Partial Differential Equations
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Lecture – 2.16
First Order Partial Differential Equations
Conservation Laws with a View Towards Global Solutions to Burgers Equation

In the last lecture, we considered Cauchy problems for Burgers equation, where the Cauchy data or the initial data was not smooth. Or even when the initial data was smooth, we found that in one of the examples that solution is only piecewise smooth. So, such functions cannot be solutions in the usual sense which we described as classical solutions. So, we asked the question.

Is there a framework under which we can admit such functions also as solutions to Burgers equation and in general for a first order partial differential equation. So, we look into that in this lecture.

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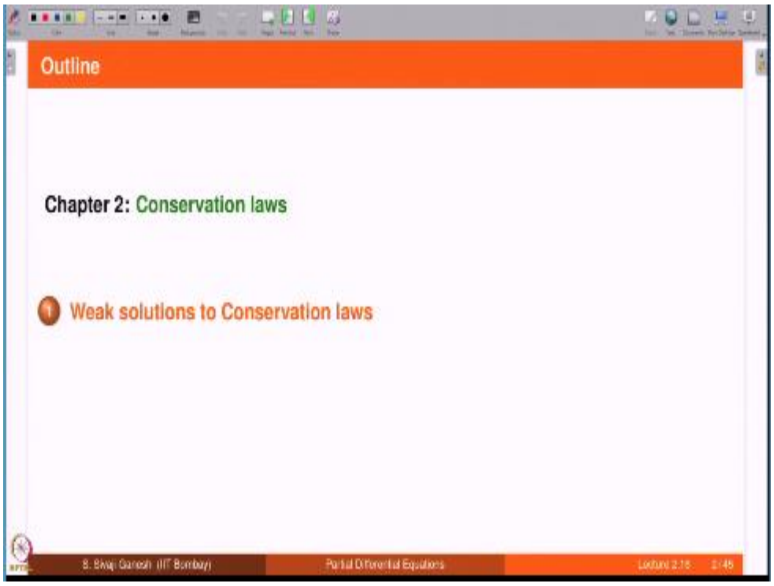

So, do not be afraid by the word conservation laws. We are not going to study too much about it. It is only in the context of Burgers equation that we are going to discuss.

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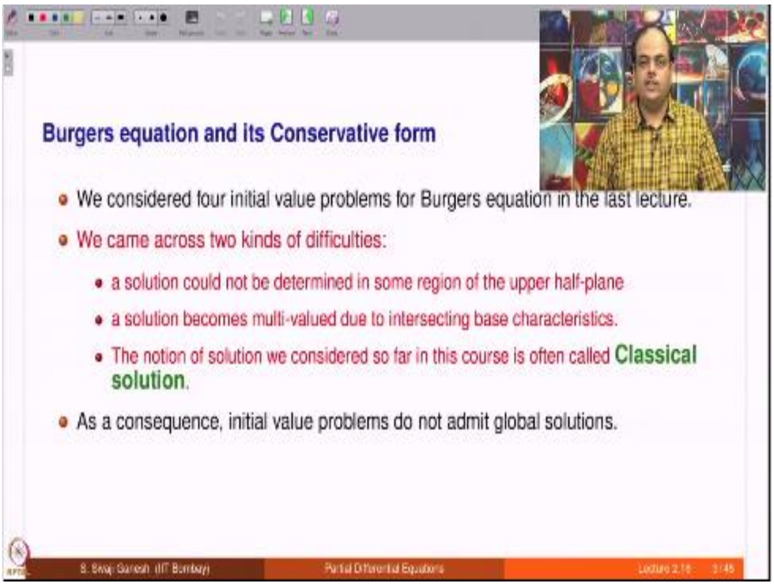

Burgers equation can be written in what is called a conservative form. We will come to that. So, this is a brief recall from the last lecture, we considered for initial value problems for Burgers equation. We came across 2 kinds of difficulties. First one was a solution could not be determined in some region of the upper half plane because there were no base characteristics.

And a solution becomes multivalued due to too many base characteristics entering a particular region. That is another reason why we could not define what is the solution there. There was some ambiguity. So, the notion of solution we considered so far prior to the Burgers equation is often called classical solution. As a consequence, initial value problems do not admit global solutions. We already understood that.

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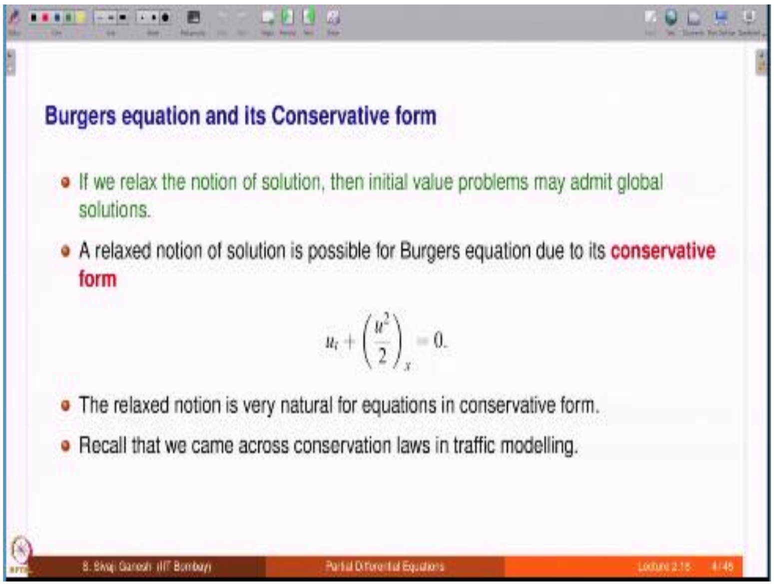

If we relax the notion of solution, then initial value problems may admit global solutions. That possibility we will get. A relaxed notion of solution is possible for Burgers equation due to its conservative form which is given by u t + u square by 2, this bracket x stands for do by do x, differentiation with respect to x. So, assuming u is smooth when you do expand this by chain rule what you get is u t + u u x = 0.

We have used the u y, when we solved it by characteristics method and I told you that y has the interpretation of time. So, therefore, it is u t. This is fine. The relaxed notion is very natural for equations in conservative form. We are going to introduce one relax notion. Of course, conservation laws are not new to us; we have seen them already in traffic modelling.

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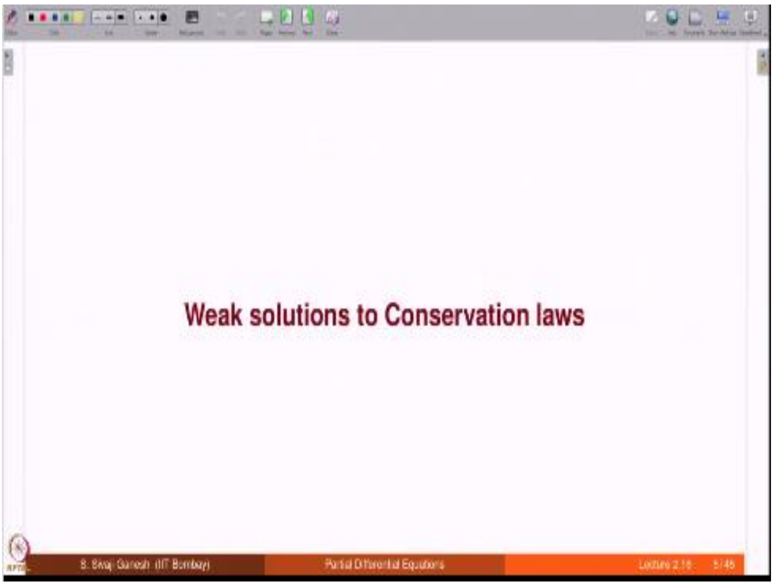

So, this notion is called generalised notion that we are talking about is called a relaxed notion is called weak solutions to conservation laws.

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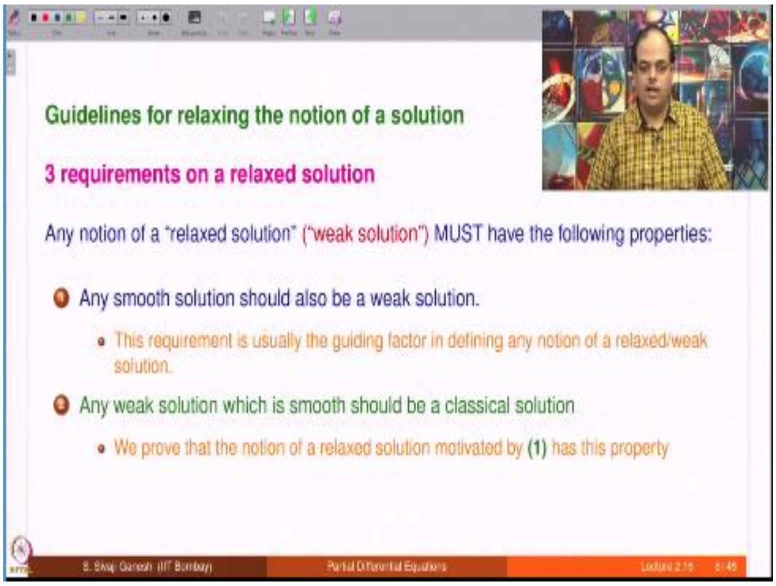

So, there is a guideline for relaxing the notion of a solution. Even for a conservation law we can demand that it should be a nice function differentiable function so that u t + u square by 2 differentiation with respect to x = 0 is actually u t + u x = 0. But we want to admit in our function, which are not so much smooth as solutions. Therefore, we would like to define a new concept of a relaxed solution and there should be some guidelines.

What are those? 3 requirements are there. Any notion of relaxed solution, we may call it weak solution must have the following 3 properties. What are they? Any smooth solution should also be a weak solution. This is to be expected. It should be there. Otherwise, you are defining some new solution. Earlier we said we want to relax because some equations may not have the smooth solutions or a classical solutions.

But if they do, if they do have classical solution, we would like that the relaxed notion also it admits as a solution. Therefore, small solution should also be a weak solution. This is usually the guiding factor in defining any notion of weak solution. We will soon see how that is going to be done. And any weak solution if it is smooth, then it should be classical solution. This is the second requirement of course.

So, we prove that the notion of relaxed solution which we are going to give motivated by 1. So we are going to define what is called a relaxed solution or weak solution concept. And then we show that any smooth solution is a weak solution. And then we also show that any weak solution which is smooth should is actually a classical solution. We will show that. These are the 2.

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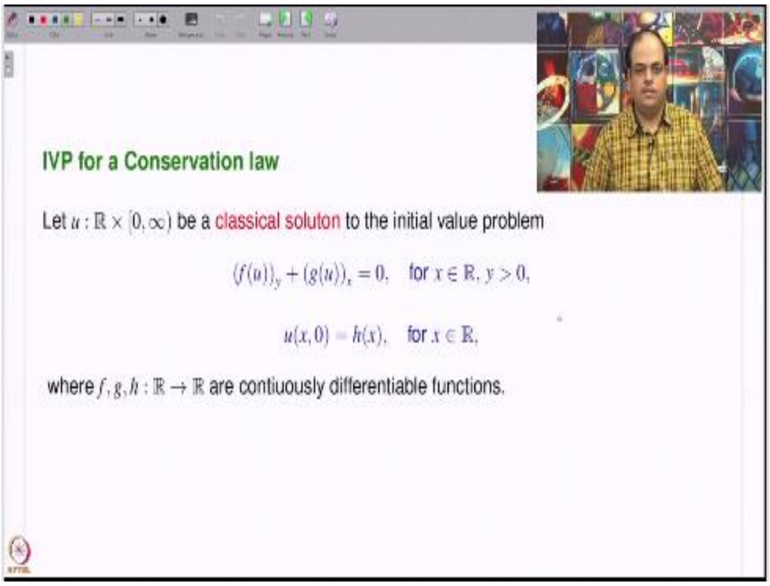

And then the third one is the most important thing and we are not going to address that. Any reasonable problem should have a solution. Otherwise, what is the use of notion of a solution, when you cannot show that such a solution exists, when the problem is reasonable? I am not expanding what is reasonable, but this is what one has to remember. These are guidelines. Discussion of this requirement for the notion of weak solution, we are not going to do.

That is beyond the scope of the course. So, let us look at the initial value problem for a conservation law. Burgers equation in the conservative form is a special example of this, where f of u is u, g of u is u square by 2. So, assume that this equation has a classical solution that means a differentiable solution so that this you can expand: f dash u into u y, g dash u into u x = 0 by chain rule and u x, 0 = h x.

This u of x, 0 makes sense and is equal to aprioiri x and h x. So, here f, g and h are apriori given to u smooth functions.

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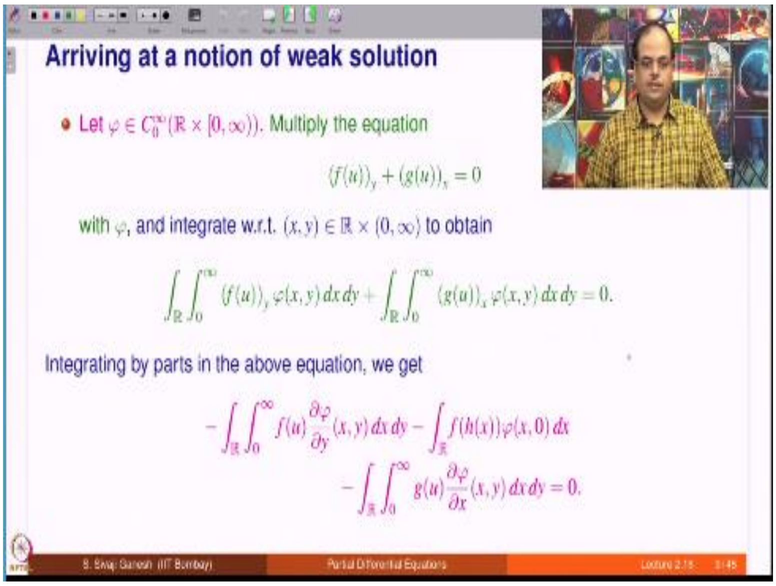

So, let us see how we arrive at a notion of weak solution. First thing is to take a function phi which is compactly supported in this domain R cross, close intervals 0, infinity, closed at 0. And C infinity that is it is differentiable any number of times. C infinity function with compact support and the domain is R cross 0, infinity, 0 closed. And first thing is your equation you have to multiply with phi.

This equation is multiplied with phi on integrate. So, this equation is simply this equation multiplied with phi. The 2 terms are separated and integrate on your domain R cross 0, infinity. So, what have we achieved? Nothing. We could not, we have not relaxed that u can be a lesser smooth function for non differentiable function etc. Therefore, first thing the moment we see a derivative here and a derivative here first thing is, idea is, to shift this derivative to C infinity function that we have.

So, therefore, we have to do integration by parts, in this integral with respect to y, in this integral with respect to x. So, with that we get from here we get this integral. And from from here, when we do we get this integral and in this integral, there is a one boundary term which is here. Because we are taking phi which are compactly supported, close 0, infinity not open 0, infinity.

If it is open 0, infinity this term will not be there, because phi x, 0 will be 0 in that case. But we are taking with this. That is because we want to account for the initial condition. That is what is going to come here. So, this is what you get at the end of integration by parts.

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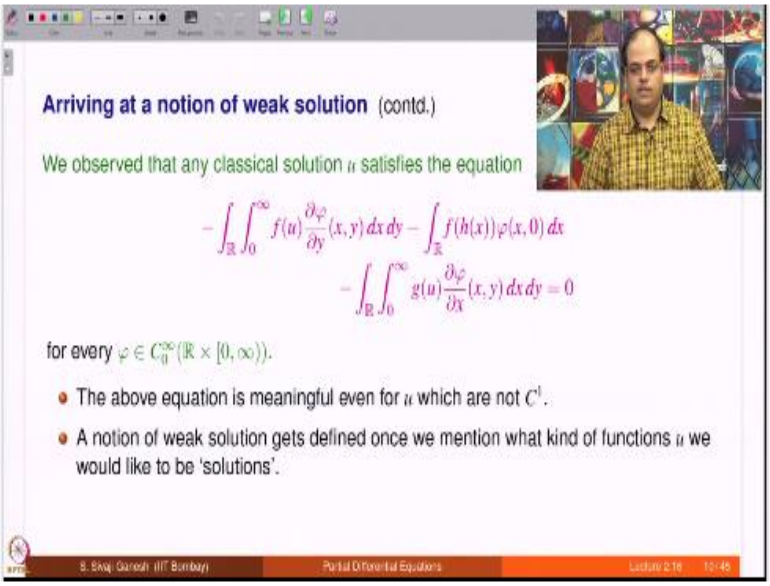

So, what we observed just now is that any classical solution satisfies this integral equation and for every phi in C 0, infinity R cross 0, infinity. And this equation, if you see it is meaningful, even for u which are not C 1. For example, u is only continuous, f of u makes sense. f of h of x of course makes sense, g of u makes sense. And these integrals are on infinite domains, but phi is having compact support.

So essentially the integrals are on f a bounded set. Therefore, when you integrate continuous functions and bounded sets, it is integrable. These are well defined integrals. So, therefore, this is meaningful even for u which are not C 1. I just use a word u is continuous; of course, you do not need even u to be continuous. What all you need is this should make sense.

A notion of weak solution gets defined once we mentioned what kind of functions u we would like to allow them as solutions. So, we have to decide which u you are going to allow as solutions for your problem then the notion gets defined. You will ask that this integral equation should be satisfied for all phi in this space. And u should lie in some space that your to identify, you have to decide.

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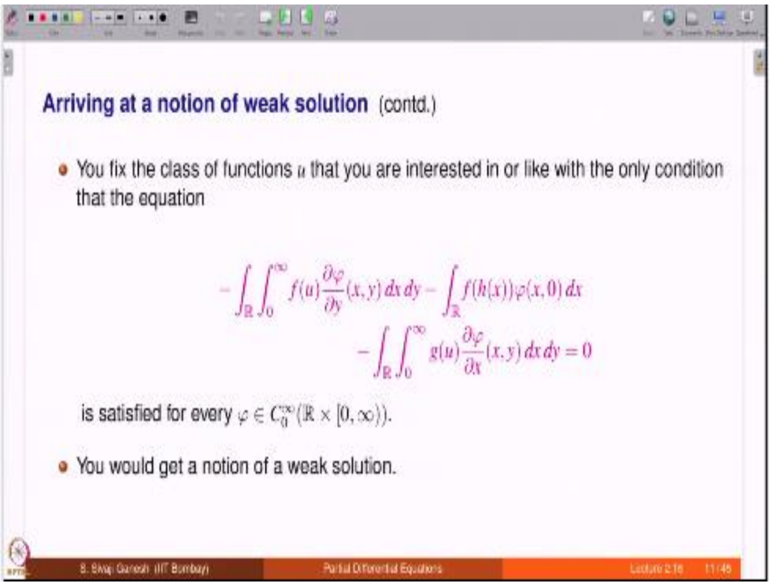

So, you fix the class of function that you are interested in or you like with the only condition that this equation is satisfied for every phi. You will get a notion of weak solution.

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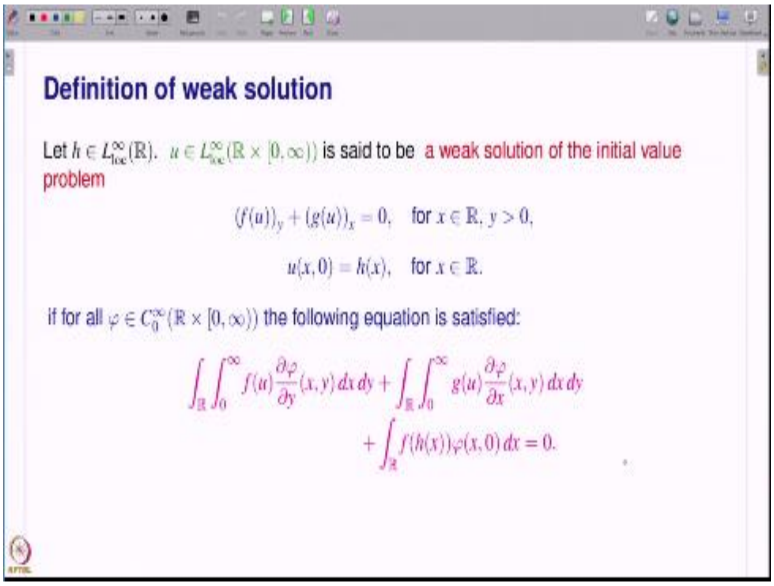

So, let us give one notion of weak solution here. Assume that h is L infinity of r, this loc should not be there, because L infinity functions are bounded functions. Or if I put L infinity loc of R, it just means that it is bounded function on every compact set. That is good enough. So, L infinity loc is good. We can keep this. L infinity just means that Lebesgue measureable functions which are bounded essentially bounded functions.

Whenever there is a loc it means that on compact sets some property holds. L infinity loc means it is in L infinity of every compact subset of R. Now, u in L infinity loc of this set, see now, we just want bounded measurable functions as solutions. Not even bounded everywhere, bounded on every compact set that is good enough for this notion. In particular non differentiable functions, all of them will come under this if they satisfy this condition.

So u in L infinity loc is said to be a weak solution of the initial value problem, which is here. If for every phi C 0, infinity, this integral equation is satisfied, this was first of all derived from this equation. Assuming that use a smooth solution multiplied with the phi coming from the space and then we found this is satisfied. Now, we forget all that and we say as long as this is satisfied I am happy.

And now I put some conditions so that this makes sense. And we are demanding it should be equal to 0. One such class is L infinity loc R cross 0, infinity.

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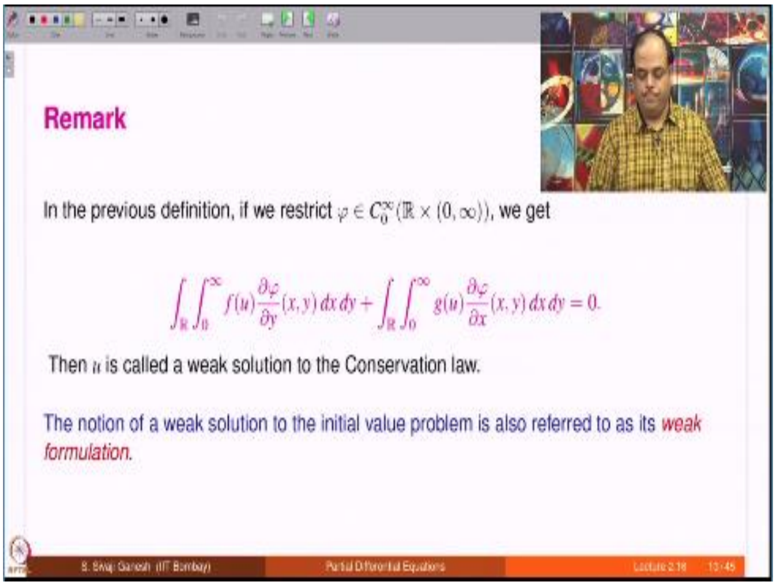

The previous definition, if we restrict phi to compactly supported functions infinitely many times differentiable R cross open 0, infinity, then the boundary integral, the integral and R, it will vanish. It will be 0. As I pointed out phi of x, 0 is 0. So, we are left with only these 2 terms, then u is called a weak solution to the conservation law. It is not Cauchy problem for the conservation law but weak solution to the conservation.

That means we are worried only about the equation and not the initial conditions. The notion of a weak solution to the initial value problem is also referred to as its weak formulation. The integral equation which we saw on the previous slide is often called a weak formulation of the conservation law.

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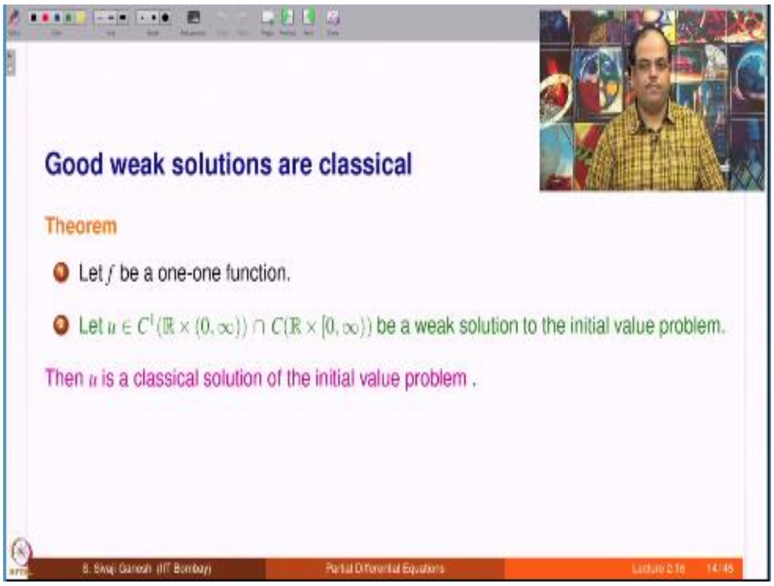

Now, this is what, we have finished the guiding principle part one. All smooth functions, smooth solution should satisfy the new formulation. That is how we have derived. Now, we will go to step 2, where we are going to show if you have a weak solution and it is smooth, it must be classical solution. That is what we are going to establish now. So, for this we need to assume slightly one extra condition on f.

Of course, for Burgers equation f of u is u. Of course that is a one-one function. So, let f be a one-one function, let u be a smooth function that is C 1. Because it is a first order PDE C 1 is required. And here I put continuity up to 0 that means u of x 0 makes sense. If you have continuity upto this 0, close 0 that means that u of x, 0 makes sense. Suppose this is a weak solution to the initial value problem, then u is a classical solution of the initial value problem.

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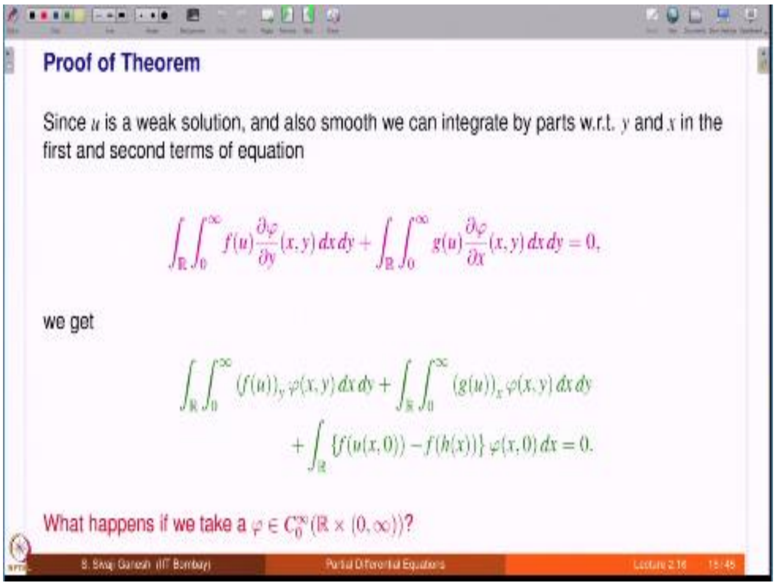

That is the conclusion. So, here we are assuming u is a weak solution and smooth, then we are going to say the classical solution of the initial value problem. So, since u is a weak solution and also smooth, what does it mean? I can go back, the strongness and the weakness, what is the connection integration by parts. So, I need to do reverse integration by parts from the weak formulation.

So, this is the meaning of what we have. Here it is a weak solution to the conservation law. Now if you do integration by parts, you will get back this. This is a boundary term that is going to come. So, what happens if I take a phi which is C 0, infinity open 0, infinity, this term will not be there.

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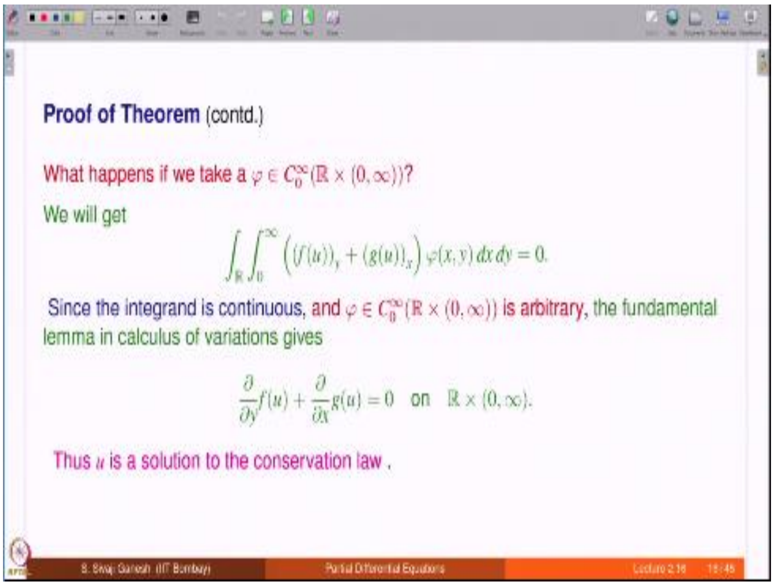

We will get this. Since the integrand is continuous, integrand is continuous. And phi C 0, infinity is arbitrary, fundamental lemma in calculus of variation, essentially it means if you integrate against C 0, infinity function, a certain function and you always get 0, then that function must be 0. So, it is essentially like this. Just imagine something like this. We have phi psi = 0 for all phi in C 0, infinity functions omega that would imply that psi is 0.

Under various assumptions psi, it is true. Definitely when psi is continuous, it is true. So, therefore u is a solution to the conservation law.

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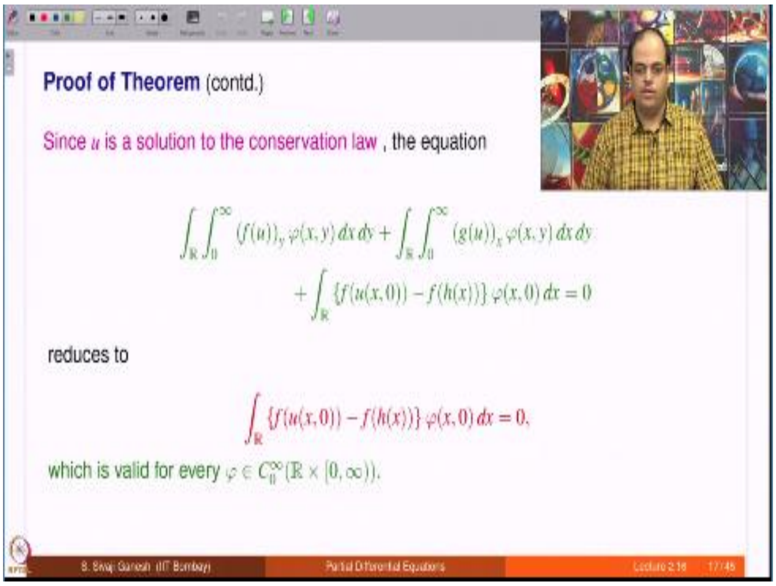

Since u is a solution to the conservation law, this equation now just becomes the last, last term, because these 2 together are 0. So, we have this. Now, once again phi is arbitrary, therefore, this must be equal to this. And if the function f is one-one inside thing must be equal to inside thing. That is idea. So, this is true for every phi in C 0, infinity R cross close 0, infinity.

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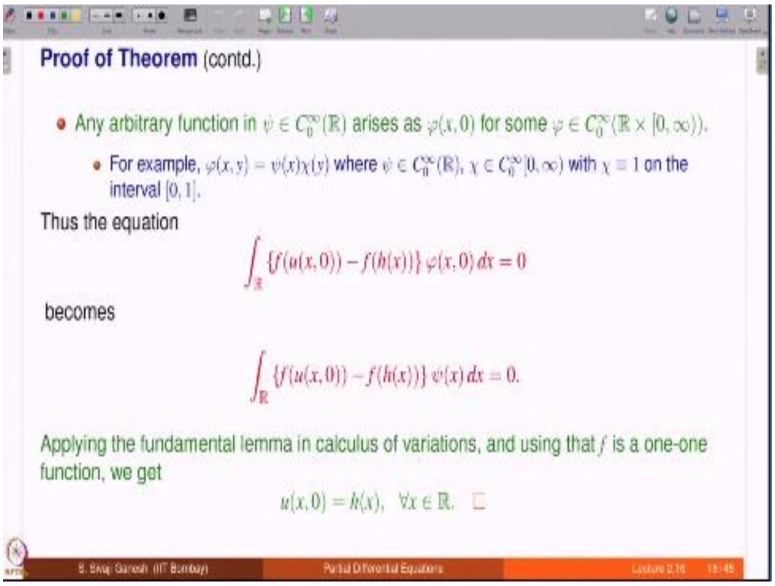

Now, if you notice, this is only function of x. Here, this is only a function of x. It is not like y, there is no y. It is just a function of x integral is an R. So, we can get any smooth function C 0, infinity function of R through this phi.

Any psi in C 0, infinity R looks like psi of x, 0 for some phi here. For example, phi of x, y = psi x into chi y, where psi is C 0, infinity R that is given to you. And chi is C 0, infinity of close 0, infinity with chi identically equal to 1 in some interval 0, 1. So, when you put y = 0, psi of 0 is 1. Therefore, you get psi of x. So that is simply this. Now, once again you apply.

This is a continuous function, integrate against any C 0, infinity of R function is 0, then this function must be 0. That result I am loosely calling it as fundamental lemma in calculus operations. So, using that f is a one-one function, we get to u x, 0 = h x. Otherwise to start with you get f of u x, 0 = f of h x. Since f is one-one, you can take away the f and you get u x, 0 = h x.

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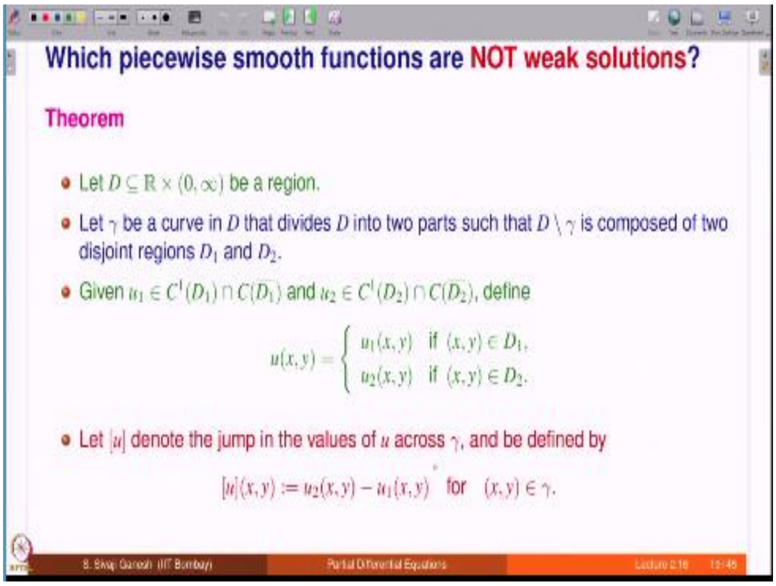

Now, we are going to look at another question. Which piecewise smooth functions are not weak solutions? The following theorem will be helpful in deciding that. Suppose you have a set D subset of R cross 0, infinity and suppose you have a curve that divides D into 2 parts. So, generally one writes this kind of picture, this is D and you have a curve gamma. And it cuts this into 2 pieces, D 1 and D 2.

And suppose u 1 is C 1 and continuous upto the boundary D 1 closure means up to boundary. That means u 1 is here. It is smooth. Similarly u 2 is here. So, u 1 is C 1 of D 1 and also continuous up to the boundary so that I want to talk about the values of u 1 on gamma. Similarly, I want to talk about u 2, values of u 2 on gamma. So, I require C 1 of D 2 and continuous upto D 2 closure, so that u 2 on gamma is also meaningful.

So, 2 condition generally people write as an intersection. So, this and this and u 2 is C 1 in the second domain and continuous up to its closure. Define a function u on D now like this. u

1 in D 1, u 1 in D 2. On gamma we are not defined. Now, let this bracket u denote the jump in the values of u across gamma. That means we had this. This is D. We had D 1 here, D 2 here. Now u 1 on gamma makes sense at points of gamma.

Similarly u 2 on gamma makes sense. So, we can look at the jump u 2 at a point p on gamma - u 1 at the point p. This is the jump, jump in u at the point p. You can also define u 1 - u 2 at point p. I have used u 2 – u 1, both are same. You have to have just a consistent way of defining it. So, these are definition, At any point x, y which is on gamma, you define the jump in u as u 2 minus u 1 at that point.

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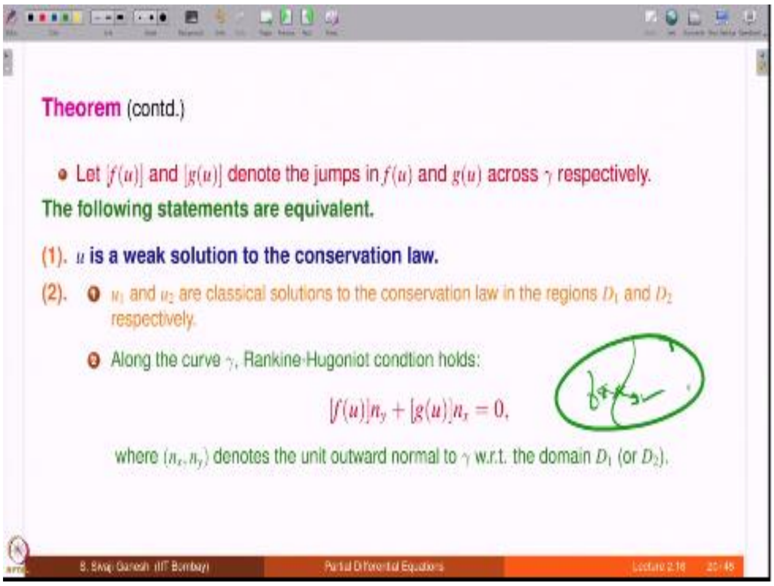

So, we have defined what is the jump at all points on gamma. Now, similarly, you define f u, jump in f u and jump in g u. Jump in f u at a point p is f of u 2 at p - f of u 1 at p. That should be the definition. Jump in f of u at a point p is on D 2, the value is going to be u 2. f of u 2 at p minus f of u 1 at p. That is a jump. Similarly, g jump in g is also defined. Then the following statements are equivalent.

Saying that u is a weak solution to the conservation law is same as saying that u 1and u 2 are classical solutions in the case of the conservation law in both the domains D 1 and D 2, u 1 in D 1 and u 2 in D 2. And along the curve gamma certain condition holds which is a few n y + g u n x = 0. What is n x, n y? It denotes the unit outward normal to gamma. That means, we had like that, we had a gamma. Take a point, so, normal direction.

This tangential direction, this is normal direction. You can take either this or this. Any one of them you choose, does not matter. That is what we are saying. So let us choose. For example, this does not matter. Because the equation anyway is equal to 0. If you replace n x, n y with - n x, - n y, it is the same.

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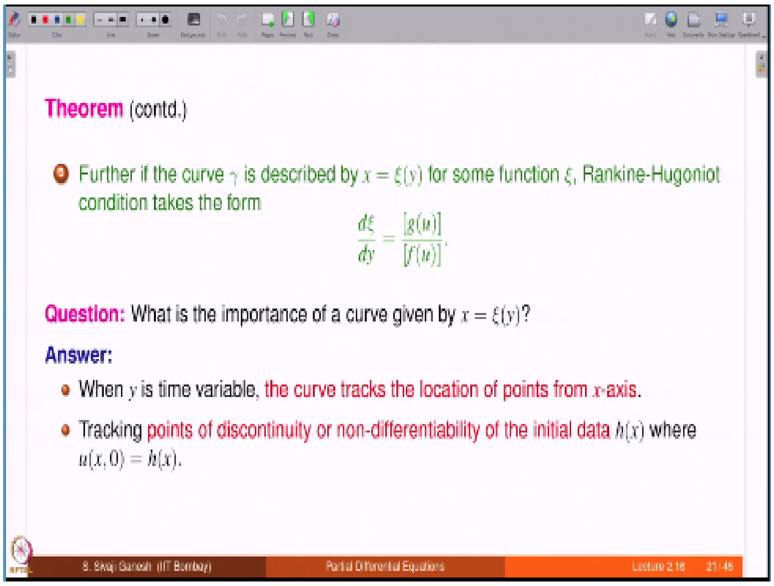

So, further if the curve is described by x = xi y, there is an interesting assumption is given by x = xi y. What does it mean? The curve looks like xi of y, y. That means it is a graph with respect to the y variable. Then Rankine-Hugoniot condition takes this form, d xi by d y = jump in g by jump in f. What is the importance of a curve given by x = xi y? See y is a time variable. Y is time variable.

So it is like xi of t,. x = xi of t. It is giving you some points in x, which are the values of somebody at t. So, it is telling you the location of somebody as time evolves. So, when y is time variable, the curve tracks the location of points. If the initial data is in x axis, the y, it will track the location of the point from x axis. Tracking points of discontinuity or non differentiability of the initial data where u x, 0 is h x. h x is the initial condition. So, these kinds of things can be tracked by this curve.

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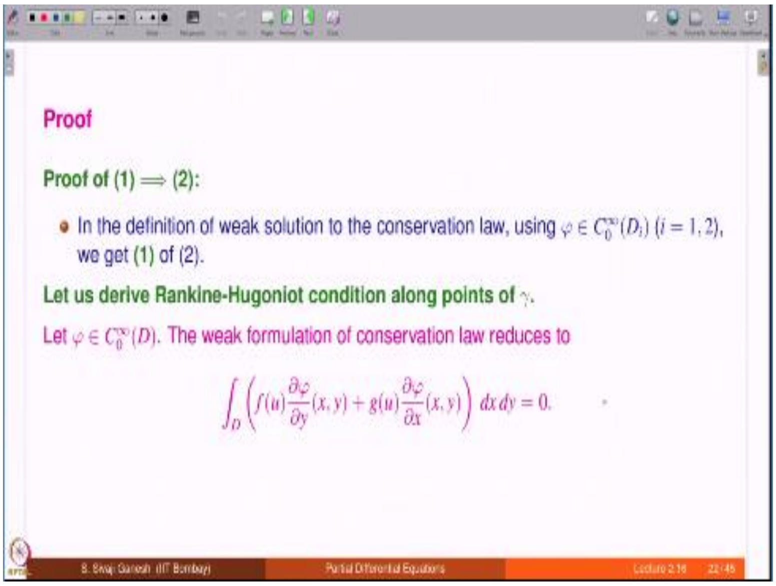

That is why we always assume x = xi y. So, proof of 1 implies 2. In the definition of a weak solution to the conservation law. Using phi, I am allowed to use any phi which is C 0, infinity of D. Now I use C 0, infinity of D 1 and D 2. We get the first part. It is a solution. Because of the smoothness we do integration by parts. Now let us derive the Rankine-Hugoniot condition along points of gamma. So, take a phi which is C 0, infinity of D.

Now I cannot take D i. If I take D i I am ignoring other other D i. If I take D 1, C 0, infinity D 1, I am forgetting D 2. Now, because the jumps are across gamma, gamma is there for both D 1 and D 2. It is a boundary. So, therefore, you take C 0, infinity of D and the weak formulation of the conservation law reduces to this.

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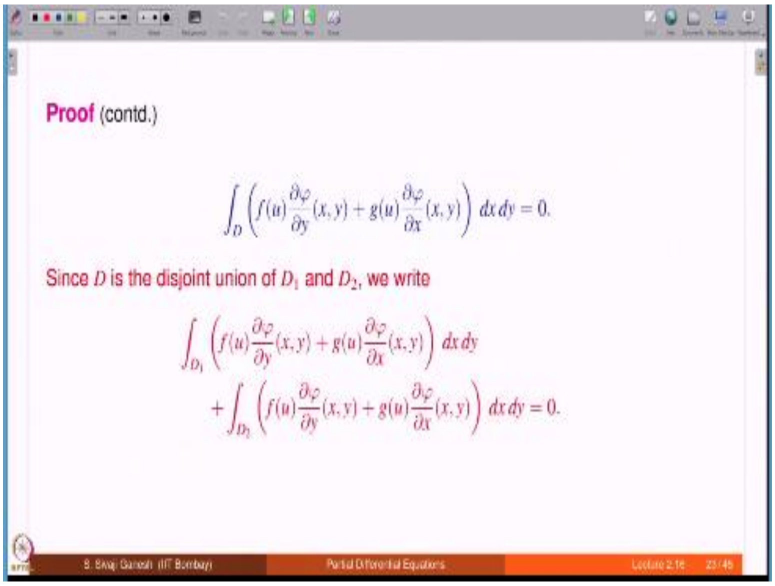

And since D is a disjoint union, you write this integral as integral over D one + integral over D 2.

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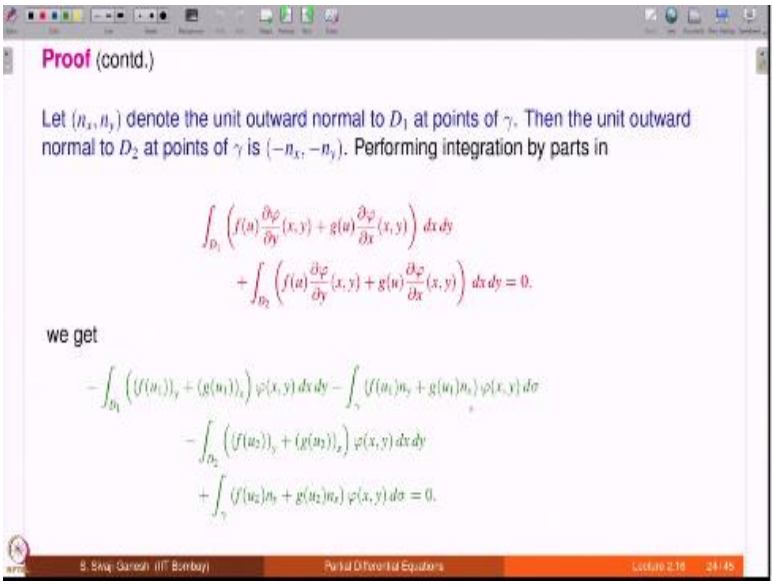

Now I think we have to do integration by parts. So, if n x and n y denote unit outward normal to D 1, then – n x, - n y denotes unit outward normal to D 2. Because this unit outward normal is what appears in integration by parts. So, performing integration by parts, we get these terms. Do by do y has gone to and D 1 u is u 1. u is u 1 on D 1. Similarly here u is u 2 on D 2.

That is why you got a f u 1 y from here g u 1 x from here and this is a boundary term coming from this integral, one with respect to x, one with respect to y. So, this is the other one.

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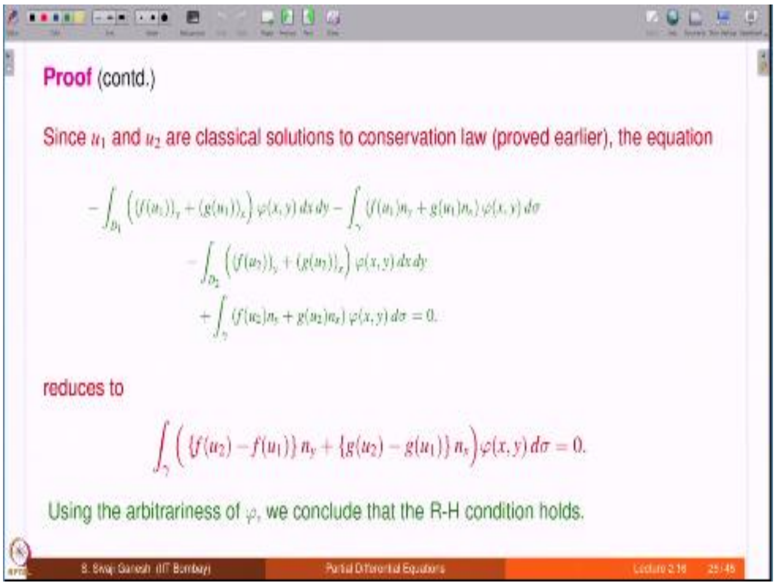

Or you can also treat this as a Greens theorem. I will leave it to you. We call integration by parts by various names depending on the context. Fine. u 1and u 2 are classical solutions to conservation law we already proved. Therefore, some things will be 0. The domain integrals will go off, what remains is the gamma integral. This is what you have now, the notation n x, n y is fixed, n x, n y is outward normal to D 1.

Then we have used – n x, - n y is outward in normal to D 2 while doing integration by parts. So, finally, we end up with this gamma integral. Now, phi is arbitrary. Therefore, integrand must be 0 and which will give you Rankine-Hugoniot condition.

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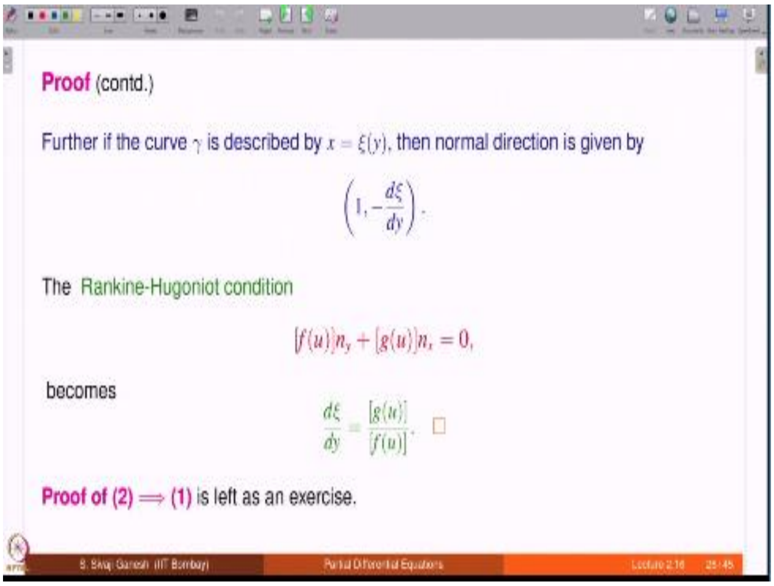

If the curve is given by x = xi y, we compute the normal. Normal direction is 1, - d xi by d y. Therefore, Rankine-Hugoniot condition becomes this equation, d xi by d y = jump in g by jump in f. 2 implies 1 is very simple. So that is left as an exercise.

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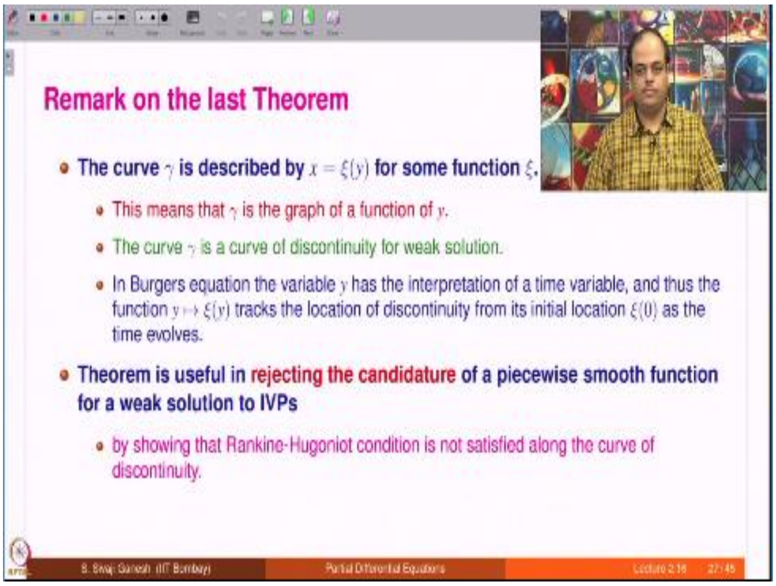

A few remarks on the last theorem. We already started with this remark when we started this theorem. The curve gamma described by x = xi y for some function xi. This means that gamma is the graph of a function of y. The curve gamma is a curve of discontinuity for a weak solution. In Burgers equation, the variable y has the interpretation of time variable. And as y going to xi y tracks the location of discontinuity from its initial location as the time evolves.

So, theorem is useful in rejecting the candidature of a piecewise smooth function for a weak solution to IVP. How you simply show the Rankine-Hugoniot condition is not satisfied along the discontinuity curve. And therefore, it cannot be a weak solution. So, it is useful in rejecting the candidature.

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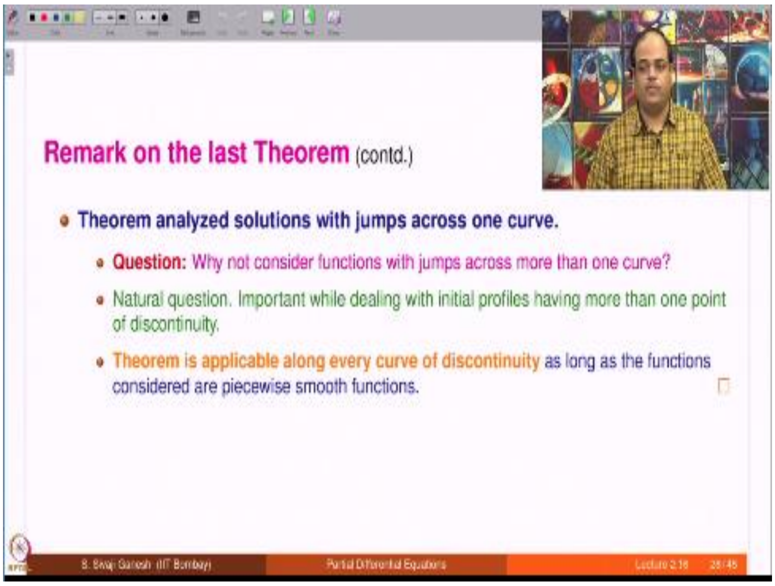

Theorem analyzed solutions with jumps across one curve. Why not consider a function with jumps across more than one curve? Very natural question. It is important while dealing with initial profiles, which have more than one point of non smoothness. Theorem is applicable along every curve of discontinuity as long as the functions considered are piecewise smooth functions.

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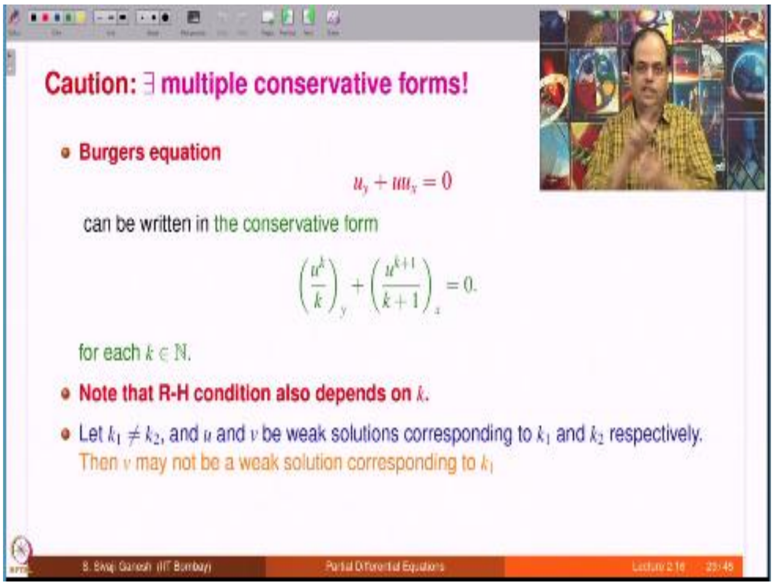

Caution is, there exist multiple conservative forms, multiple. For Burgers equation we have written 1, u y + u square by 2 of x = 0. And these are so many, as many as natural numbers. So, we have here at least, at least infinitely many are there. Now, which one will you consider? You have to decide. So, note that R-H condition also depends on k. Because it is jump in g by jump in f that is the equation for d xi by d y.

Now g varies. This is the g, new g. It depends on k. Similarly, this is f depends on care. So, our h condition depends on k. So, what does that mean? If you take 2 different k s and take u and v, which are weak solutions corresponding to the different case then the vice versa may not be true.

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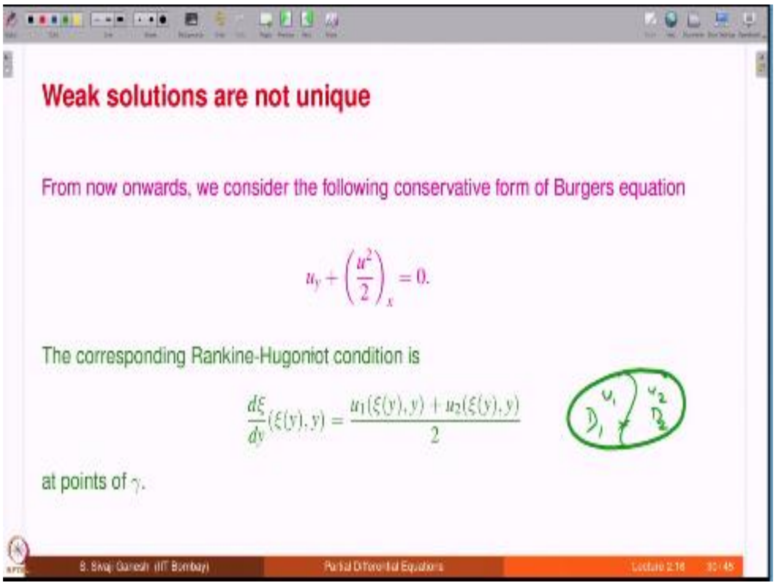

So, therefore, in applications we have to realise which one is the most correct or useful conservative form of your equation also. So, weak solutions are not unique, because we have relaxed the notion of a solution. Now a lot more functions. We do not expect uniqueness now that is what is demonstrated by these examples. Now, we are going to use the following conservative form throughout this lecture.

From now onwards, u i + u square by 2 x = 0. Let us write down what is the Rankine-Hugoniot condition. It will turn out to be u1 at point of gamma + u 2 at the point of gamma by 2. That means the average of u 1 and u 2. So, for example, this is the 1, D 1, D 2, u 1 here, u 2 here. Take a point. Take the value of u 1 and take the value of u 2 and take the average. That should be the d xi by d y, the slope with respect to y here.

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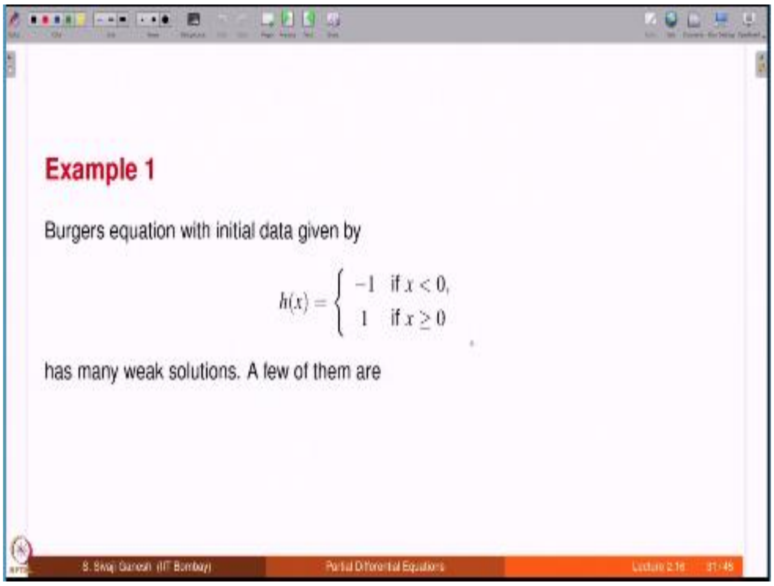

This is example 1 that we considered in the last lecture, with initial data - 1and 1. - 1 upto x less than 0, 1 for x greater than or equal to 0. It has many weak solutions. Few of them are being given here.

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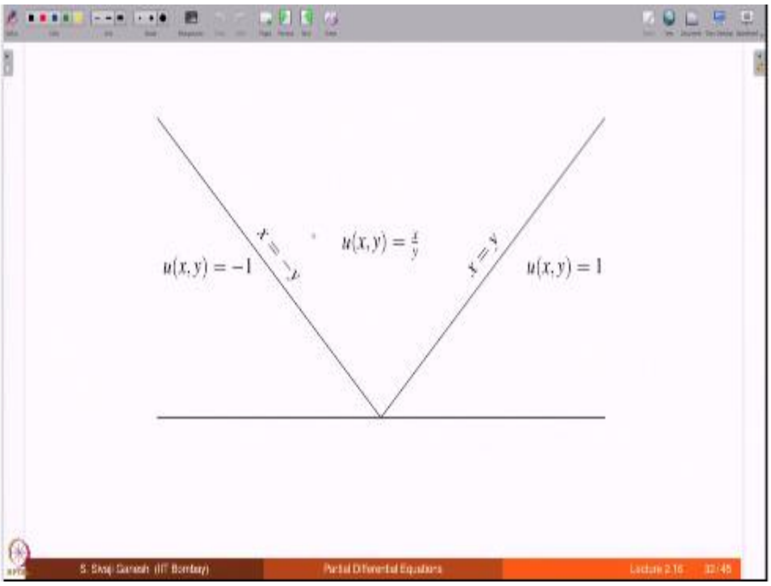

For example, here we have 2 curves of discontinuity. One is the line x = - y, another one is a line x = y. Now, let us see whether our R-H condition is satisfied here. What is the average of I mean this is D 1, this is D 2 here. So, u 1and u 2 average. x by y on this line is – 1. Because the line itself is x equal to minus y therefore, x by y is – 1. Therefore, what is the average of – 1 and – 1? It is -1.

So, x = xi y. That is the curve we are considering and we have to look at d xi by d y. So, in this example, d xi by d y is - 1 and which is also equal to the average. Therefore, R-H condition is satisfied across this line. Now, let us discuss across this line. Across this line one side the value is 1 other side is x by y, but on this line x by y is 1. Therefore, average is 1 and that is same as x = xi y, derivative of xi y is 1.

So, Rankine-Hugoniot condition is also satisfied across this line. Therefore, this is a weak solution.

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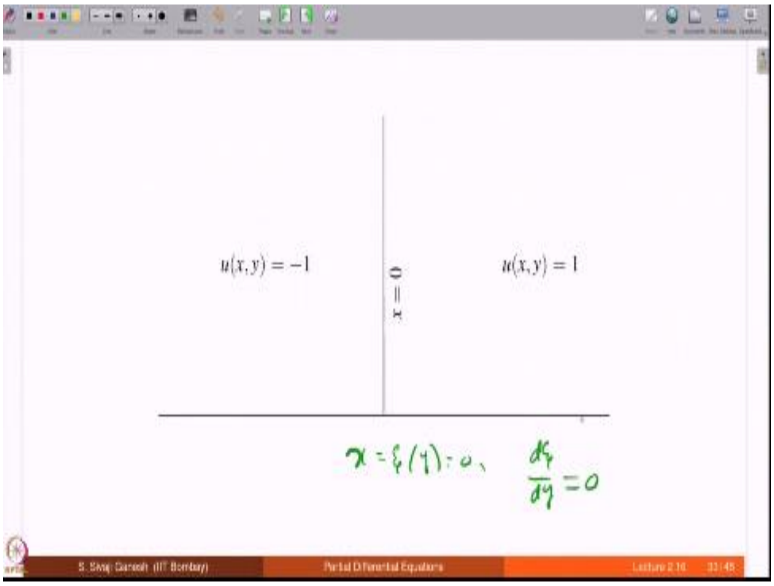

Let us look at this. Here the curve of discontinuity is x = xi of y = 0. So therefore, d xi by d y, of course is 0. And that is same as the average of - 1 and 1. – 1 + 1by 2 is 0. So, this is also a weak solution.

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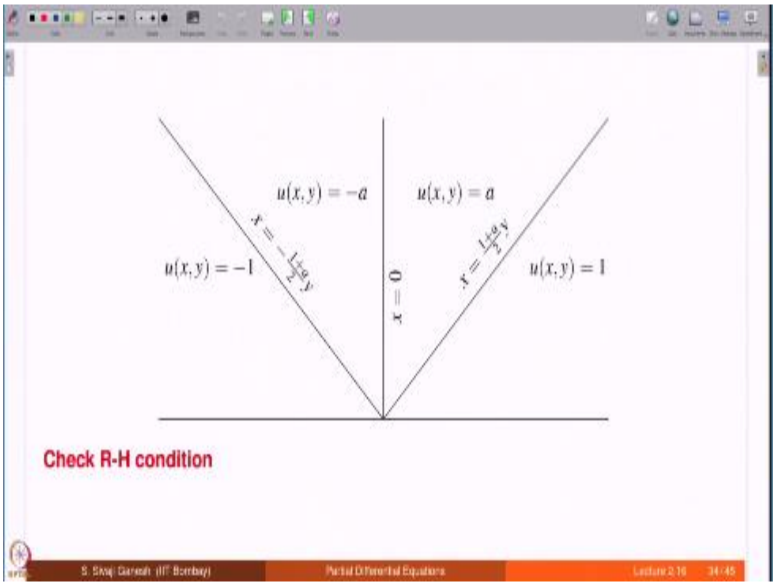

Let us look at this. In this case d xi by d y is this quantity of -1 + a by 2 - 1 + a by 2 and the average is also that, - 1- a divided by 2, which is precisely this. Here d xi by d y is 1 + a by 2 and that is precisely the average 1 + a by 2. Here xi y is 0, d xi by d y is 0. For this curve d xi by d y is 0 and that is also the average, - a + a by 2 is 0. So, here also R-H condition is satisfied across all the 3 discontinuity curves.

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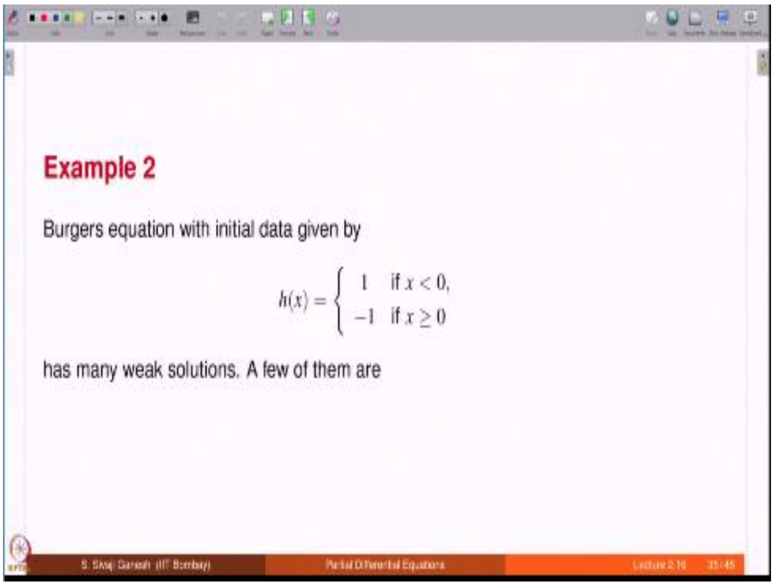

Let us look at the example 2 which is 1 and – 1. This also has many weak solutions.

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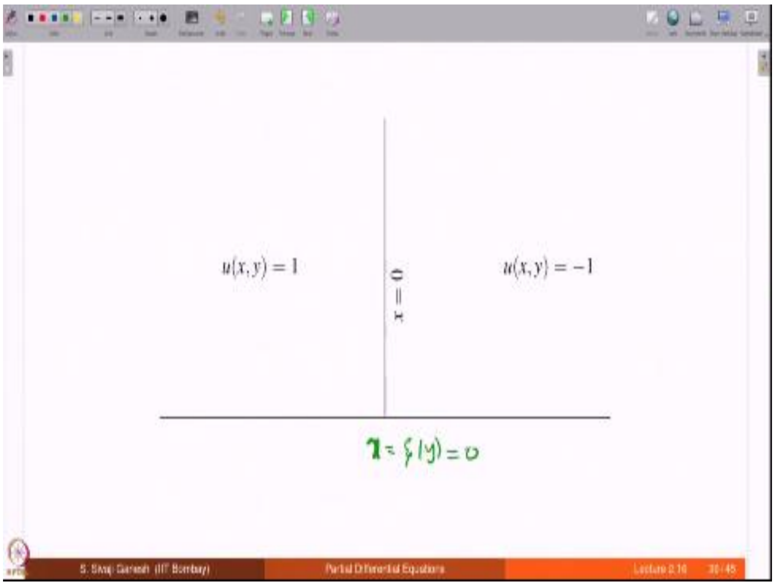

This is one of them. Here d xi by d y is 0. Because this equation x = xi of y = 0. Therefore, d xi by d y is 0 and that is the average of 1 and minus 1. So R-H condition is satisfied.

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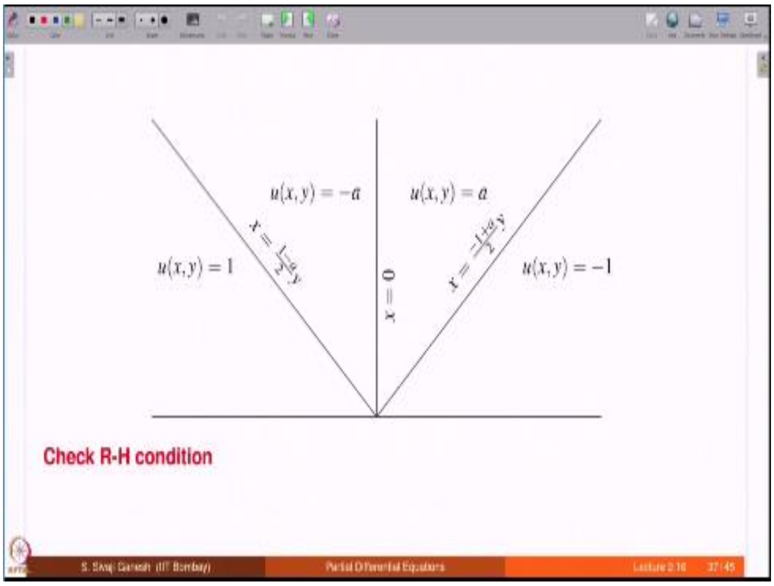

Here it is 1 - a by 2 and that is precisely d xi by d y. This is xi of y, d xi by d y, Here - 1 + a by 2 and that is d xi by d y. So, R-H condition is satisfied across this line also. What about this line? Also. Because - a + a by 2 is 0 and that is xi dash of y. Xi of y is 0. Therefore, xi dash is also 0. So, R-H condition is satisfied across all the 3 lines.

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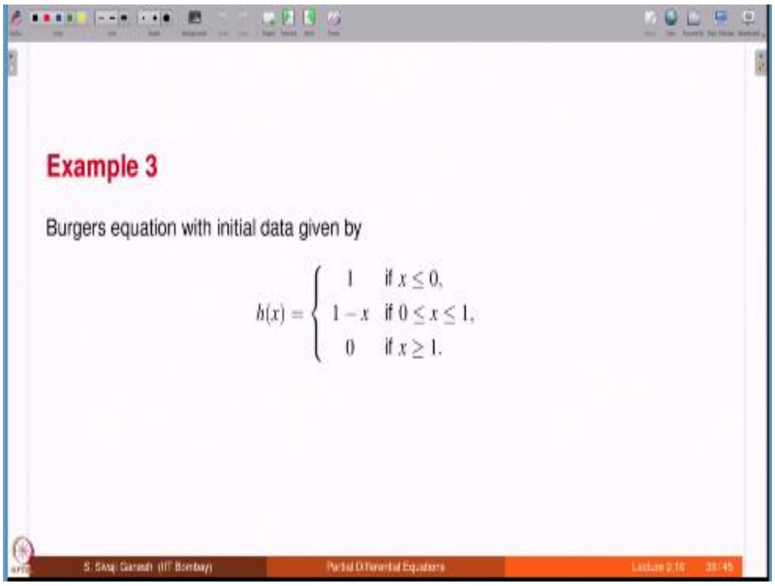

Let us look at the example 3.

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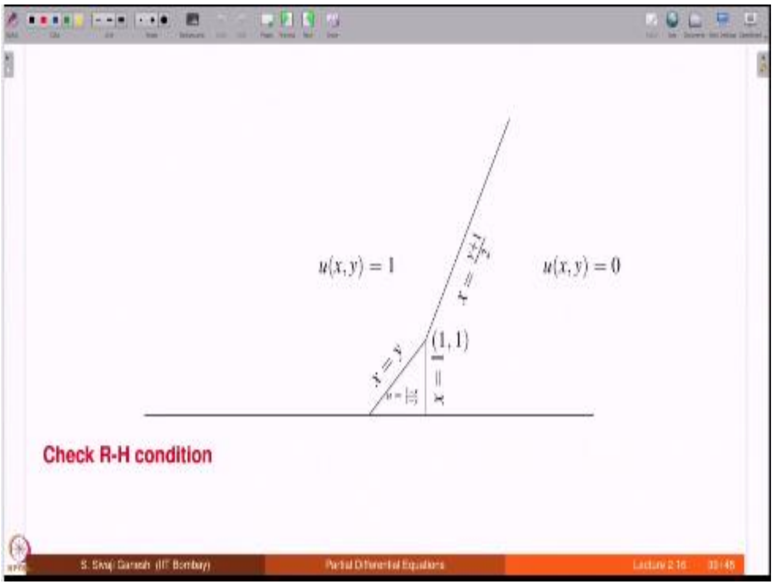

You got this. Here, of course, there are many pieces. This is 1 line of discontinuity. This is another line, this is another line. Let us look at the top line, this line. x = y + 1 by 2. What is xi dash for this line? Half, y by 2. If xi of y is y by 2 + 1 by 2, derivative will be 1 by 2. And that is precisely the average 1 + 0 by 2. So, across this line R-H condition is satisfied.

Let us check whether across this line R-H condition is satisfied. Across this line 1 - x by 1 - y is nothing but 1. Because on this line x = y. So, therefore, 1 - x by 1 - x, u is 1 from this side, u is 1 from this side. Therefore, average is 1 and that is also xi dash of y. Xi of y is 1, therefore, xi of y is y. Therefore, xi dash of y is 1. Therefore R-H condition is satisfied here also.

Now, let us look at this line x = 1. On this line x equal to 1, this u is 0, because it is 1 - x by 1 – y. x = 1 means it is 0 by 1 - y. So u is 0 this side. u is 0 this side. So average is 0. Now, what about xi of y, its derivative. Xi of y here is 1 and its derivative is 0. Therefore, R-H condition is satisfied across all the 3 line segments.

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Now, the concept of entropy solutions, we are not going to study too much about it, but just to get some awareness. So, we saw that there are too many weak solutions.

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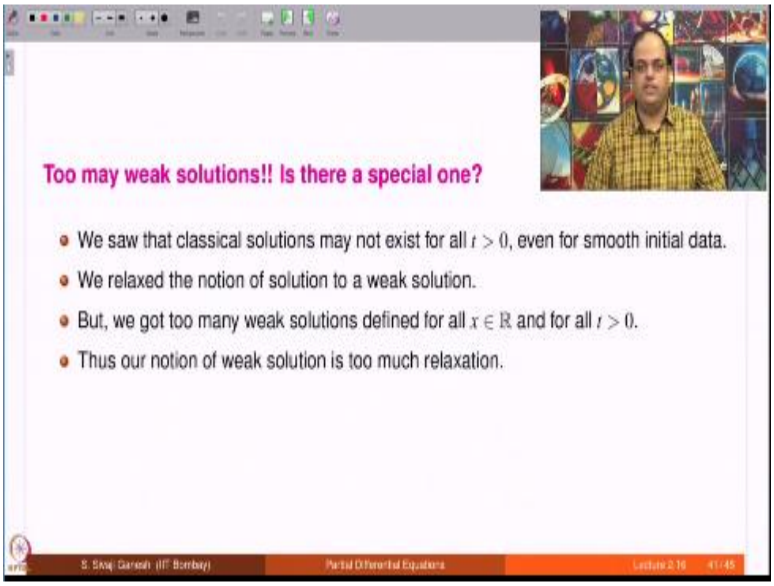

Is there a special one? Somehow we are longing for a unique solution. So, we saw that classical solutions may not exist for all t positive even for smooth initial data, for Burgers equation we have seen. We relaxed the notion of solution to a weak solution. We did that. But we got too many solutions, too many weak solutions defined for all x in R and for all t positive. We just saw. Thus our notion of weak solution is too much relaxation.

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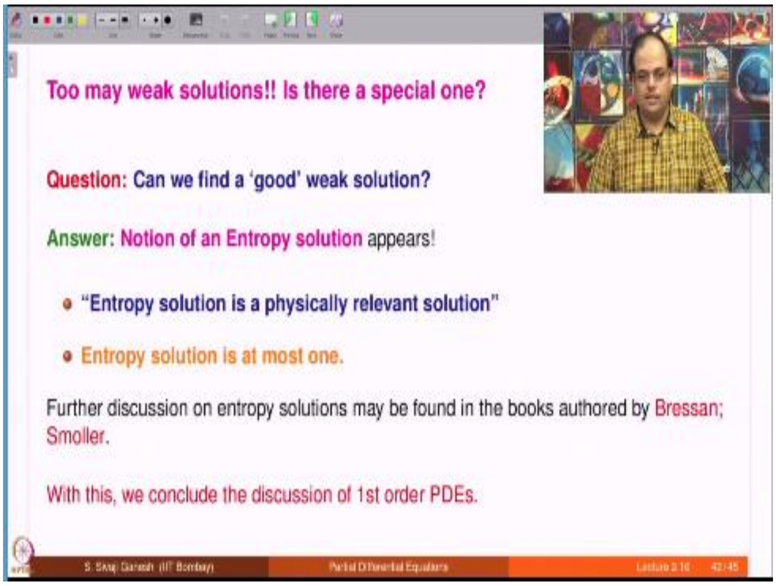

Can we find a good weak solution? This is where the notion of an entropy solution appears. Entropy solution is, people call it a physically relevant solution. In examples of physical importance, they have introduced the entropy, concept of entropy solution and that is indeed a physically relevant solution. And even for a mathematical problems, which are very far from obligations, any entropy condition people call it physical irrelevant solution.

There is no physics behind that. Entropy solution is what is a desirable solution because it fixes some solution uniquely even in a mathematical problem. So entropy solution is at most one. That is how the notion get developed. So, further discussion on entropy solutions may be found in the books by Bressan and also the book of Smoller that I have earlier referred to in the last class. So, with this we conclude the discussion of first order PDEs.

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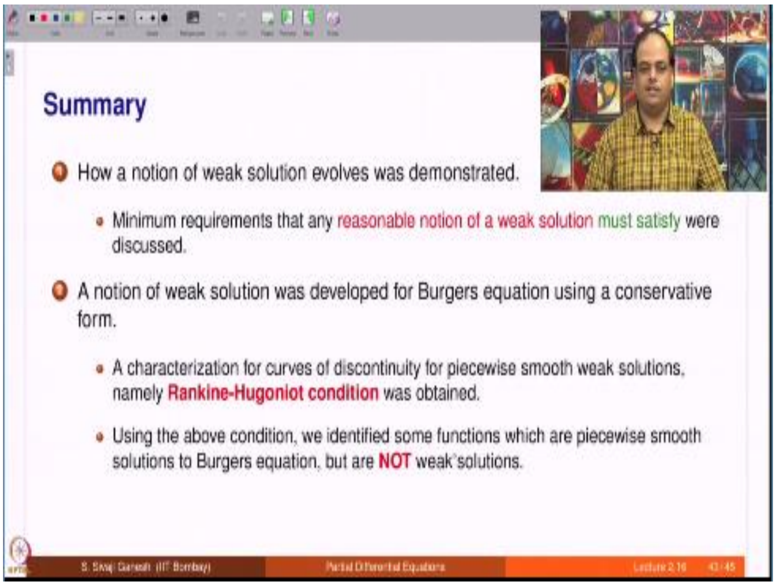

So, what did we see in this lecture? How a notion of weak solution evolves? It was demonstrated. Minimum requirements that any reasonable notion of weak solution must satisfy. They were discussed. A notion of weak solution was developed for Burgers equation using a conservative form. A characterization for curves or discontinuity for piecewise smooth weak solutions was obtained.

That was called Rankhine-Hugoniot condition. Using the above condition we identified some functions which are piecewise smooth solutions to Burgers equation. And, in fact, we have all the examples that we have seen, they are all weak solutions only. We found them.

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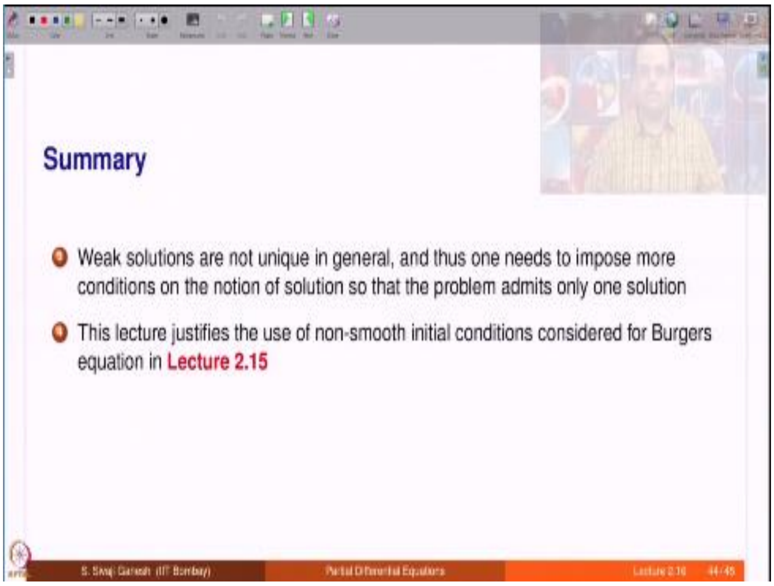

So, weak solutions are not unique in general. And thus one needs to impose more conditions on the notion of solution so that the problem admits only 1 solution after this imposition of new rules. And this lecture is intended mainly to say that whatever initial conditions and solutions that we considered in 2.15 lecture, of course, they are not classical solution. But still they have some meaning and they can be given meaning in the sense of weak solutions and further continuation is entropy solutions.

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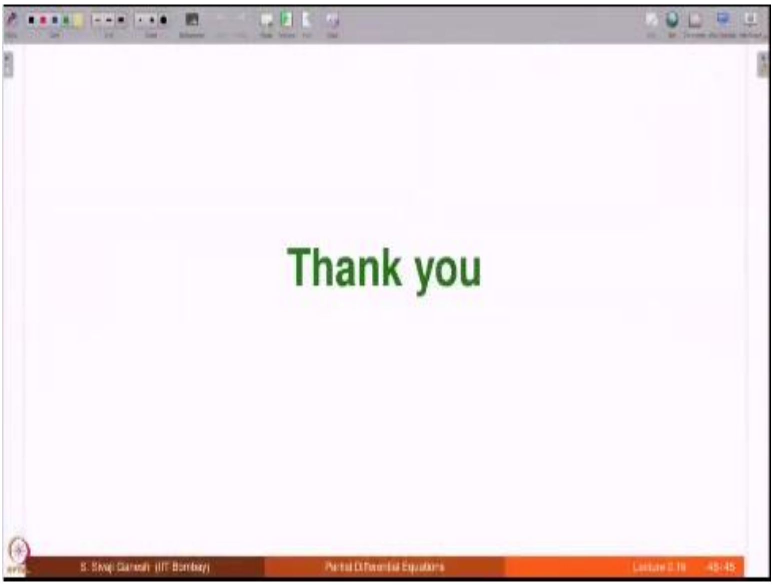

Thank you.