# Partial Differential Equations

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## Lecture – 3.2
## Second Order Partial Differential Equations
## Curves of Discontinuity

In this lecture, we are going to discuss about certain curves of discontinuity associated to second order Quasilinear partial differential equations. So, we start with a brief review of the lecture 3.1 and then we move on to discuss curves of discontinuity.

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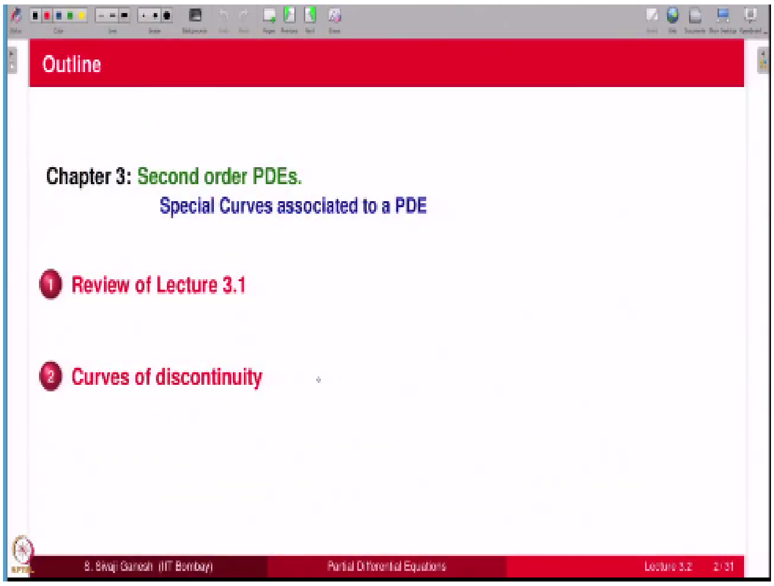


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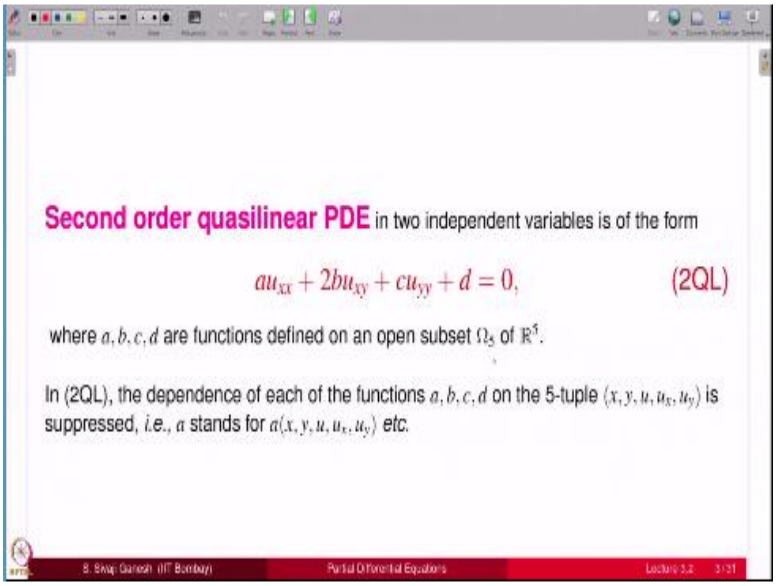

So, as you know the secondary Quasilinear equations we are denoting by 2QL and it stands for au xx + 2 bu x y + cu yy + d = 0, where a, b, c, d are functions of this x, y, u, u x and u y. So, they are defined on some subset of R 5 which is called omega 5.

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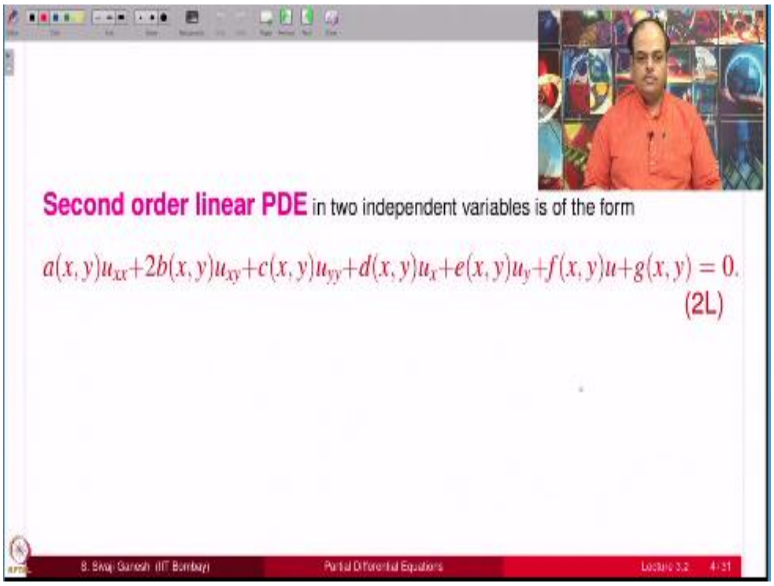


And a special case would be a linear equation, second order linear equation. It has the general linear equation looks like this.

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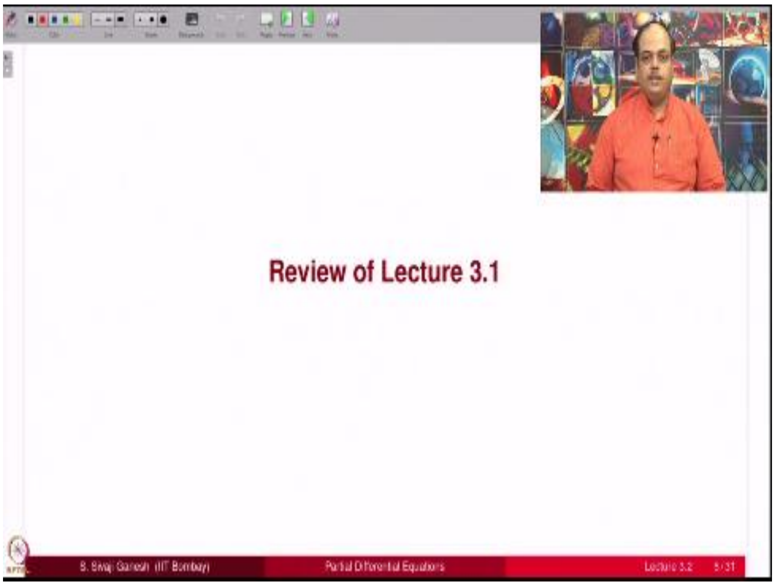


So, let us start with the review of the previous lecture.

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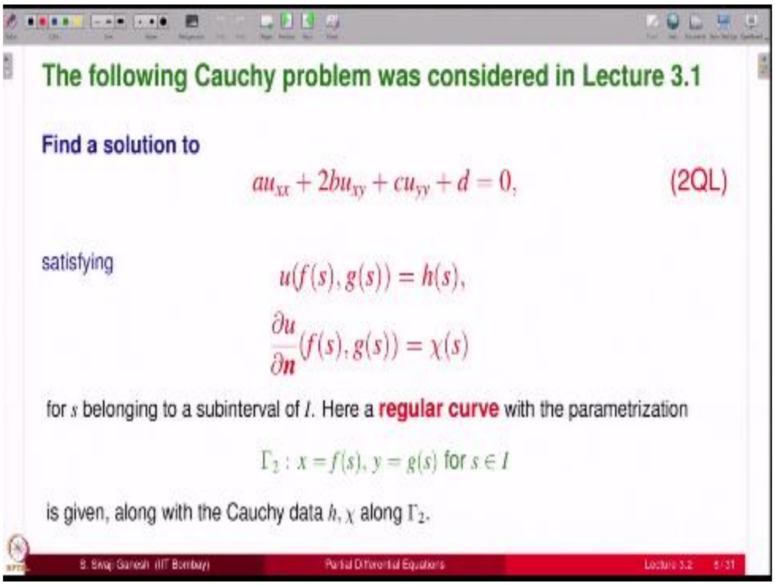


We have considered the following Cauchy problem in the lecture 3.1. What is that? Find a solution to the second order Quasilinear equation satisfying u of f s, g s = h s. And the normal derivative of u at a point f s, g s is chi s. What is f s, g s? It is a curve parametrically given in a plane gamma 2 and the curve is a regular curve. That means, f dash and g dash do not vanish simultaneously.

So, along this curve we are specifying the value of u as h and the normal derivative of u as chi. So, these functions h, chi are given. So, this is often called Cauchy data along gamma 2.

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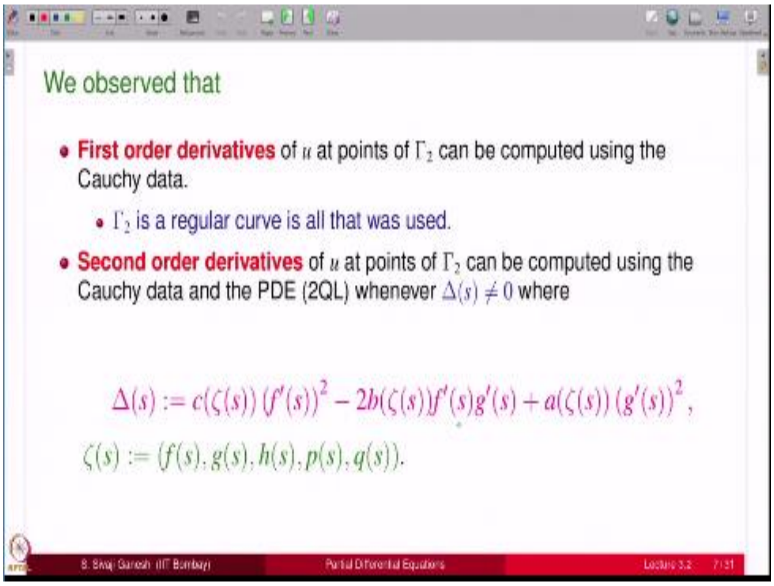


So, we observed that the first order derivatives of u at points of gamma 2 can be computed using the Cauchy data. For that what all was needed is that the curve gamma 2 is a regular curve. That was good enough. Then we also tried to compute second order partial derivatives of u again along the curve gamma 2. It can be computed using Cauchy data and now, we have to involve the partial differential equation as well.

So, the second order Quasilinear equation using these 2 we can solve, but one more condition was to be met. So, that was delta of s is not = 0. What is delta of s? It is this expression, c f dash square - 2 b f dash g dash + a g dash square. We know what f and g are. These are the functions which describe the curve gamma 2 parametrically. Then what is zeta s? It is a 5 tuple which is f s, g s, h s, p s, q s.

We have already solved for p s and q s using chi and h. Therefore, this makes sense and we can ask this condition. If this is nonzero delta s is not 0, then we can compute second order derivatives. And we have also shown that higher order derivatives can also be computed no further extra conditions are needed.

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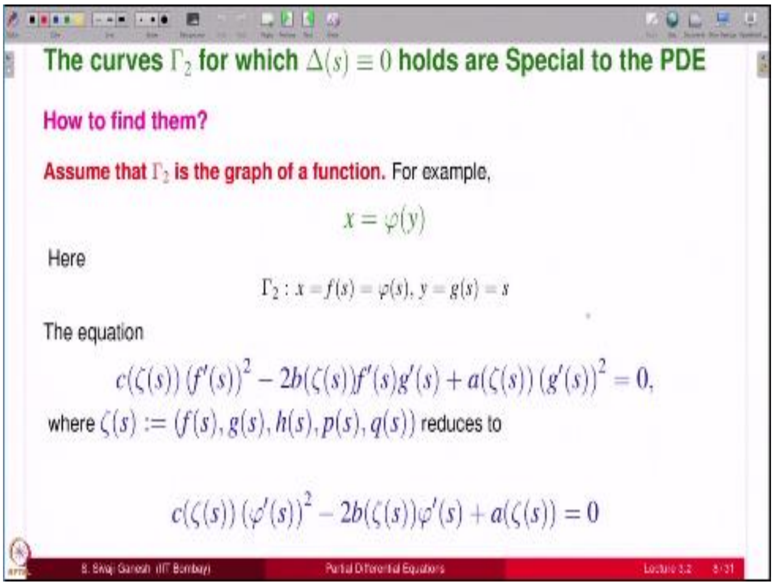


Now, we would like to understand this condition in more detail. What is this curve? What is this? Where delta s identically = 0, how the curve looks like? So, let us assume to start with that the gamma 2 that we consider is not parametrically considered defined. But let us say it is a graph of a function of the type x = phi y. It is the graph of a function of the variable y. x = phi y.

Then what happens to gamma 2? x = f s that will be phi s. y = g s which is s. Why we parameterize s, then x becomes phi s. So, these are parametric representation. And s belongs to that side where y belongs to. Wherever this function phi is given the domain s belongs to the same domain, we are not writing that right now. Now, what will happen to this equation? Now, we need to substitute what is h s, p s, and q s.

h s is still general, p s and q s are to be determined from h s and chi s. So, these are still unknown, but what is known is f dash, f dash square is phi dash square, g dash is 1. So, this equation becomes this equation. Of course if you want to know what function satisfy this condition, we need to still know this h, p and q. That is a problem with this equation, because the equation is Quasilinear equation.

So, for a Quasilinear equation, if you are interested in this question, then you must be given a solution of the equation then you can ask what is that curve which will have this property? That can be now determined because solution is given. Therefore, you know what is p and q? So, it makes sense. On the other hand if the equation is actually linear, this a, b, c s are not functions of zeta s at all. Zeta s is a 5 tuple.

They are simply functions of the first 2 variables f s and g s. And what is f s and g s? That is phi s and s. So, therefore, you know explicitly c, b and a only in terms of phi. So, it will be a differential equation for phi. So, it further reduces in the case of a second order linear equation to a simpler equation. For a Quasilinear equation however, we need to be given a particular solution for which we can ask.

What are the curves x = phi y along which there is a problem in finding higher order derivatives.

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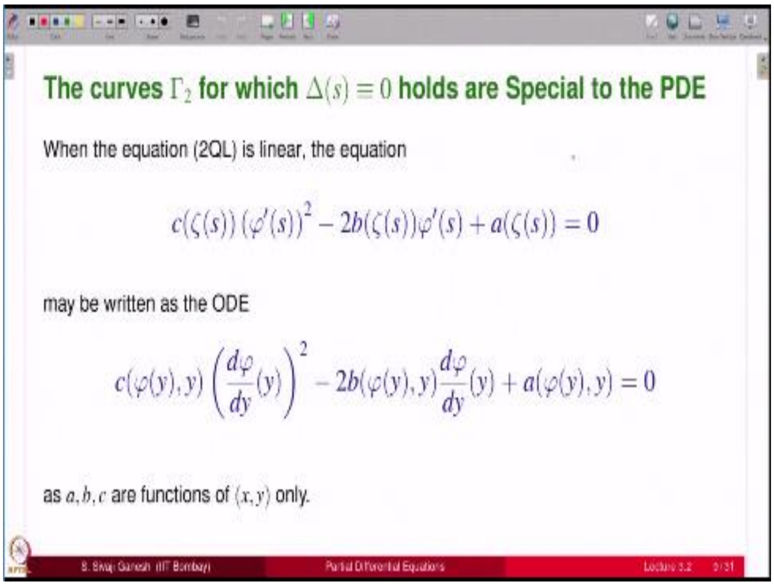


Problem in the sense we are unable; our scheme of things fail. Our scheme to compute the second order derivatives onwards fails because delta s is 0 there. So, as we saw these are special to the PDE because the definition of course involves the equation. For the linear equation, this gets simplified to this. x = phi y. So, this is a differential equation, nonlinear, first order, degree 2, nonlinear and coefficients will involve only 2 variables now.

Because we have a linear equation. So, one can hope to solve this for phi then we would have got the curve x = phi y where phi is the solution of this ODE is a curve along which delta s is identically = 0. Or there are troubles in solving for higher order derivatives. In other words, along these curves, there is some problem to determine higher order derivatives of the solution or of a possible solution.

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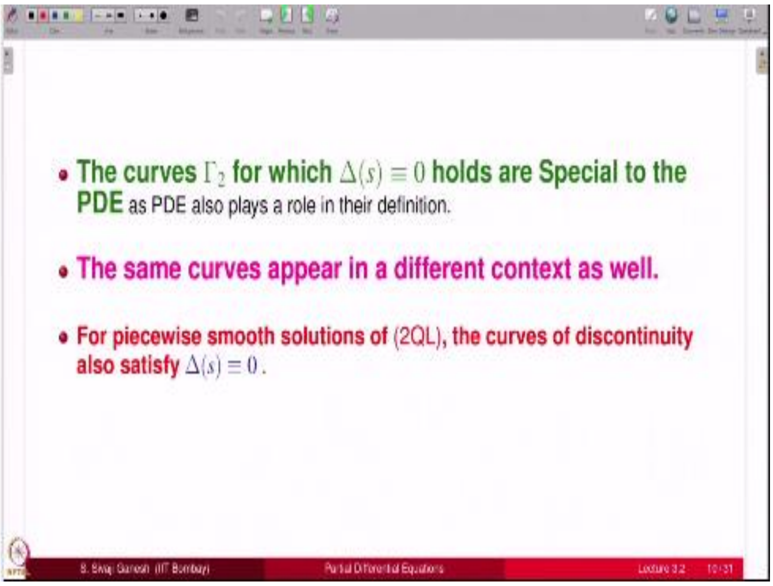

So, even if you no solution, there is a problem to determine that because of delta has been 0. Now, the curve gamma 2 for which delta s identically = 0 holds are special to the PDE as PDE also plays a role in their definition apart from gamma to itself. The same curves appear in a different context also. So, for piecewise smooth solutions of the second order Quasilinear equations, the curves of discontinuity also satisfies delta s identically = 0.

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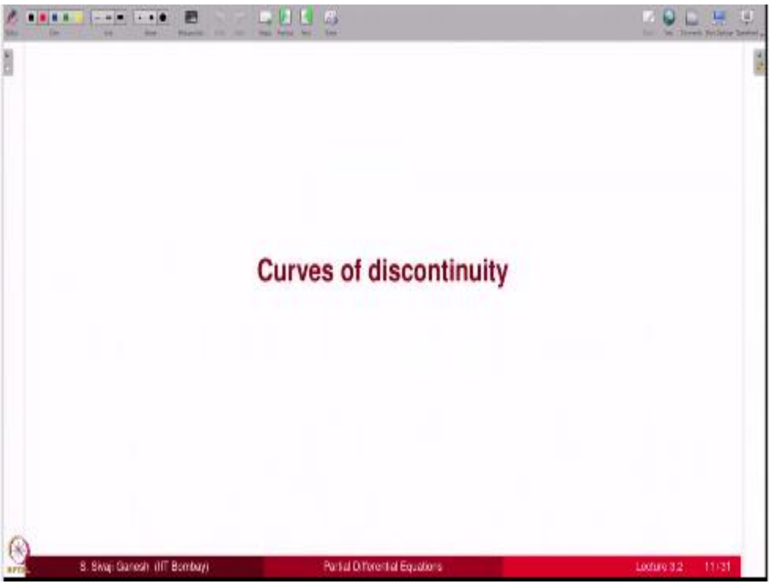


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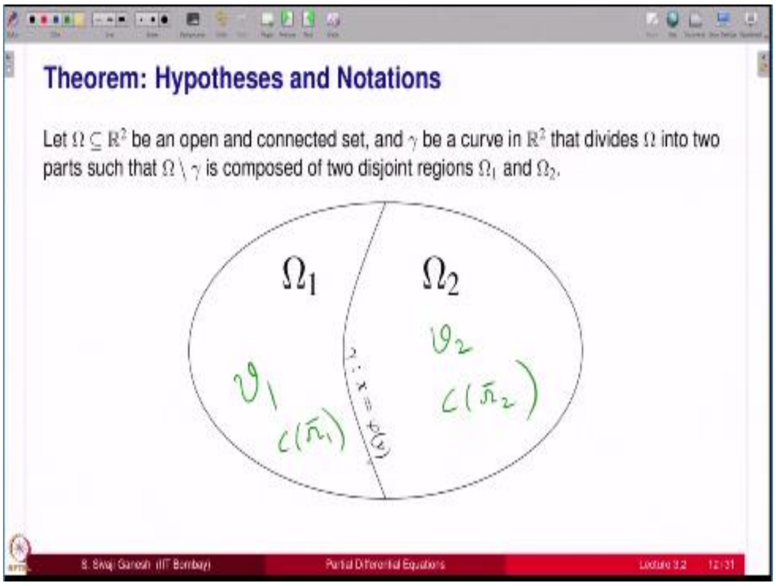


So, let us, we are going to state a result. In the form of a result it will be easier to remember. So, hypothesis and notations: Suppose you have an open and connected set in R 2 and take a curve in R 2, gamma that divides omega into 2 parts. So, that means that we have this. Let us say this is omega. And we have a curve gamma, which cuts this into 2 pieces, one is omega 1 other one is omega 2.

So, what is omega now? Omega consists of omega 1, omega 2 and this curve gamma, on the part of the curve gamma. We are going to assume that the curve gamma is given by x = phi y. We have already seen its interpretation in lecture 3.1. Why do we consider x = phi y? Once again we will give at the end of this result.

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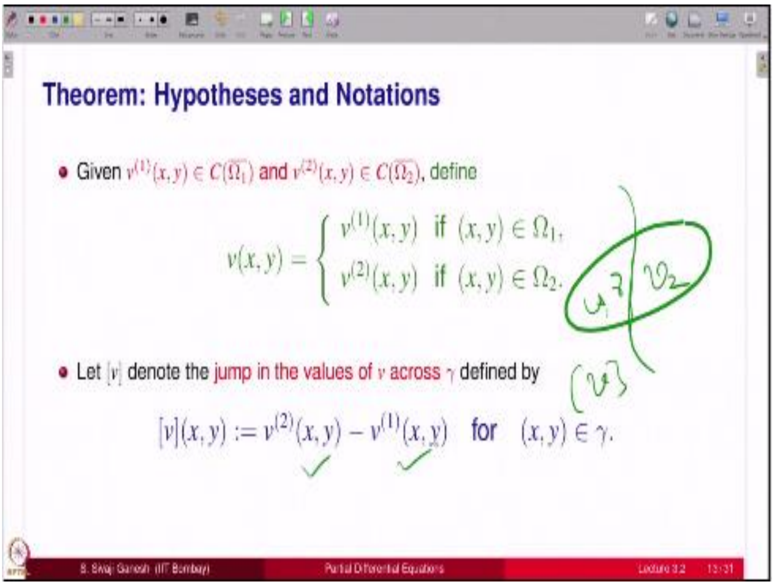


So, given v 1 in C of omega 1 bar, which means, that you are given a function v 1 here, which is continuous on omega 1. Of course, omega 1 open set, but it should be continuous upto the boundary of omega 1. In other words, what actually we want is its meaning on this curve. So, therefore, v 1 on the curve gamma makes sense. Similarly, v 2 is in C of omega 2 bar means that the values of v 2 on the curve gamma makes sense.

It will guarantee that apart from of course are on this boundary also they make sense because it is continuous up to the closure of the domains. In this case omega 1, in this case omega 2. If such functions are given, we can look at the value of v 2 on gamma and v 1 on gamma, it makes sense, meaningful. Therefore, we can look at the difference. Let us see. Let us define a function v, in omega 1 it is v 1, in omega 2 it is v 2.

Let box of v denote the jump in the values of v across gamma. That means, the definition is take a point x, y, take a point x, y on gamma 1 in omega 1, this is where v 1 is defined, this is where v 2 is defined. We have seen v 2 of x, y makes sense, when x, y is in gamma. Let us call this part as a gamma may not be the outside part. v 2 of x, y makes sense, v 1 of x, y makes sense because v 1 is also continuous upto the boundary.

There in particular its values in omega 1 make sense. And we can consider the difference. You could have even taken v 1 – v 2 there is no problem and we are considering v 2 – v 1. So, this is called jump in the function v. That means you have a function v defined on omega 1. Here we have a function here we have a function which is continuous, so that the values make sense along this curve gamma. Then you look at the jump in v as v 2 – v 1 at points of gamma.

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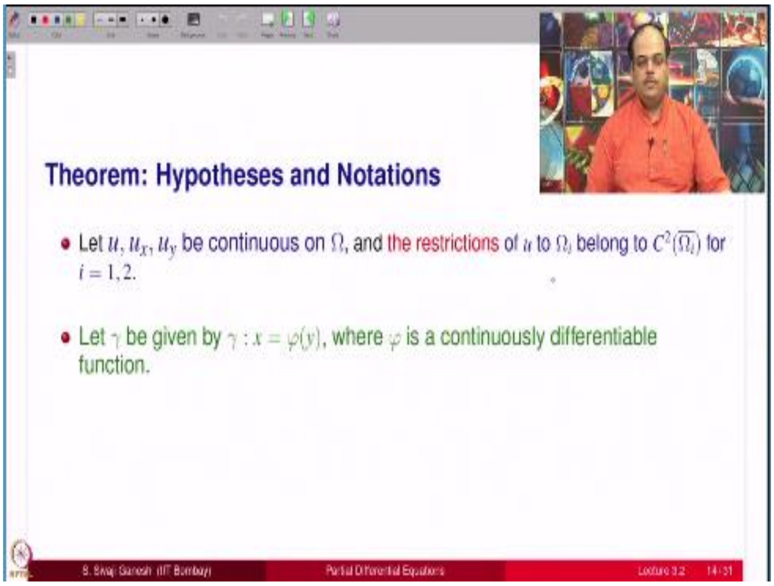


Let u, u x and u y be continuous on omega and the restrictions of u to omega i, they are C 2 functions because we want them to be solutions to the PDE. So, that is why the C 2 ness and we want omega i bar because we are going to consider the jumps in second order derivatives. First of all derivatives should be meaningful on the curve gamma. So, if I assume u is C 2 of omega 1 bar, we may put 1 on the head, u 1 is C 2 of omega 1 bar. It means that second order derivatives of u are defined on gamma because they are continuous in omega 1 bar. That is the reason why we have this condition. So, let gamma be given by x = phi y where phi is a C 1 function.

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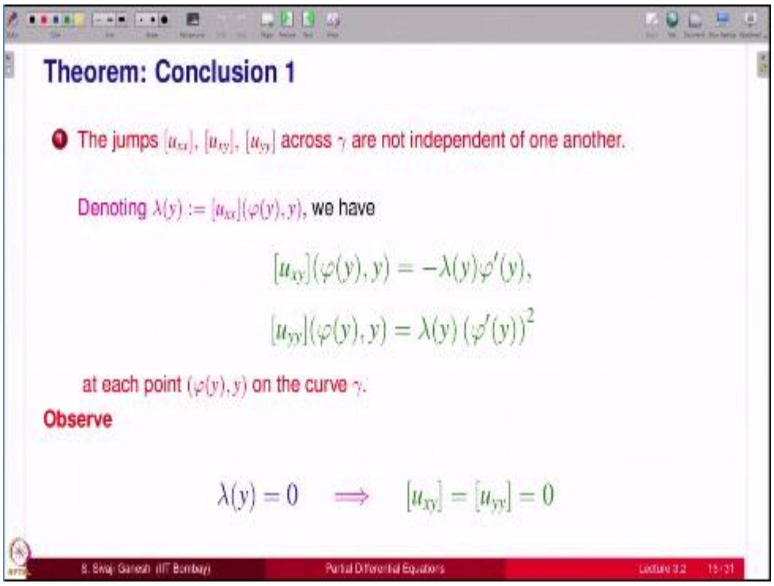


Now the jumps in u xx, u xy and u yy are not independent of one another. It means they are interrelated. To start with it looks like yes, u xx is a jump in second order u xx derivative, this is jump in xy derivative; this is jump in yy derivative. Why should they be connected? So, let us denote lambda y as a jump in u xx at a point on gamma. A point on gamma looks like phi y, y.

So, then this is the result. Jump in u xy is in terms of lambda y and phi dash. And jump in u yy is lambda into phi dash square? We are going to prove this. So, they are related. Observe that if lambda y is 0 what will happen? Lambda y is 0 means the hand sides are 0 here, which means jump in u xy and u yy are zeroes. That means, if jump in u xx is 0, then jump in u xy as well as uyy are 0.

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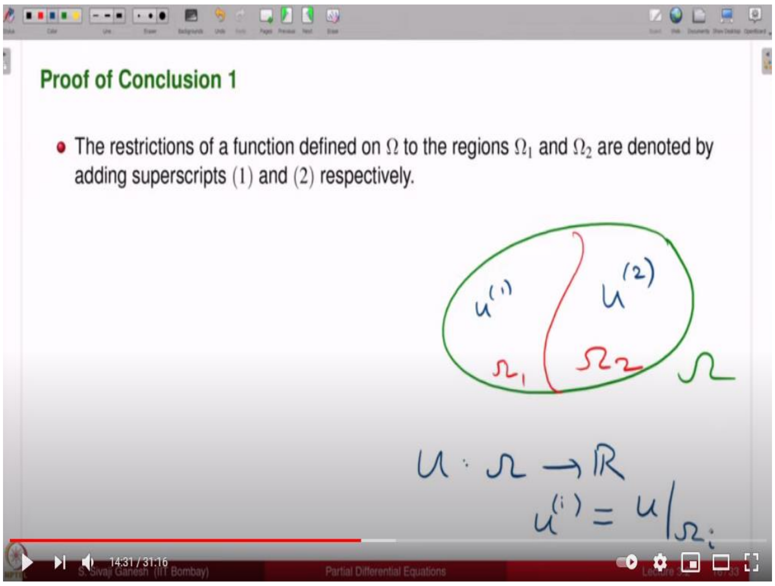

Let us prove the conclusion 1. The restrictions of a function defined on omega to the region's omega one omega 2 are denoted using superscripts. That is, we had this as the omega and we had a curve which is making it into 2 parts omega 1 and omega 2. And suppose we have a function u defined on omega or any function u, u x, u y we denote the restriction of u to omega 1 by u 1 and here by u 2. So, u i is u restricted to omega i.

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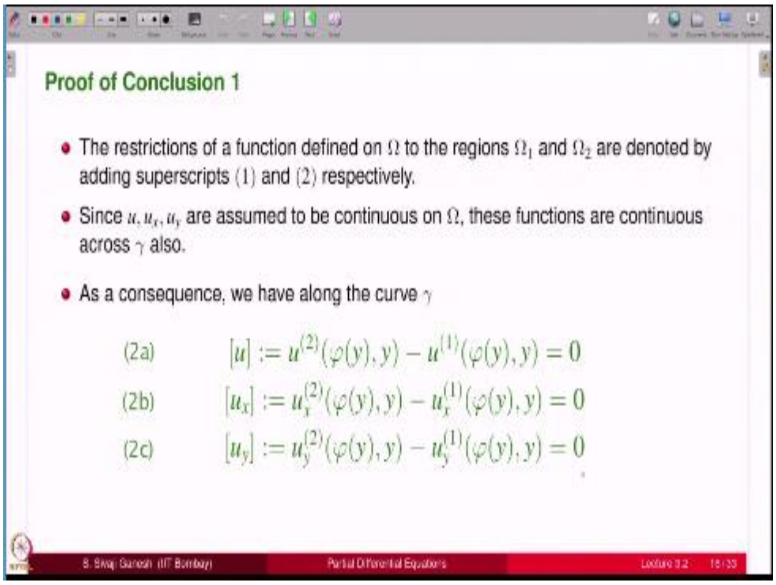


Since u, u x, u y are assumed to be continuous on omega, these functions are continuous across gamma also. In other words, there are no jumps across gamma. So, jump in u is u 2 - u 1, phi y, y is a point on the gamma. Similarly, the jump in u x defined by u x 2 – u x 1 that is also 0. Jump in u y is also 0. So on differentiating the equations 2b and 2c, the last 2 equations, differentiate them. Because differentiating the first equation will not give you anything. Because if you differentiate this with respect to y, what will you get? Nothing useful because we want to now consider jumps in second order derivatives. That is the reason.

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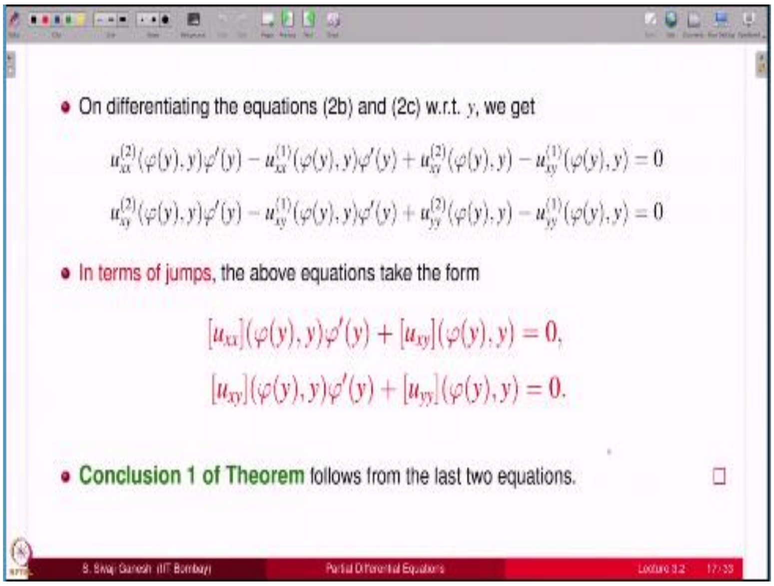


So, we differentiate the last 2 equations on the previous slide. We get this by differentiating the first equation and this by differentiating the second equation. Now, in terms of the jumps, this equation is nothing but jump in u xx, into phi dash + jump in u xy = 0. That is the first equation. And the second equation is jump in u xy, which is this into phi dash and jump in u yy = 0. So conclusion 1 of the theorem follows from the last 2 equations, because we called u xx jump as lambda then jump in u xy is - lambda phi dash.

Therefore jump in to u xy is - lambda phi dash that will give you jump in u yy is - lambda phi dash into phi dash. That is a conclusion 1.

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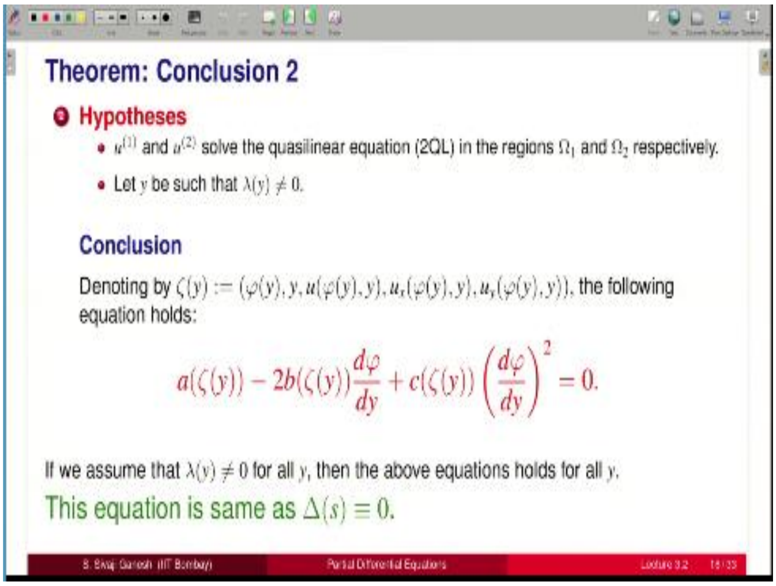


Let us look at the conclusion 2 of the theorem. For that we need to assume the hypotheses are u 1 and u 2 solves the Quasilinear equations in the region's omega 1 and omega 2 respectively. Let y be such that lambda y is not = 0. Recall lambda is the jump in u xx at the point phi y, y. Conclusion: Denoting by zeta y = this quantity phi y, y, u at the point phi y, y, u x at the point phi y, y, u y at the point phi y, y.

Recall the point phi y, y means x = phi y. This is how a point on gamma looks like. The following equation holds. We are going to show this equation holds. Whenever lambda y is not = 0 at that point this equation holds. So, if you assume lambda y is not = 0 at all the points, then the equation holds for all y. Then this equation is same as delta s identically = 0.

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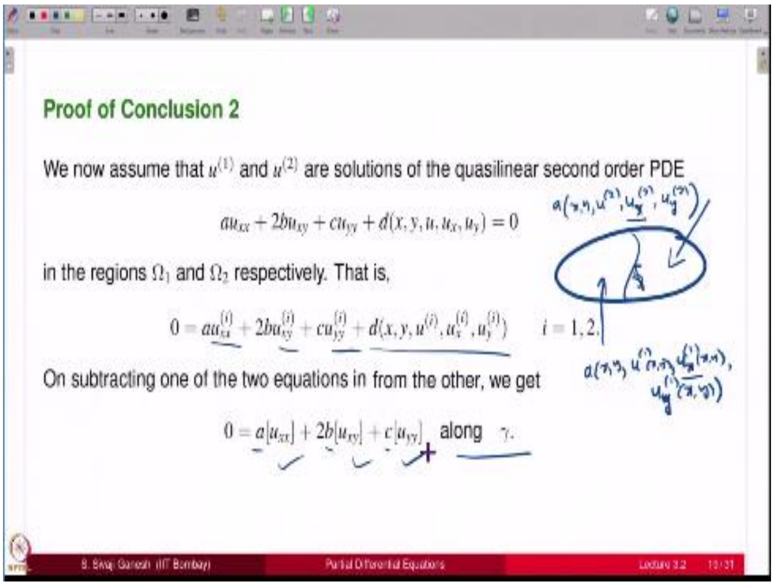


Proof is very simple we are going to assume that u 1 and u 2 are solutions. Therefore, we can write down this equation same omega 1 and omega 2 with the appropriate superscripts. So, we have this. Now, we have 2 equations, we have to subtract one from the other. And we have assumed that there are no jumps in u, u x and u y. Therefore, the coefficients a, b, c, d they all are like this. They are functions of x, y, u, u x, u y. There is no jump in them.

So, this a will come out to be a. So, only u pick up the jump in u xx. So, 2b will remain as it is because the curve it is a continuous function. And it is the same in both. Therefore along gamma, what I mean by saying both is this. You have an expression for let us say here a x y, u 1 of course of x, y, u x 1 x, y, u y 1 x, y, we have this. And here what we have is exactly same, but with 2.

Now, on this curve, if x, y is here then both are same. Because there is no jump u x 2 is same as u x 1, when x, y is on this curve gamma. That is why when you subtract; they come out as they are and the d term gets cancelled because the same in both of them when you are looking at points of gamma. So, you pick up only the jumps. Jumps in u xx, jump in u xy, jump in u yy they come here. So, we have this.

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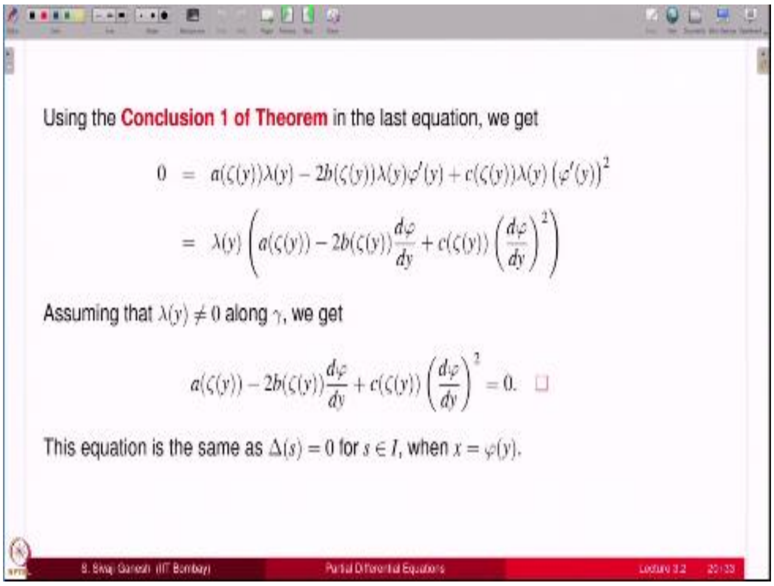


So, using the conclusion 1 of theorem in the last equation, we get 0 = a of zeta y lambda y - 2 b of zeta y lambda y Phi dash + c of zeta y lambda y Phi dash y square. So, if you take lambda y common, what you get is this, a - 2 b phi dash + c phi y squared. Assuming that lambda y is not = 0 along gamma, we get this, the one in the brackets must be 0. If lambda y is not = 0 for all y, then the one in the parenthesis must be 0.

So, we get this differential equation and this equation is the same as delta = 0 for s in I. Same equation when x = phi y. So, we have assumed that the curve is of the form x = phi y.

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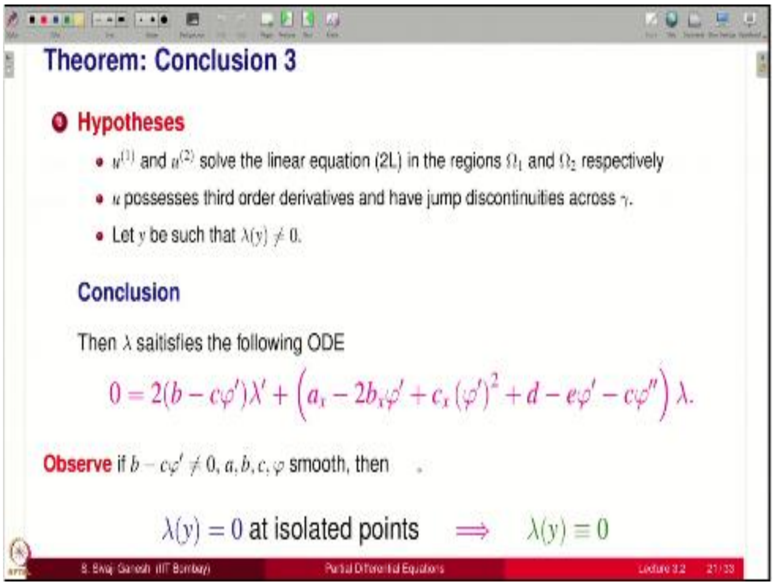


The conclusion 3 of the theorem is as follows: Hypothesis, assume u 1 and u 2 solve the linear equation in the regions omega 1 and omega 2 respectively. The linear equation and u possesses third order partial derivatives that is required for stating this conclusion and have jump discontinuities across gamma. Let y be such that lambda y is not = 0 then lambda satisfies the following ODE.

0 = 2 into b - c phi dash into lambda dash + this quantity into lambda. So, this is a linear ODE with variable coefficients because they depend on y. This depends on y. So, it is a linear ODE with variable coefficients. Of course, if b - c phi dash = 0, then this will be a singular ODE otherwise, it is a linear ODE. Now, observe that if b - c phi dash is not = 0 and a, b, c, phi are smooth functions, then lambda y = 0 at isolated points implies that lambda y is identically = 0.

Let us understand this carefully. If lambda y = 0 at some point this ODE may not hold, but we are here hypotheses lambda y = 0 at isolated points. That means, the points where lambda y = 0 can be reached by the points where lambda y is not = 0 at which the ODE holds. This ODE makes sense, because whenever lambda is not 0 at some point, the body happens. The lambda satisfies this ODE.

And you see this is an equality as people say equality is a closed condition. Therefore, it follows that even if lambda y = 0 at isolated points, this ODE continues to hold. And if we are assuming all these functions are smooth and this is nonzero, then it turns out that the quantity in these brackets and the quantity in these brackets are smooth functions and they are locally Lipschitz functions.

Then, we can apply uniqueness theorems for the initial value problems and conclude that lambda y is identically = 0.

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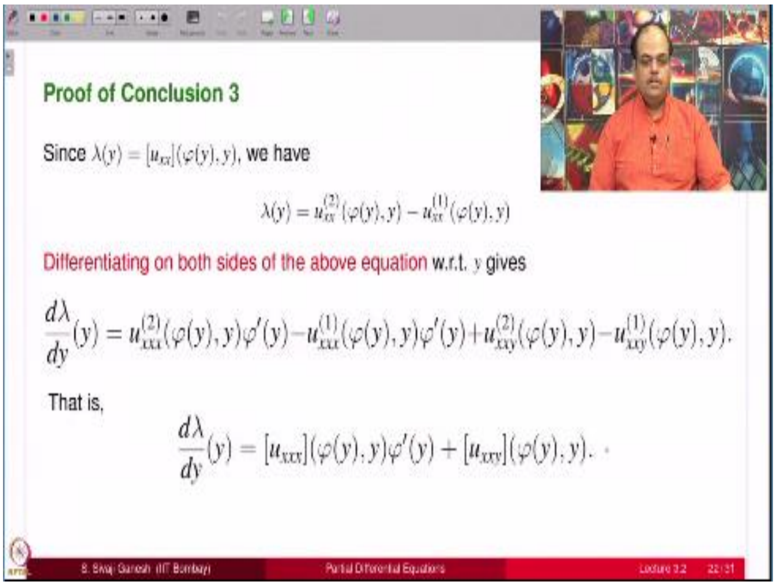


So, lambda y by definition is jump in u xx which is this. Differentiate both sides of this equation you get lambda dash with respect to y, therefore you differentiate this with respect to x you get you triple x. But phi is there therefore phi dash and from here you get u triple x into phi dash. Now, differentiate u xx with respect to y variable you get this. Similarly, this. You get this. Therefore, we can express d lambda by d y as jump in u triple x into phi dash + jump in u xxy.

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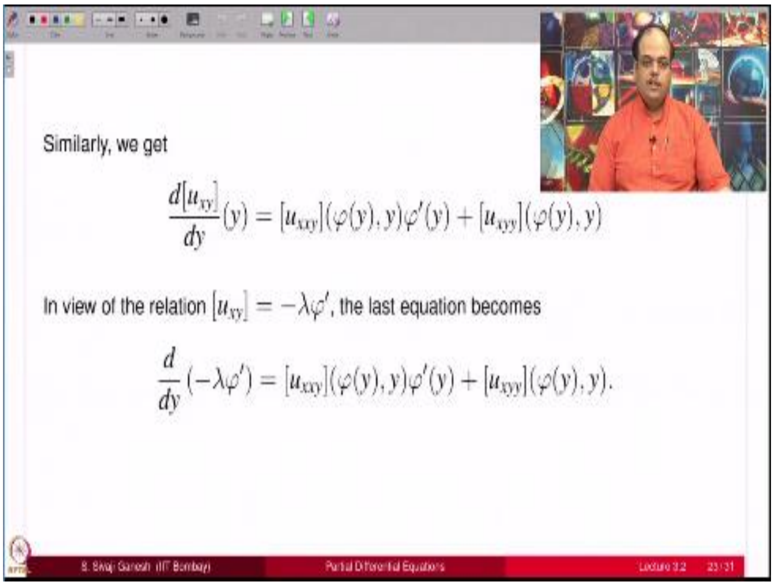


Now, similarly, we can also get derivative of u xy, you write the equation for u xy jump, differentiate with respect to y, you will end up with this relation. Now, we know that u xy jump is - lambda times phi dash. That is a conclusion 1. Therefore, this equation I can write it as this.

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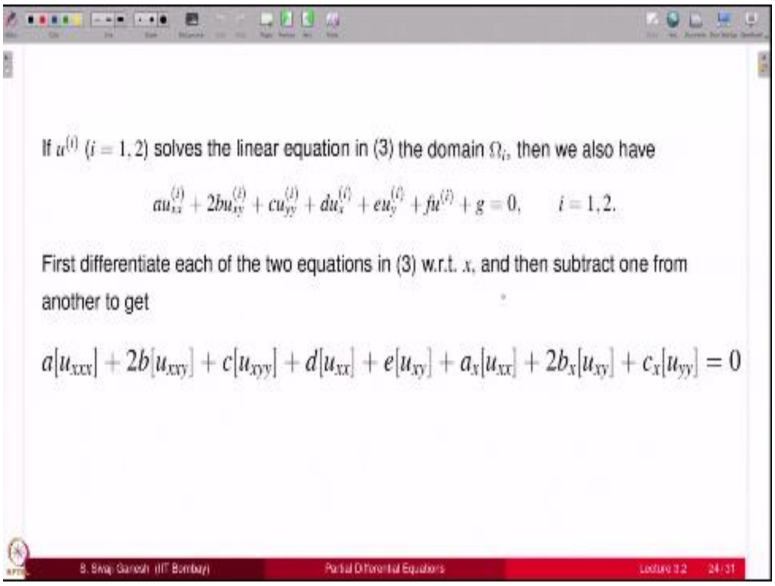


Now, we are going to use u i solve the linear equation in the domain omega i. So, we have these 2 equations one in omega 1, one in omega 2. In gamma we will consider because we are going to take jumps. So, firstly is to differentiate this with respect to x because we are interested in third order derivatives, differentiate with respect to x, then subtract one from another, we get this expression.

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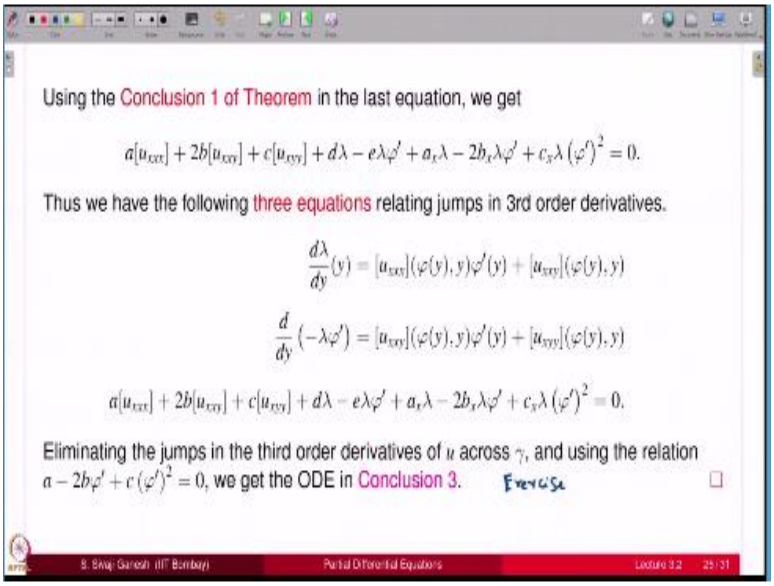


Now, once again use the conclusion 1 of the theorem which expressed basically the relations of jump in u xx, u xy and u yy in terms of lambda which is actually jump in u xx. So, we get this relation. So, we have 3 equations, this, this the one we just obtained this. Now eliminating the jumps in the third order derivatives of u across gamma and using this relation that a – 2b phi dash + c phi dash square is 0. That is the equation phi has to satisfy.

Even for Quasilinear equation, therefore, for linear equation also, so this equation is satisfied. So, how do we eliminate it that is what you have to look at. So, we want to remove these conditions, we have 3 relations in them, maybe solve for them; that is what it is. And substitute and we get the ODE in conclusion 3. So this is left as exercise to you. This is a simple algebra algebraic exercise.

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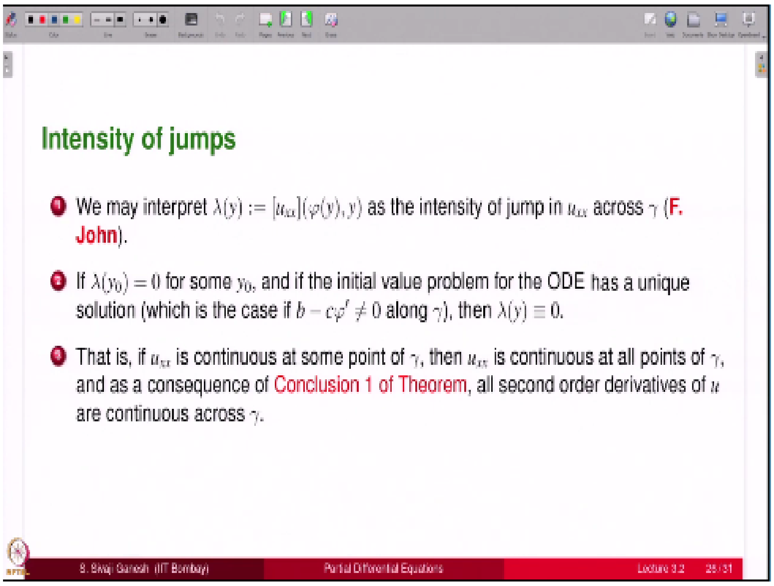

Now, we may interpret this lambda as the intensity of the jump in u xx, intensity of the jump in u xx across gamma. This interpretation is taken from the book of F John on PDEs. So, if lambda y 0 is 0 for some y 0, that is lambda is 0 at some point. And if the initial value problem that we had for the ODE has a unique solution, what is the ODE that we were considering is the linear equation for lambda which we saw: 2 into b - c phi dash into lambda dash + a huge expression into lambda?

If that has a unique solution, of course, it is a homogeneous linear equation, 0 is always a solution. And if it has uniqueness also, then it must be that lambda is identically equal to 0. We already did this conclusion. So, if u xx is continuous at some point, this interpretation is in terms of u xx. What is lambda? Afterall, it is a jump in u xx. If jump is 0 means what? u xx is continuous at that point, then it is continuous at all points of gamma. And therefore, once you have u xx jump is 0; jump in u xy and u yy is also 0 from conclusion 1.

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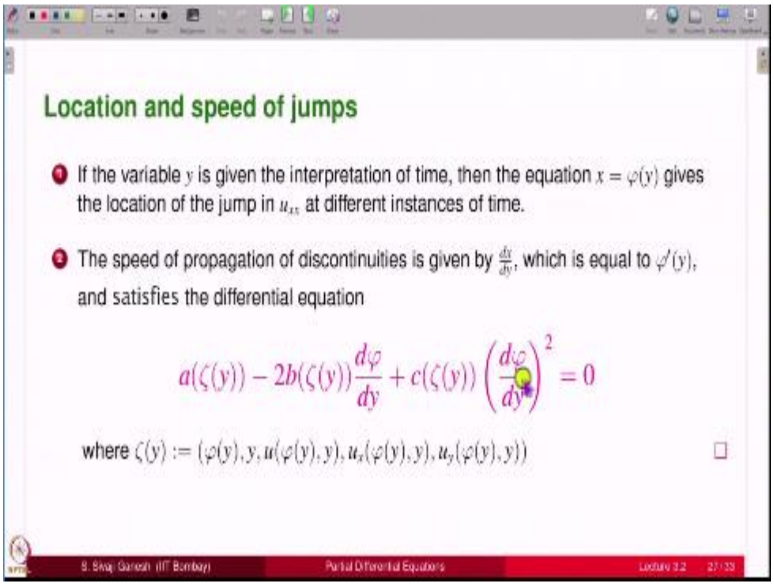


So, location and speed of jumps: If the variable y is given the interpretation of time, the equation x = phi y gives the location of the jump in u xx for various time instances. At different instances of time, the speed of propagation of discontinuities is given by dx by dy which is = phi dash of y. dx by dy is phi dash of y and satisfies the differential equation a of zeta y - 2 b zeta y phi dash + c of zeta y phi dash square, where zeta y is given by this 5 tuple.

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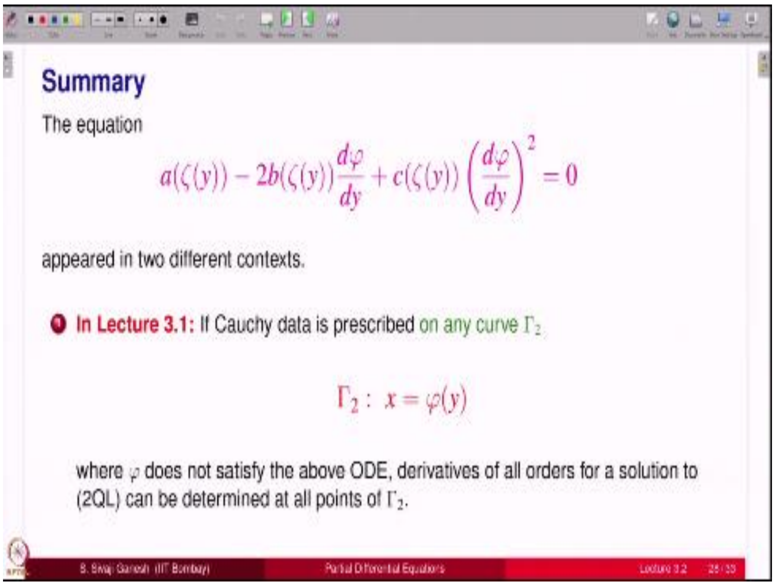


So, let us summarise this equation which is the ODE which is there on this slide. It appeared in 2 different contexts. In lecture 3.1 how it appeared? If the Cauchy data is prescribed along the curve gamma to x = phi y, where phi does not satisfy the above ODE. Derivatives of all orders for a solution to 2QL can be determined at all points of gamma.

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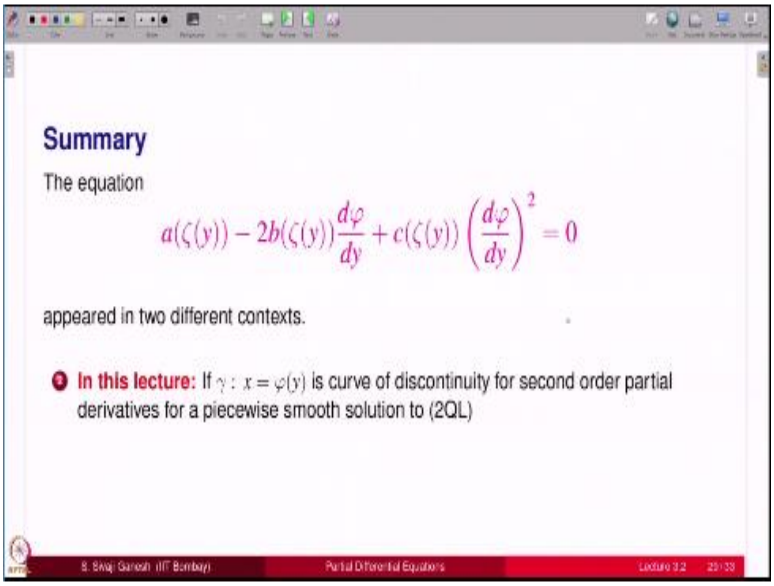


In this lecture, if gamma given by x = phi y is a curve of discontinuity for second order partial derivatives for a piecewise smooth solution of 2QL, as in the theorem that we have stated today in this lecture, then phi solves the above ODE.

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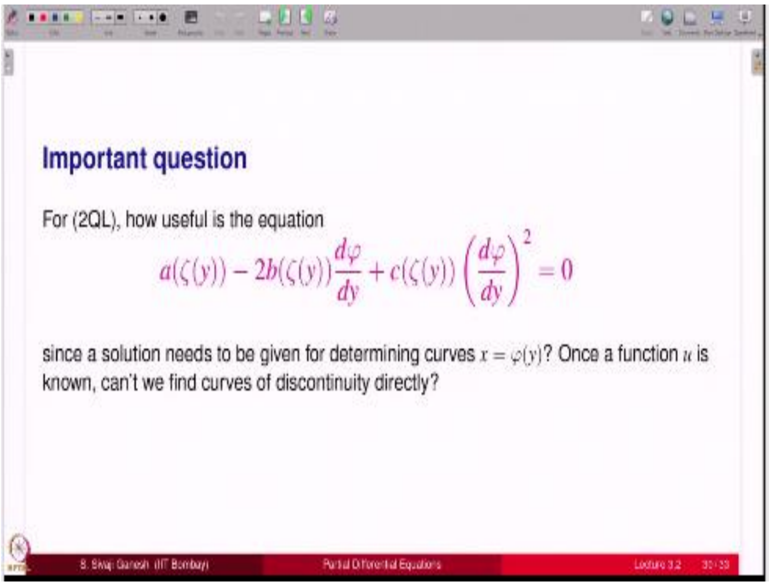


Important question is, for a second order Quasilinear equation, how useful is this equation? Since, to write down this equation, zeta of y it requires the knowledge of a solution to the 2QL that needs to be given. So, how is it helpful in identifying these curves x = phi y, where phi solves this equation? Once the function is known, can not we find cause of discontinuity directly?

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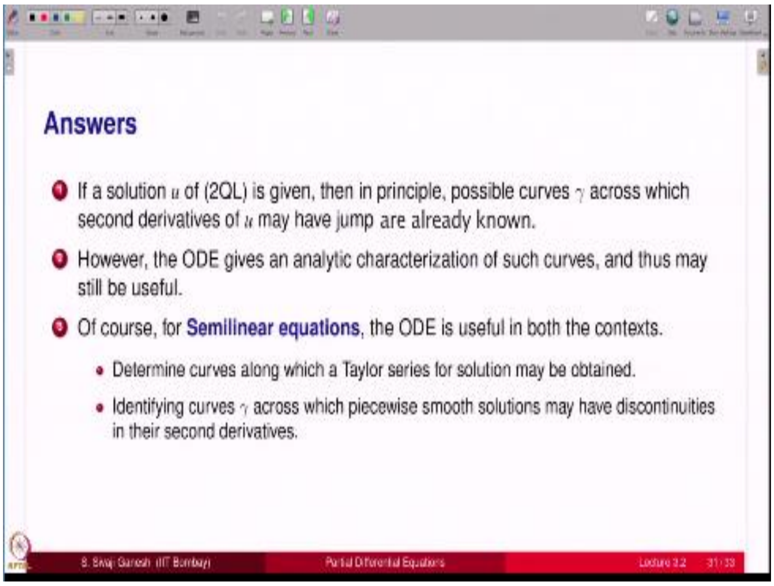


Do we still need to solve this ODE? Answer: If a solution is given then in principle possible curves gamma across which second order derivatives may have jumped are already known. However, the ODE gives an analytic characterization of such curves. And thus may still be useful. Of course, for semi linear equations, the ODE is useful in both the contexts. Determine curves along which a Taylor series for a solution may be obtained.

Or identifying curves across which piecewise smooth solutions will have discontinuities in the second order derivatives.

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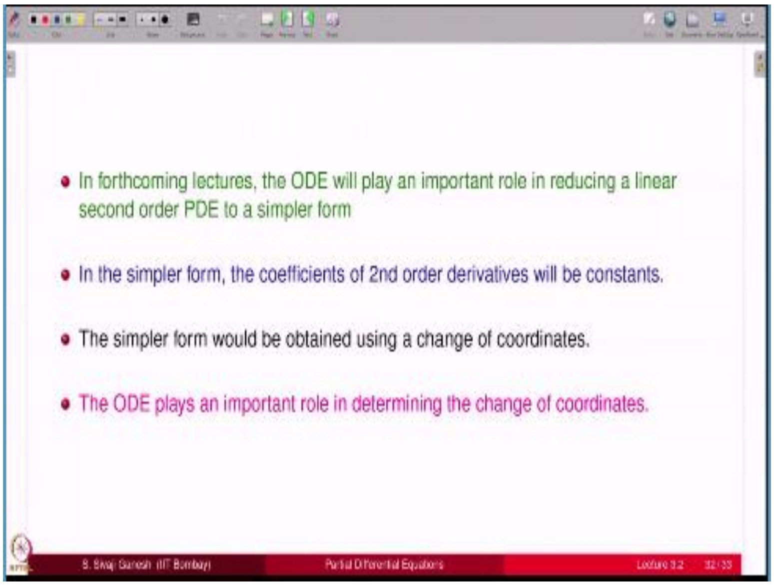


So, in forthcoming lectures, the ODE will play an important role in reducing a linear second order PDE to a simpler form. In the simpler form, the coefficients of second order derivatives will be constants. To start with in a second order linear equation coefficients are functions of x and y. But this ODE will help us in obtaining a simpler form of these equations in which the coefficients of the second order derivatives are constants.

The simpler form would be obtained using change of coordinates. The body plays an important role in determining the change of coordinates.

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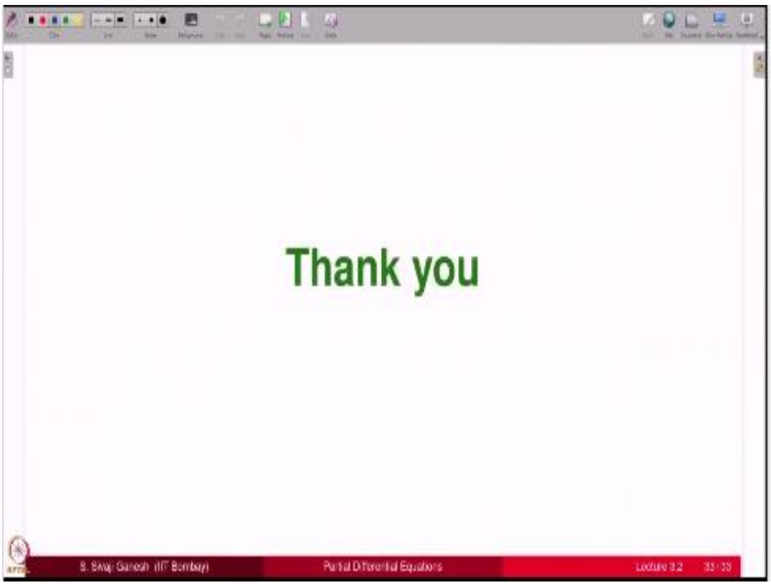

Thank you.