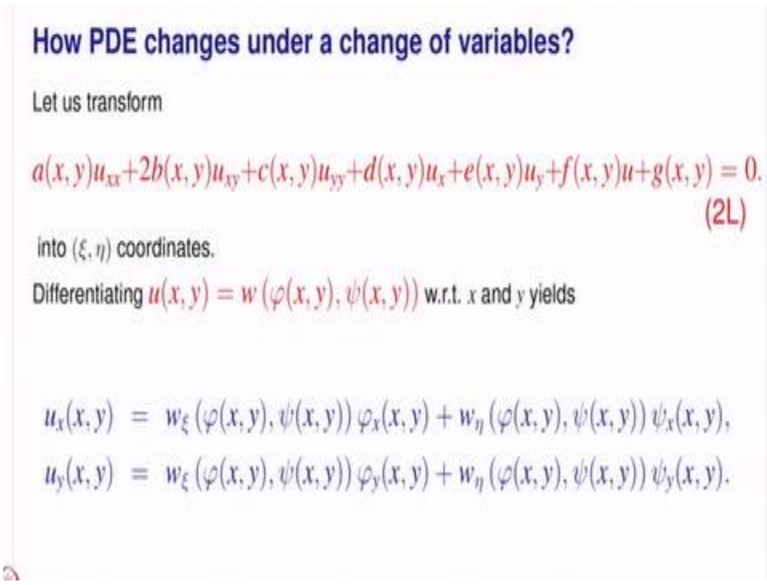

Now, how PDE changes under a change of variables that needs to be understood. So, let us transform the given equation which is the second order linear equation into psi eta coordinates that requires for us to, we know what is x and y in terms of psi and eta. So, we need to compute u xx, u xy, u yy and u x and u y. So, differentiating u xy = w of phi x, y psi x, y with respect to x and y gives you an expression for u x and u y.

Let us see what is u x. So, differentiate w with respect this variable which is psi and then differentiate phi with respect to x. Differentiate w with respect to eta and then differentiate psi with respect to x, this is the chain rule. Similarly, we get u y. Please do all these computations, pause the video, do the computations by yourself because the computations are spread over many slides. So, it is very difficult to show you all the time all the equations.

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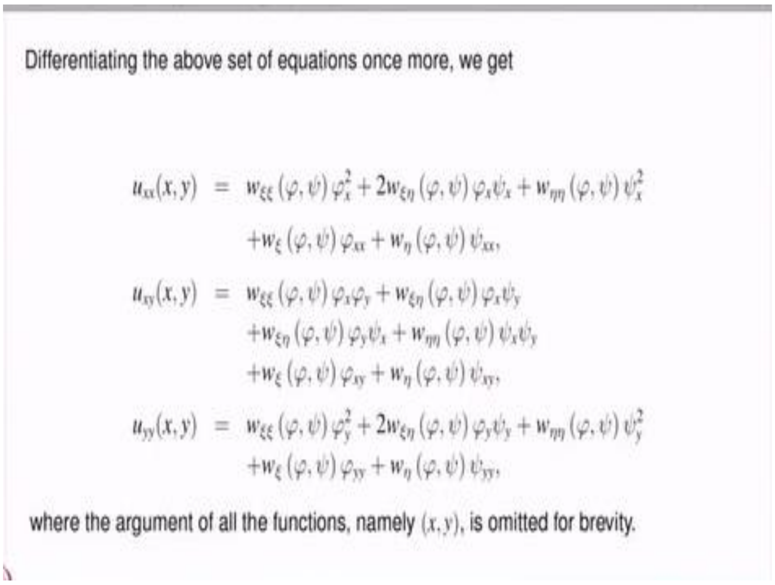

So, differentiate once again because we need to compute u xx, u xy and u yy. So, that we can go back and substitute in the given linear equation. So, compute u xx, u xy and u yy.

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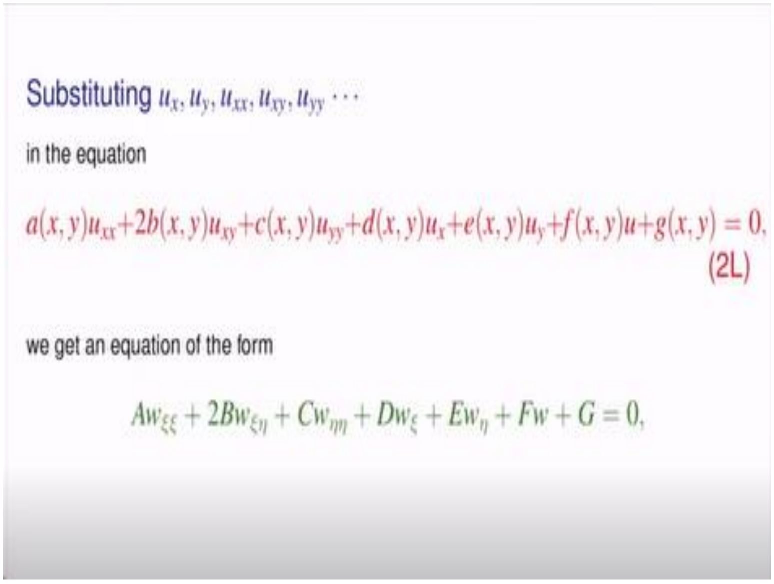

Substitute them in the equation then you get an equation of the following type capital A w psi psi + 2 times capital B w psi eta capital C w eta eta + Dw psi + Ew eta + Fw + G = 0. Now, we have to tell what the coefficients A, B, C, D, E, F, G are. But we want to say what is the assertion that we wanted to make, the type of the equation does not change. And the type of the equation is determined by b square - ac.

Therefore, it is enough to know what is is capital A, capital B and capital C, we do not have to know what the E, F, G, R. Therefore, we do not present.

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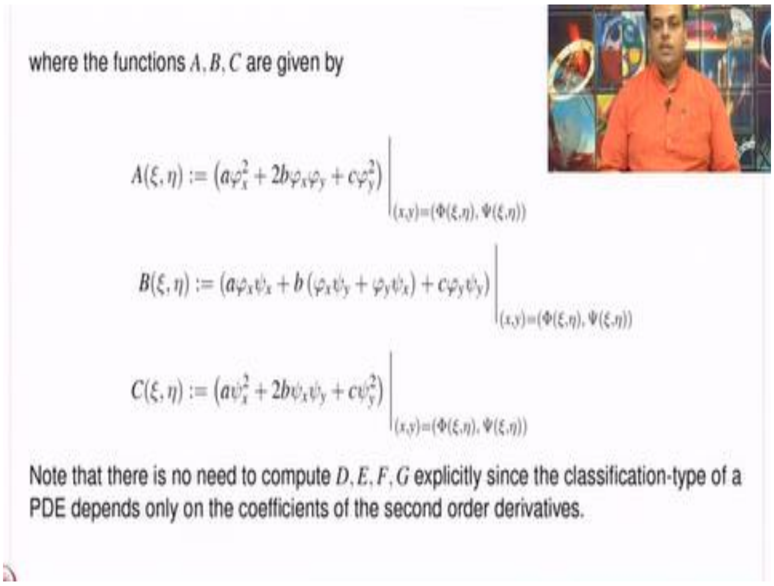

So, we just write what are the expressions for A, B, C which are this please do the computations on your own. As I pointed out, there is no need to compute other coefficient D, E, F and G because the classification type of a PDE depends only on the coefficients of these second order derivatives.

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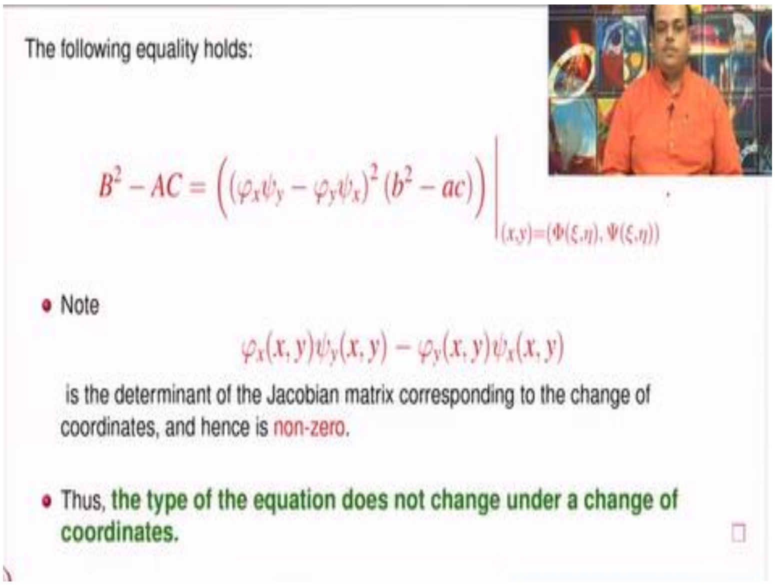

So, b square - ac, you get this expression. This is the new B square - AC capital letters. These are small letters, b square - ac multiplied with somebody square. So, as long as this is non zero, the sign of B square - AC here, he will be the same as the sign of the b square - ac here, if this is negative, this will also be negative. If this is positive, this will also be positive. If this is 0, this will also be 0. So, we need to know that phi x psi y - phi y psi x is non zero.

But it is non zero because it is the determinant of the Jacobian matrix corresponding to the change of coordinates, therefore, it is non zero. Therefore, the type of the equation does not change under a change of coordinates.

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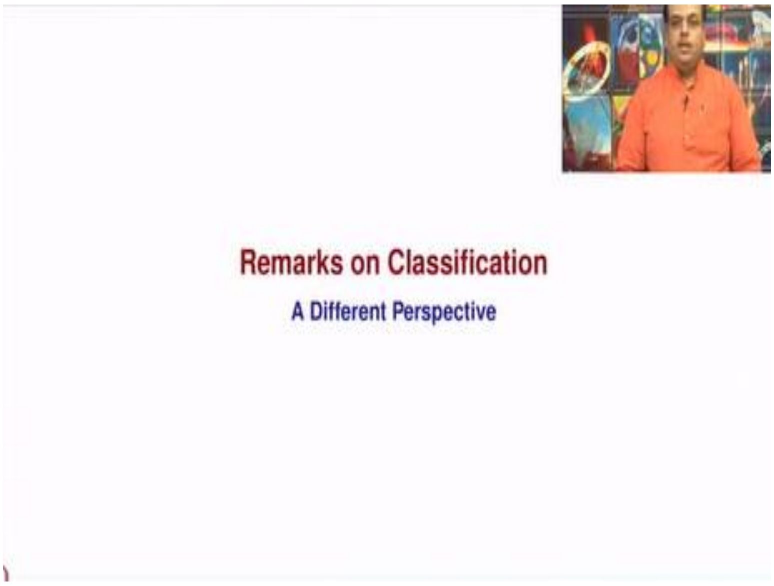

Now we give some remarks on classification. It is a different basically a different perspective.

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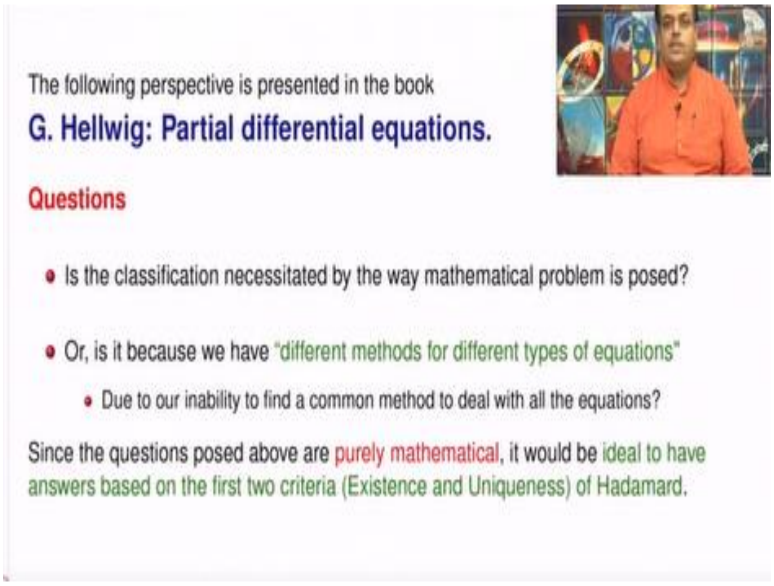

The following perspective is presented in the book of Hellwig on partial differential equations. Let us ask some questions, Is the classification necessitated by the way the mathematical problem is posed? Is it inherent to the problem itself? Or is it because we have different methods for different types of equations? Is it because you know how to solve a certain type of equation therefore, you give it and classify it and then solve.

Another type of equation you know how to solve separately. So, you give it another name. In other words, you put it in a different classification and do it is that the reason or is it something inherent to the way mathematical problem is posed. So, second one could be due to your inability to find a common method to deal with all equations. Because we are unable to deal with all the equations in my a common method.

We compartmentalize the equations and then solve them separately. Is that the case that is a question posed here. These questions are purely mathematical. Therefore, it will be ideal to have answers based on the first two criteria of Hadamard. Hadamard gave a criterion For a problem to be well posed, there are three aspects to that the first criterion is the existence of a solution. Second aspect is uniqueness of a solution.

Third aspect is continuous dependence on the data in the problem. Often the third one is desirable, because the equation that we are going to deal with the problems that we are going to deal with are coming from some physical models and measurements in the physical situations are only approximate. Therefore, by making some errors in the measurements that namely the data in the equation are in the problem.

We do not get solutions which are very far from the actual solutions. Therefore, the third property of Hadamard is desirable, but Hellwig can we justify using just existence and uniqueness. Because these are mathematical questions, how many solutions are there? Whether there is a solution. Using only these two can you justify the classification. Often people give an explanation for classification by saying that different kinds of problems are well posed for different kinds of equations.

There is a justification people give. So, Helwig does not seem to agree with such a justification. Therefore, he asked himself this question, can I justify using only existence and uniqueness parts because these are mathematical.

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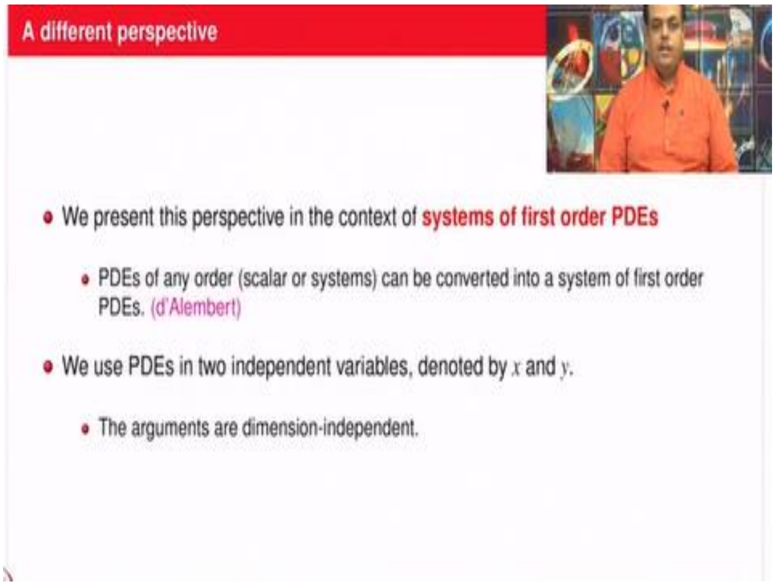

So, we present this perspective in the context of systems of first order PDEs. PDEs of any order scalar or a system of equations can be converted into a system of first order PDEs. In fact, the same statement was made even for ODEs. This is due to the d'Alembert. We use PDEs in two independent variables denoted by x and y. Because arguments are dimensioned independent.

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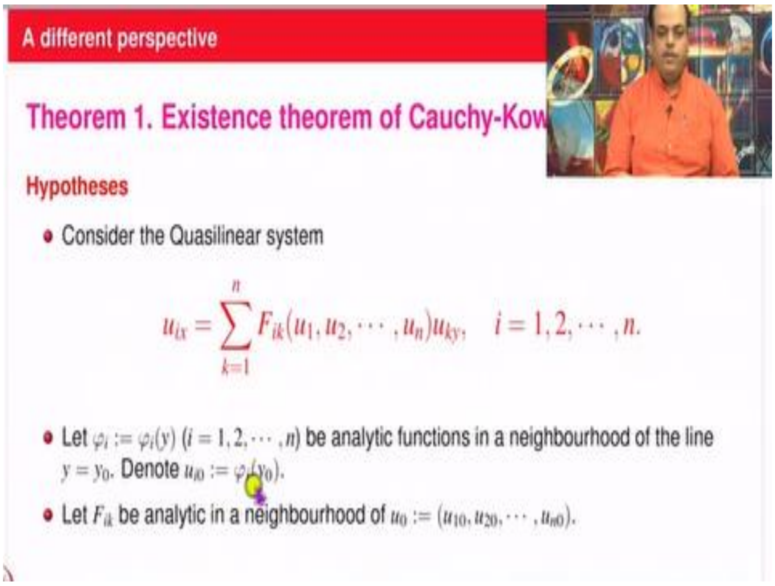

In fact, we do not give arguments, I request you to consult the book of Hallwig for more details. My idea was to bring this kind of thinking to your notice. So, theorem one existence theorem of course, Cauchy Kowalewski this we have hinted at in lecture 3.1. Now, we are going to pause the theorem for a very special quasi linear system. Consider this quasi linear system in these unknowns are u 1, u 2, and u n and there are n equations, u i x means derivative of u with respect to x, u i with respect to x, i equal to 1 to n.

There are n equation, there are n unknowns, these are quasi linear, because the u k y coefficient depend on the unknowns also. In fact, the x is not considered here, x and y could also be included, but in this theorem, it is not included. So, consider this system. What is if you look at what is that is different from the kind of equation that we are considering? Here the derivative of x is explicitly given in terms of other derivative, that is the difference.

There is no coefficient sitting in the front of this. Let phi i that is phi i of y be analytic functions in a neighbourhood of the line y = y 0, denote u i 0 = phi i of y 0. Now, let F i k be analytic in a neighbourhood of this vector u 0 which is given by u 10, u 20 and u n0 which is nothing but this phi 1 y 0, phi 2 y 0 up to phi n y 0.

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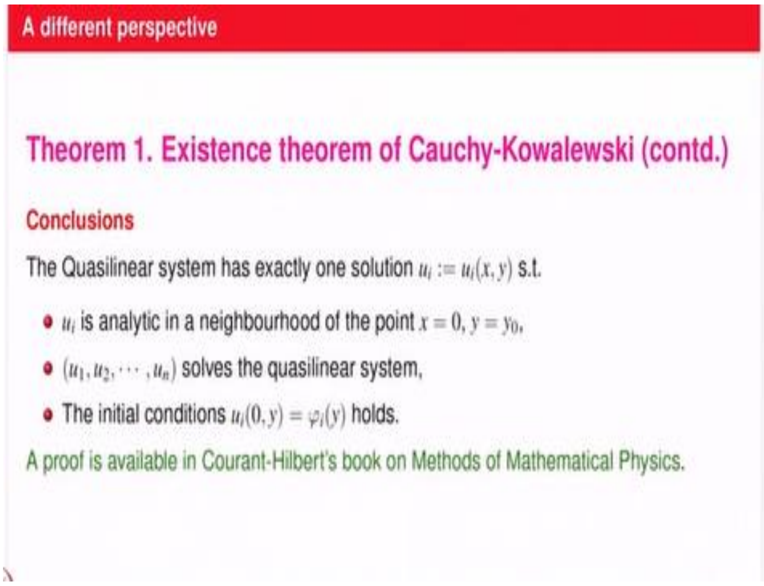

These are the hypotheses, then the quasi linear system has exactly one solution with the property, that is analytic in a neighbourhood of this point 0, y 0 and this vector solves a quasi linear system and initial conditions are satisfied. A proof is available in Courant-Hilbert's book on methods of mathematical physics.

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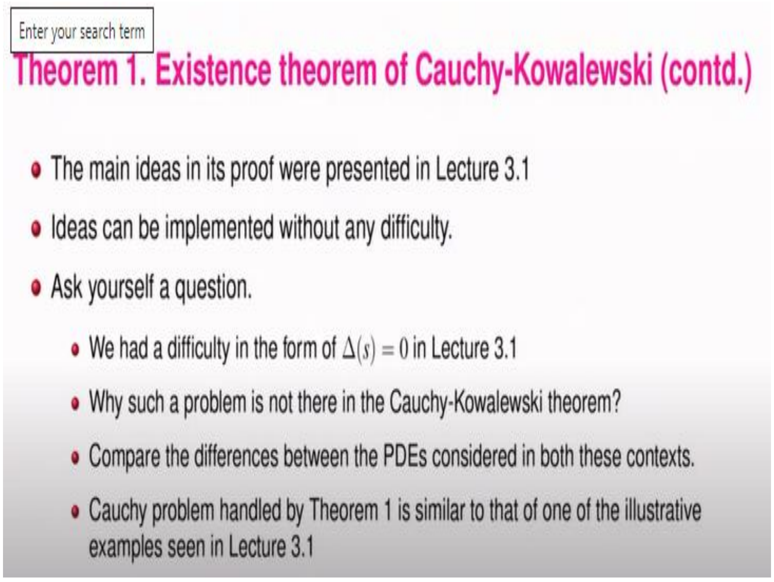

The main ideas in the proof are already presented in lecture 3.1. Ideas can be implemented without any difficulty, we need to ask the following question we had a difficulty in the form of delta s = 0 in lecture 3.1. We try to determine all the derivatives first derivatives came up very easily using the Cauchy data second derivatives there was a problem delta s = 0 we could not determine. So, there was a difficulty that is not there here.

So, you have to remember the Cauchy-kowaleski theorem and the theorem that we stated. So, compare the differences between the PDEs considered in both these contexts, I already give a hint, what is the difference. Cauchy problem handled by theorem one looks similar to one of the illustrative examples that we have presented in lecture 3.1.

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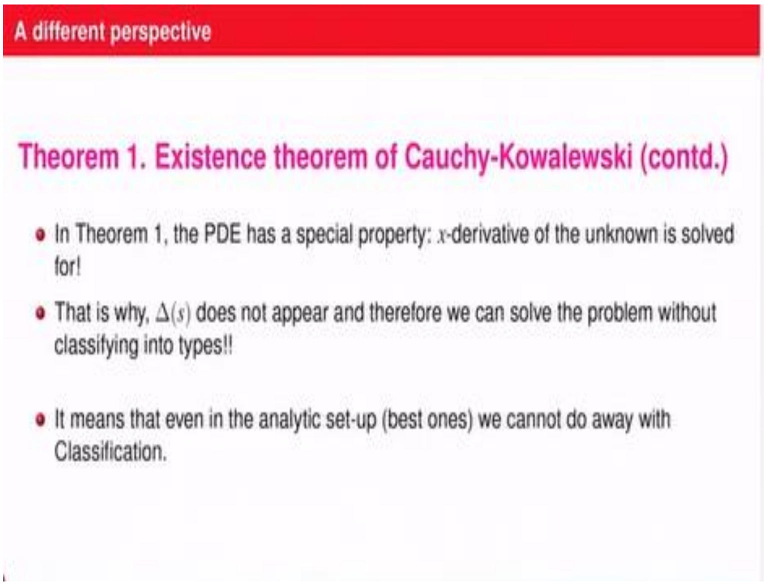

Go and find out what that is and what the similarities are. In theory 1 the PDE has a special property x derivative of the unknowns u i is solved for explicitly, that is why delta is does not appear. And therefore, we can solve the problem without classifying into types. It means that even in the analytic setup, we use the best of things, because you have nice series for everything that you have, we cannot do away with the classification. That is what we observed in lecture 3.1.

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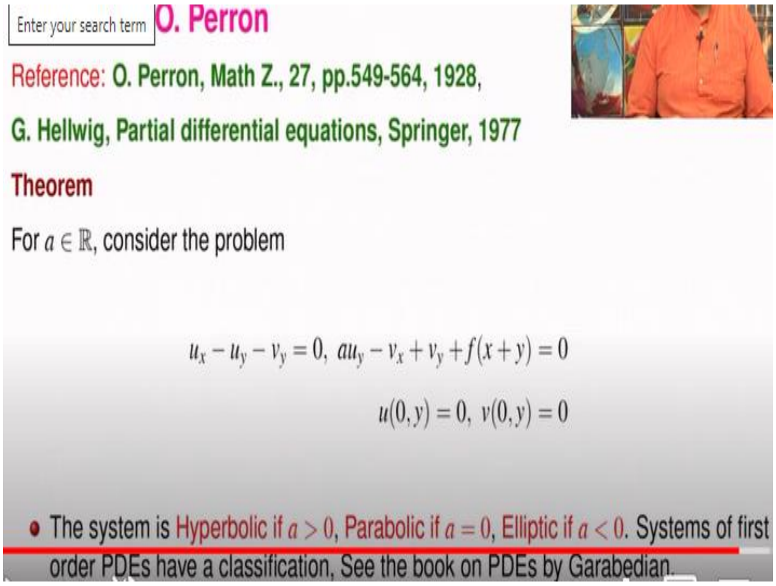

Now, we are going to look at one example of Perron this is reference. If you wish, you can consult this journal or it is very much explained in the book of Hellwig. For a real number a consider this problem it is a system of two equations, u x - u y - v y = 0, au y - v x + v y + f of x + y = 0. Here f is a given function and you are given this condition you have 0, y and we have 0, y are 0. So, the unknowns are u and v.

There is a way to classify first order systems or PDEs which we are not introduced. But you can ignore for the moment these names. What matters is these conditions? So, this is a number depending on whether it is positive 0 or negative, the properties of this term system changes we are going to give this assertions. So, the system is hyperbolic if a is positive parabolic if it is 0 a 0, eliptic a is negative.

System of first order PDEs have a classification. See the book on PDEs is by Garabedian that book has.

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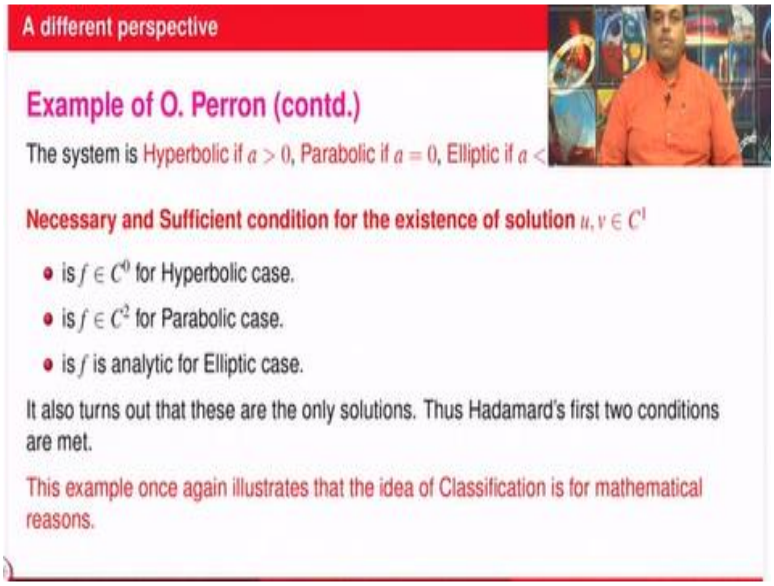

So, the system is hyperbolic, if a is positive parabolic, if a equals 0 elliptic, if a less than 0 do not worry, if you do not know the definition of hyperbolic parabolic elliptic for a system of first order PDE because assertion will not need that. So, this is about necessary and sufficient condition for the existence of solution u v in C 1. It is a first order PDE expert solutions to be C 1. What is the condition?

In the hyperbolic case that is when a is positive, you require that the f to be C 0, continuous C 0 means continuous function. For parabolic case, that is when a is 0, you require f to be C 2 for the same assertion. What is the assertion? Necessary and sufficient condition for the existence of a solution which is C 1, u and v are C 1 functions that changes from case to case. If a is positive, you require f to be C 0.

If a is 0, you require f to be C 2, if a is less than 0, you require more smoothness on f that is analytic. So, it also turns out that these are the only solutions that is Hadamards first two criteria are met. So, this example once again illustrates that the idea of classification is for mathematical reasons not because of the third requirement of Hadamard.

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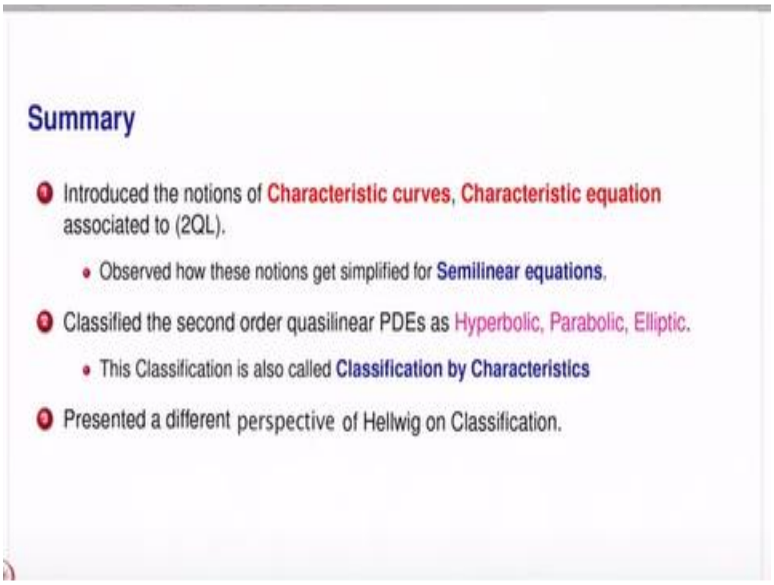

Let us summarize what we did in this lecture. We introduced the notions of characteristic curves and characteristic equations for second order quasi linear equations. We observed how these notions get simplified for simply the equations. We classified the second order quasi linear PDE says hyperbolic, parabolic and elliptic. This classification is also called classification by characteristics.

We presented a different perspective of hellwig on classification. Of course, we are also shown the classification type does not change with change of coordinates. And in the next lectures, we are going to take wave heat and Laplace equations as inspiration and ask the question, whether given any PDE can I do a change of coordinates? So, that at least the part which involves second order partial derivatives looks like that of one of these three equations.

Those are called canonical forms. We will discuss them in the forthcoming lectures. Thank you.