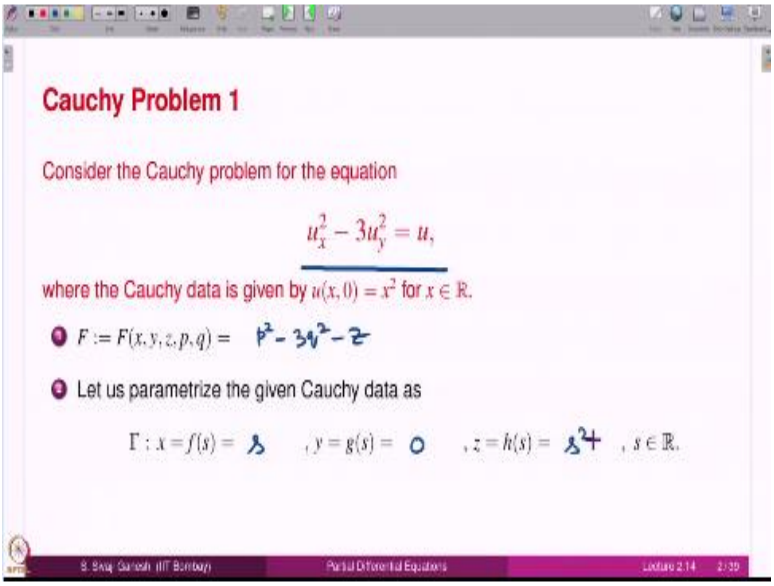
Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture – 2.14
First Order Differential Equations
Tutorial on General Nonlinear equations

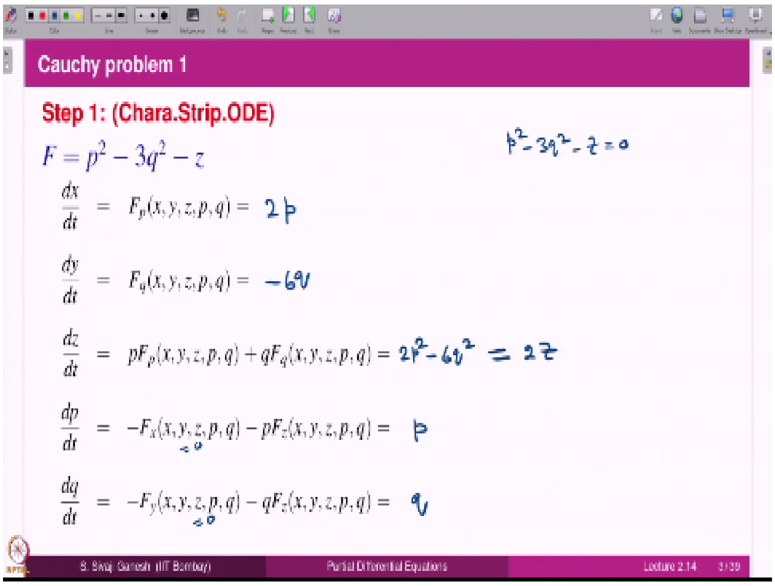
Welcome to the tutorial on general nonlinear equations. In this tutorial, we are going to solve 4 Cauchy problems for general nonlinear equations. And we will highlight some of the important and unique features of each of these examples. So, let us move on to the first problem.

(Refer Slide Time: 00:35)



So, let us solve the Cauchy problem for this equation. So, first thing we have to identify what is the F which defines this equation, so, that is p square – 3 q square – z and next second step is to parameterize the given Cauchy data. Cauchy data given here is u x 0 = x square for x in R; therefore, x = f s = s. This is in the notation that we have used in the proof of the existence and uniqueness theorem. It is helpful to stick to the notation, done.

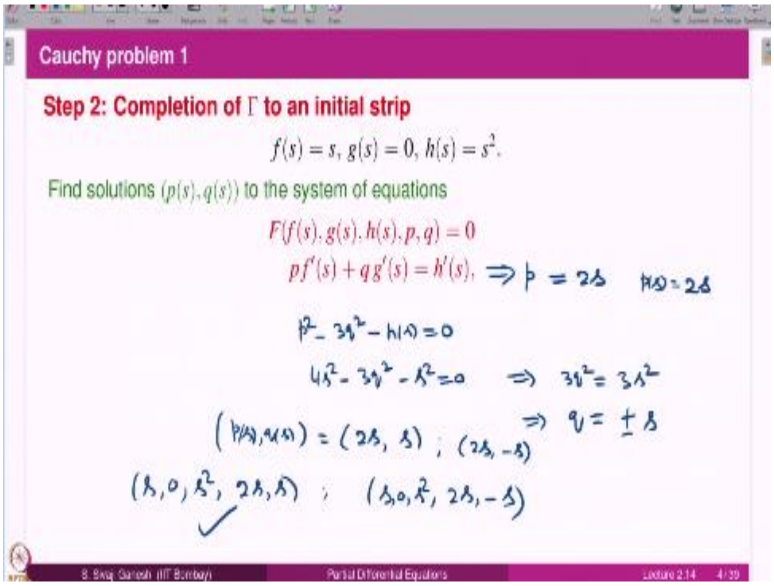
(Refer Slide Time: 01:17)



Now, we need to solve the step 1 is to solve characteristic strip ODE. F is given like this. So, what is F p. It is 2p. F q is –6q and p of p plus q of q is 2p square – 6q square and using the equation, equation recall is p square – 3q square – z is equal to 0; p q and z should satisfy that equation. So, this is equal to 2z and the equation does not depend on x; therefore, the derivative with respect to F x is 0.

Similarly, derivative with respect F y is 0. They do not; F does not depend on x and y. So, what we are to compute is F z. F z is –1 into –p is +p similarly, –1 into –q is q. So, this is the system of characteristic strips, equation for characteristic strips. Now, we are to solve this characteristics ODE with some initial data. But, as you know the Cauchy data gives us only initial conditions for x y z, we need to find for p and q and that is what is called finding an initial strip.

(Refer Slide Time: 02:33)

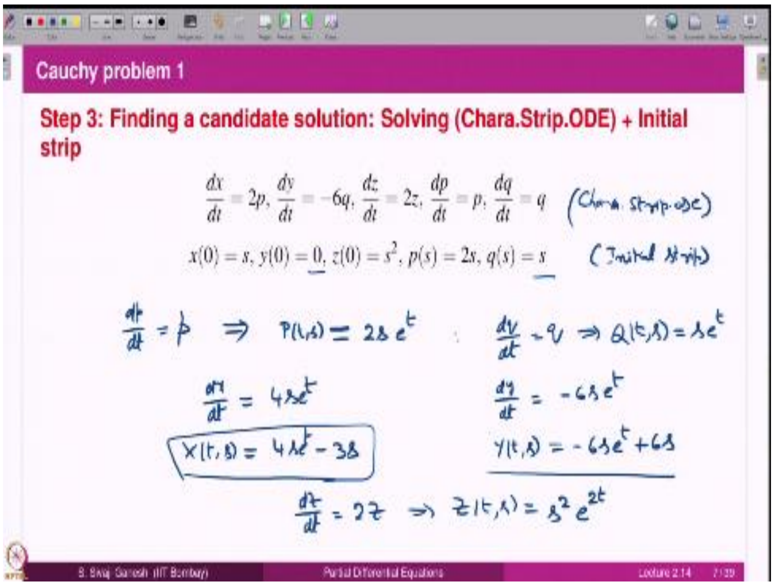


So, our completion of gamma to an initial strip. Gamma is given by here and then we need to find p s q as solutions to the system. Often, it is the case that the second equation is gives us information more quickly; first equation generally is a nonlinear function. And second one most of the times because the nature of the f and g that we choose one of them is 0 for example, f prime g prime in this example, g prime is a let see what. That means, in this example, it is p f prime is 1 + q into, g prime is 0.

So, I do not write that equal to h prime that is 2s which means, we know now p s, so, that is p of s = 2 s. Now, we need to find q s. Now, I will use this equation. The equation is p square – 3q square – z; in the place of z, it is h s = 0, but now, I have found what is p square that is 4s square – 3q square; h s square = 0. So, that will give us 3q square = 3s square and that will give us q is equal to + or – s; therefore, we have got a 2 pairs s; p s, q s, p sof course, is always 2s; other one is q s is s, or – s. So, thus, we have a 2 initial strips.

What are those? s, 0, s square, this is f g h part and then p q that is 2s, s; second strip is, second initial strip is s, 0, s square, 2s, – s. So, what we do is, we will solve the Cauchy problem using this initial strip; using the second one goes similarly and we will give the final answer for the second strip, but we will take this as the initial strip.

(Refer Slide Time: 04:39)



Step 3 is a finding a candidate solution which means solving characteristic ODE with the initial strip. I have recalled here what is chara. strip ODE and this is the initial strip, initial conditions given by the first initial strip, I have chosen q x = s here. So, we have to identify which equation easy to solve. For example, the equation where p and q involve only p and q therefore, dp by dt = p. This, I will demonstrate here clearly the solutions.

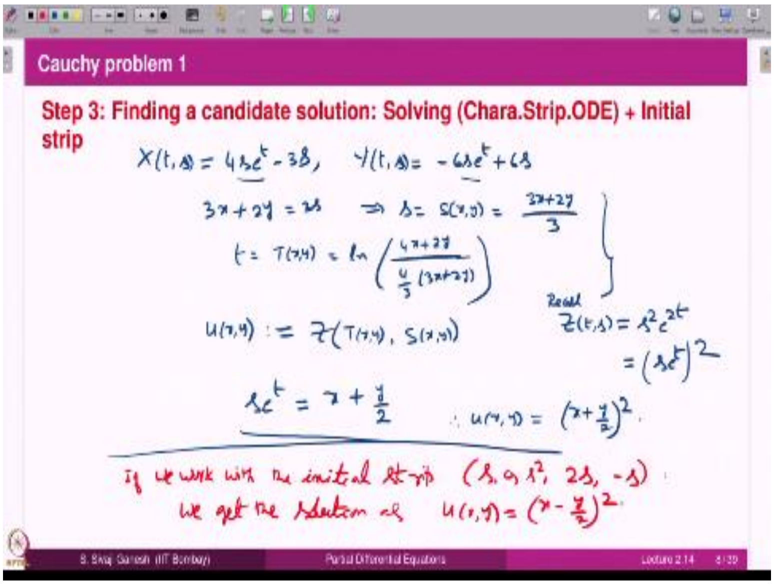
Later on, I will try to write down answers, dp by dt = p. Therefore, the solution looks like it will be a constant times e power t; dp by dt = y mean solution is e power x into constant, but now, when t = 0, I want the initial condition to be satisfied. So, A must be 2s sorry. So, this equal to 2s e power t. Similarly, we have dq by dt = q that will give us that q of t s = s into e power t. So, we know p and q.

So, we can substitute these values here for x and y in the equation for x and y. So, therefore, dx by dt = 2p that is 4s e power t and here dy by dt is – 6 q that is equal to – 6s e power t. Now, we can integrate and we get x of t s is equal to 4s e power t. Integral of this is same plus the constant; it should be such that x of 0 s = s, when I put the t = 0, I get 4s from here. So, I need to subtract 3s from here. So, now this satisfies the initial condition.

Similarly, for y, this is –6s e power t; integral will be again e power t so, –6s e power t + constant, but at t = 0, I want the initial condition to be 0. Therefore, –6s e power t, I add 6s; now, this satisfies. So, this is x t s; this is y t s. Now, we need to find s and t in terms of x and y. Before that, let us also solve dz by dt. To solve dz by dt, we do not need any information, because it was only z that gives us that z t s is equal to e power 2t into constant and that constant has to be s square.

So, therefore, what I need to know to find a solution is t and s, in terms of x and y, that is what the theorem says. Solve for t and s from these equations. So, how do we solve that?

(Refer Slide Time: 08:11)



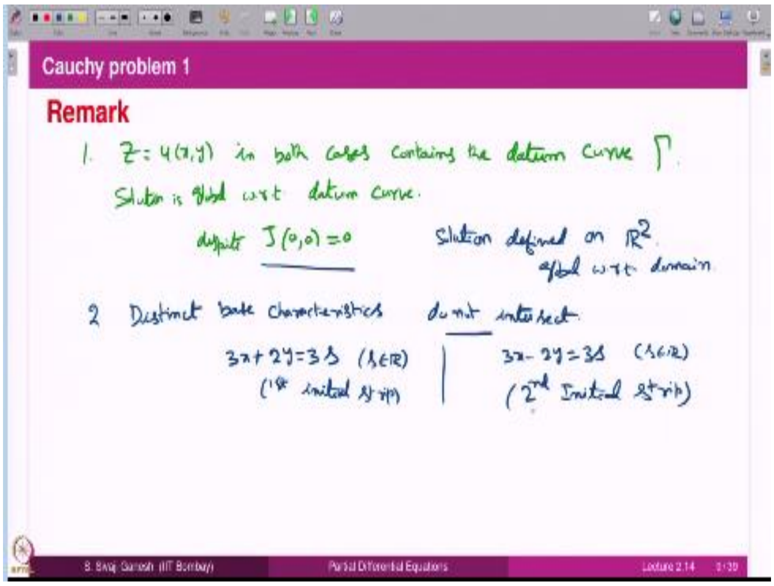
Let me recall what is X t s, Y t s. We have; from here, you are going to observe that 3x + 2y = 3s. So, that implies s = s of x y = 3x + 2y by 3. This is done. Now, what about t? t is equal to T x y that is equal to logarithm of 4x + 2y divided by 4 by 3 into 3x + 2y. Of course, only when this makes sense. This is the how the s and t turned out to be. Now, if I want to find my solution u of x y, what I need to do?

Our definition of u of x y z of T of x y, S of x y. What is our Z of t s recall? That is s square e power 2t. So, e power 2t is what we want. We have found t here. So, why do not we compute e power 2t from here? Or sometimes, it is much easy to compute s e power t because that is what is there in our equations for x and y if you notice, this combination comes s e power t, s e power t.

So, if you multiply x and y, suitably and add and do something, then you can get rid of this. So, you get an expression for s e power t directly. Then you can substitute here and get the answer. So, let us do that. So, what we get is s e power t is actually x + y by 2. So, y by 2 will make it –3s e power t + 3s. When you add, 3 s, 3 s gets cancelled. Minus 3 s e power t, 4s e power t. So, you get to s e power t.

So, this is what the expression you got. Therefore, you have u of x y is equal to x + y by 2 whole square. This is the solution. So, if we chose, that is what it is done. Now, suppose you chose the other initial strip. So, this, I leave it for you to work out. What is that initial trip? s 0 s square, p s is equal 2s, q s is –s. We get the solution as u x y is equal to x – y by 2 whole square.

(Refer Slide Time: 11:32)



Let us have a remark. First point is that z = u of x y. So, what is that? One is x + y by 2 whole square; other one is x – y by 2 whole square. So, z is equal to u of x y in both cases contains the datum curve gamma. That is what solution is global with respect to datum curve. This is true. So, if you actually compute the Jacobian at 0,0 that is actually 0 so, this happens despite that. That means, if you have one point of singularity, it may happen that you have a global solution.

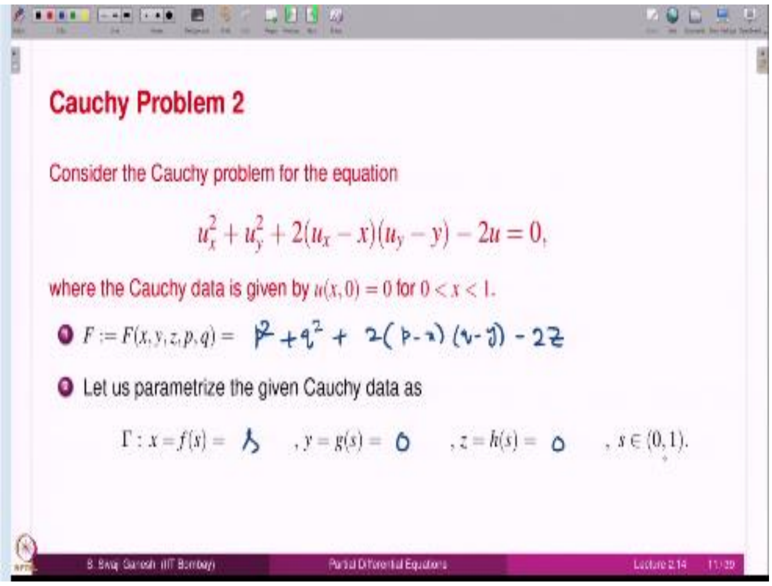
And what is the domain of the solution? Domain of the solution is R2. What better we can expect? So, this is also global with respect to domain. Equation is nonlinear, but still nice things happen for nonlinear equations also. It can happen. So, with this observation, what I want to say is that tools used may not be applicable, but it does not prevent the equations to have or from having global solutions.

Tools, of course, when J 0 = 0, you cannot apply our theorem. Existence uniqueness theorem, you cannot apply because transversality condition fails. But despite that we had global solution, means tools were failures but something else actually happens. And because tools are only sufficient conditions, is not it? Application of inverse function theorem or implicit function theorem, they guarantee you certain things can be done.

They do not say it cannot be done if you do not satisfy these conditions. So, therefore, nice things can happen. In this example, distinct base characteristics. Next, distinct base characters do not intersect. They do not intersect. So, 3x + 2y = 3s. This is the equation of a family of base characteristic curves. They are all parallel lines. This is in the case of the first initial strip.

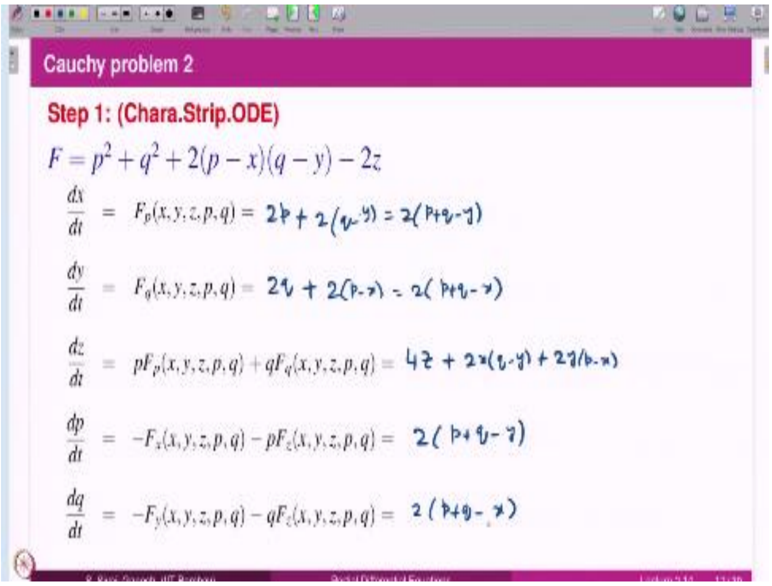
And in the case of second initial strip, it was 3x – 2y = 3s. This is the second initial strip. So, they do not intersect. Let us move onto the next problem.

(Refer Slide Time: 15:27)



This looks slightly more complicated. So, what is F here? p square + q square + 2p – x into q – y – 2z and the datum curve u x 0 = 0. Therefore, this is s; g is 0; this is 0; s is 0,1.

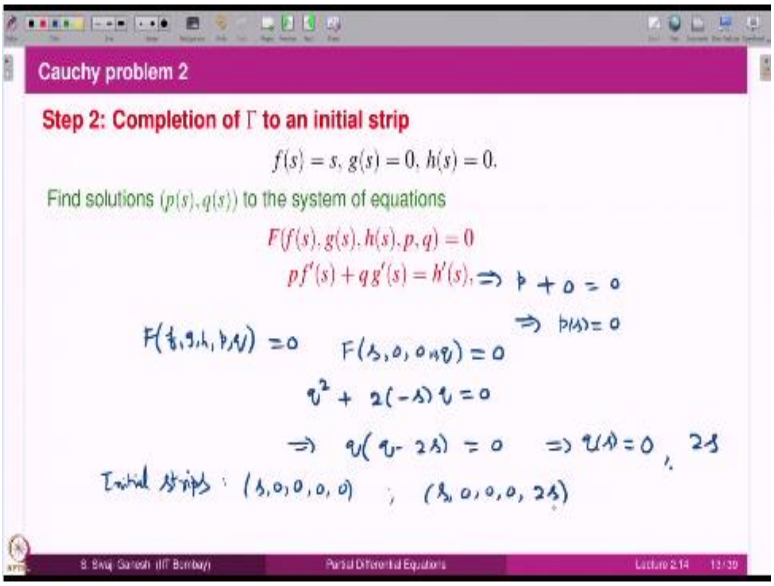
(Refer Slide Time: 15:58)



So, what is the F p? From here, it is 2p on first term; from here it is 2p into q – y which is 2 into p + q – y. What is the F q? It is 2q from here and from here, it is 2q into 2 into p – x. No, there is no p here. 2p that is coming from p square + 2 into q – y. Now, here is 2q + 2 into p – x that is 2 into p + q – x. Now, dz by dt after some simplifications will become, I am not doing a computation, please do it, 2x into q – y + 2y into p – x.

And dy by dt will turn out to be 2 into p + q – y, 2 into p + q – x. So, this is the characteristic strip of ODEs for characteristic strips.

(Refer Slide Time: 17:37)



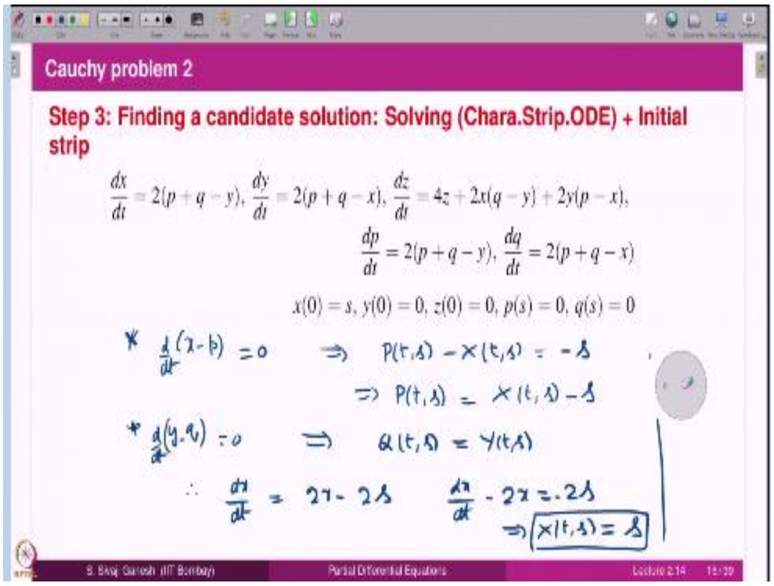
Now, completion of gamma to an initial strip, so, as before, this equation gives us p into, f prime is 1, + q into, g prime is 0, equal to h prime is 0. So, that gives us straightaway p of s is 0. So, once p of s is 0, we are to substitute in the equation and get the equation for q x. So, next thing is F of f g h p q = 0, we need to solve. But, what is that? It is F of f s is s, g 0 h 0, p, we just found 0 and q = 0.

So, we have to substitute these quantities in the equation what we get is q square because p is 0, q square + 2 into – s into q = 0. So, that implies if you take a q common, q – 2s = 0 that implies q of s, it has 2 solutions s and 2s. Therefore, what are the initial strips? 2 initial strips are possible. So, as we discussed earlier, initial strips have to be smooth functions otherwise our procedure does not go through and of course, initial strips are going to be restriction of the strips coming from the equation.

Therefore, they must be smooth if the solutions are smooth. So, we will not take for example, we could also take, q s will s if s is rational and 2s is irrational sorry not irrational, it is 0. We can do this but we will not do that because that bad functions. They are not smooth functions. So, f g h that is s 0 0 here one more 0 should be there, this p is also 0. So, comma p, q also 0 is one script.

Another thing is 0 0 f g h p 0 and it is 2s. These are the 2 strips which are possible. So, let us work with this strip, working with this is similar. So, we will not do this; we will do with this.

(Refer Slide Time: 20:17)

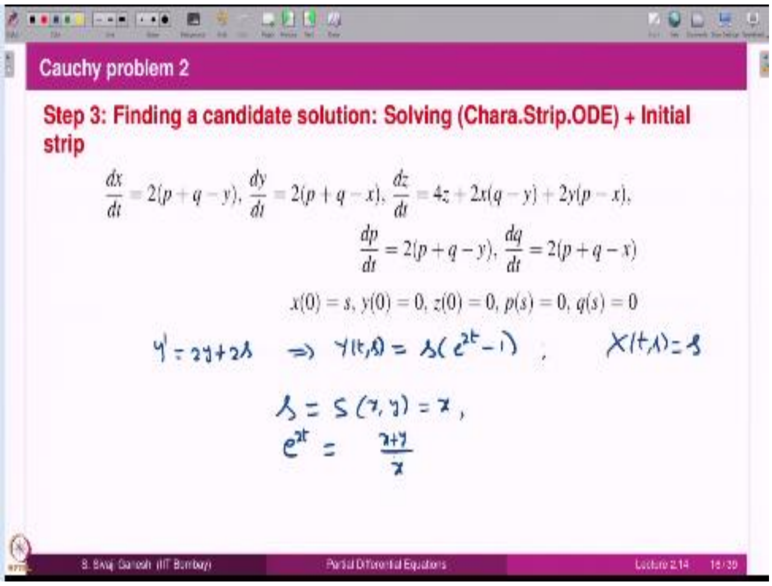


So, we have to solve this system of equations visibly this system is very complicated; everything is coupled like anything ; p q y appears, p q x appears and so on, but you observe dx by dt is same as dp by dt that simplifies slightly because dx by dt = dp by dt that means, d by dt of x – p is 0 that means, x – p is always constant that means x and p differ by a constant.

So, similarly for y and q. So, we had to make some simplifications. So, first observation is, I did x – p d by dt is 0. So, that gives us p t s – s t s, I do not know both of them, but this relation is there; is constant and what that constant should be. At t = 0, what is it? At t = 0, p 0 and x is s. So, this is –s. So, therefore, that gives us p t s = x t s – s. Similarly, we need to find about q. So, y – q d by dt is 0.

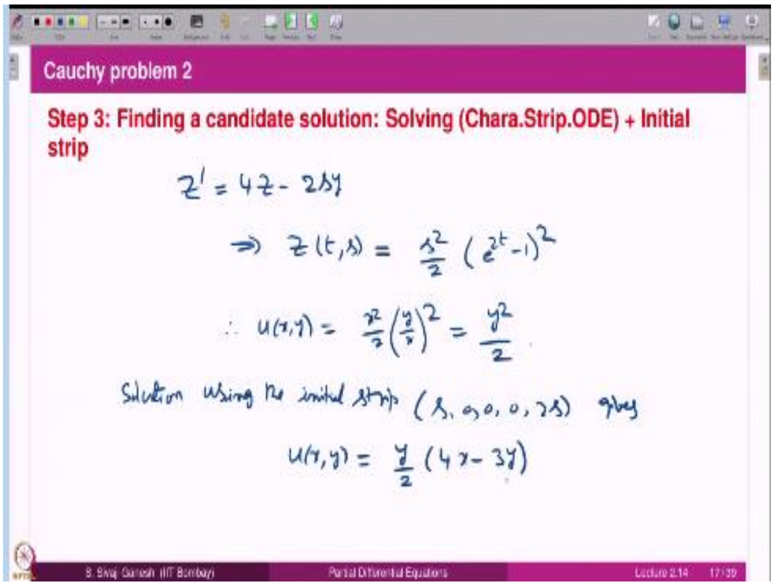
So, that will give us y – q is constant and what that constant should be. Here, I am taking q = 0 so, it must be 0. So, therefore, I get q t s = y t s. Therefore, the equation for x, we may write it as 2x – 2s. So, that is nothing but dx by dt – 2x = 2s and integrating this, what we get is X t s = s, this is – 2s, x t s = s. So, please do this computations thus, integration of this ODE. Similarly, the equation for y, y dash. Let me do on the next page.

(Refer Slide Time: 23:12)



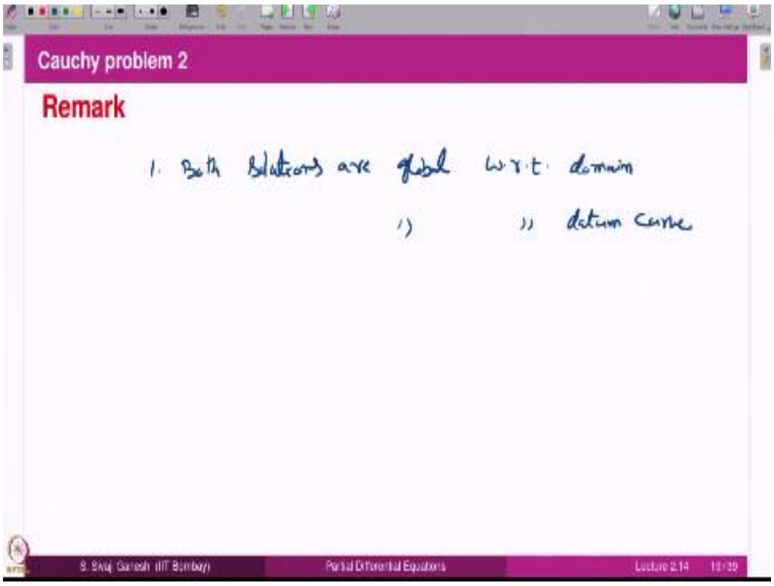
Equation for y is y dash = 2y + 2s and integrating this we get to Y t s is equal to s into e power 2t – 1. So, from these 2 expressions, what is the expression for X? X t s = s. This will give us an expression for s function of x y is actually X and what about Y. Y, I do not know but let me write e power 2 t using this s = s. So, y by x + 1; e power 2t x + y by x. Now, equation for z.

(Refer Slide Time: 24:16)



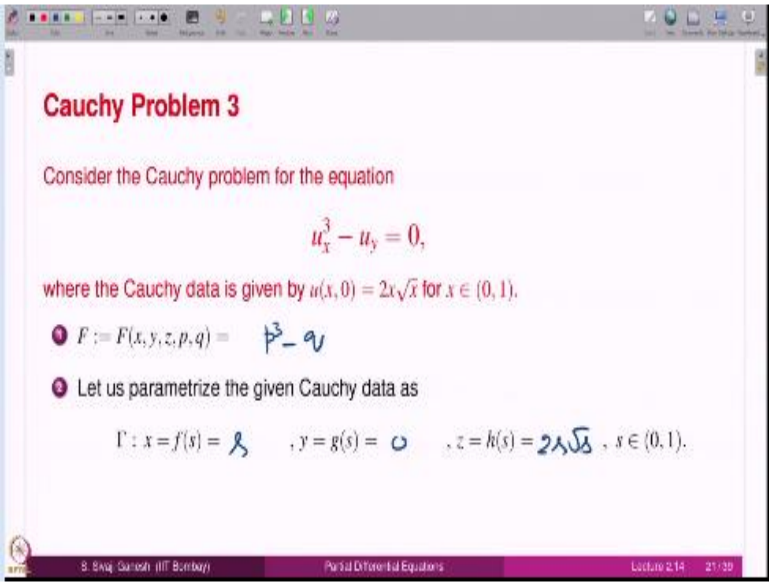
z prime = 4z – 2s y. On integration, what we get is; z t s is equal to s square by 2 into e power 2t – 1 whole square. Therefore, u of x y = x square by 2 into y by x whole square that will give us y square by 2 that is a solution and solution using the other initial script, what was that s 0 0 p was 0, q was 2s, gives u of x y is equal to y by 2 into 4x – 3y.

(Refer Slide Time: 25:18)



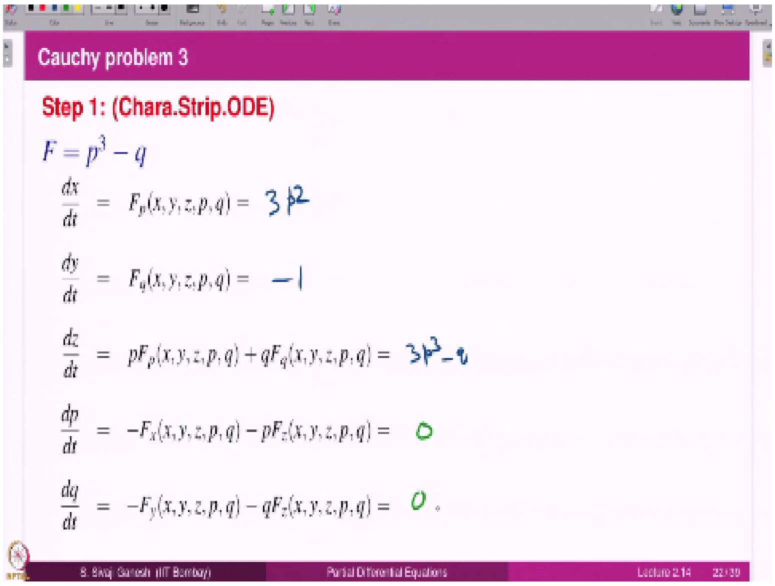
So, here both solutions are global with respect to domain and of course, once it is global with respect to domain, it is also global with respect to datum curve, nonlinear equation global solutions. So, message is that nonlinear equations does not mean things are always going to be bad.

(Refer Slide Time: 26:14)



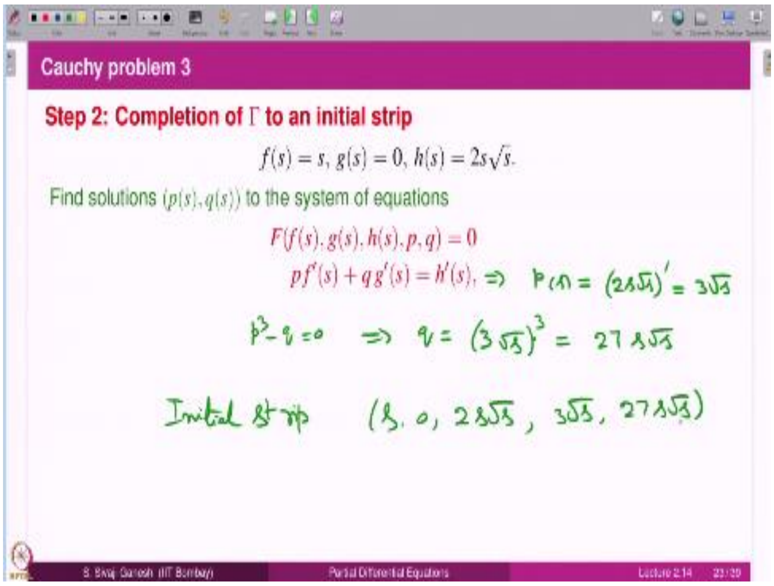
Let us look at this Cauchy problem, u x cube – u y = 0. So, F is p cube – q. This is s, 0, 2s, root s that is the initial Cauchy data.

(Refer Slide Time: 26:31)



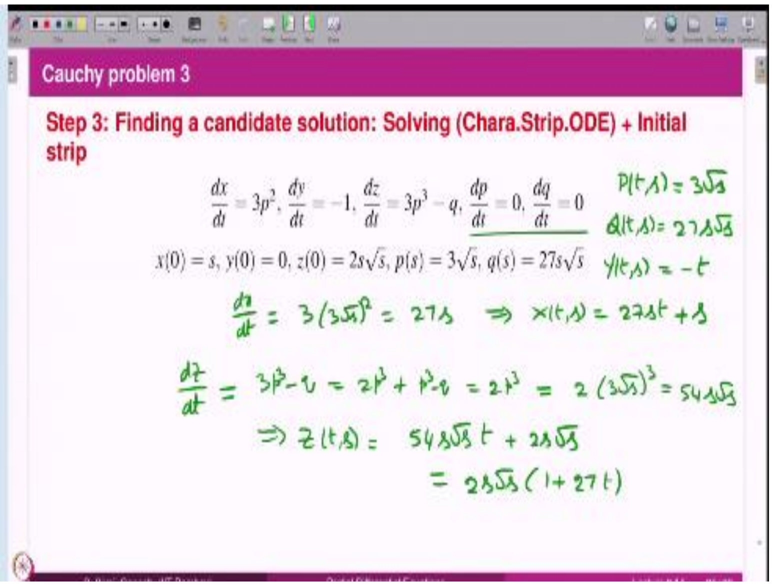
Now, this is very simple, because F depends only on p and q, F p is 3 p square, F q is –1, p of p + q of q is 3 p cube – q that is equal to; we can keep it as it is for now. And F x is 0; F z 0 sorry, this is 0; this is 0.

(Refer Slide Time: 27:05)



Now, we need to do this extension. So, always concentrate on this p, f prime is 1 + q into g prime is 0. So, do not even write equal to 2s root s derivative with respect to s. On that you can see is actually 3 root s. Now, what is q? By our equation, p q – q is 0 that implies that q is equal to p cube that is 3 root s whole cube that is 27 s root s. So, what is the initial strip? We have got only one initial strip here, s, 0, 2s root s, p s is 3 root s, q s is 27 s root s.

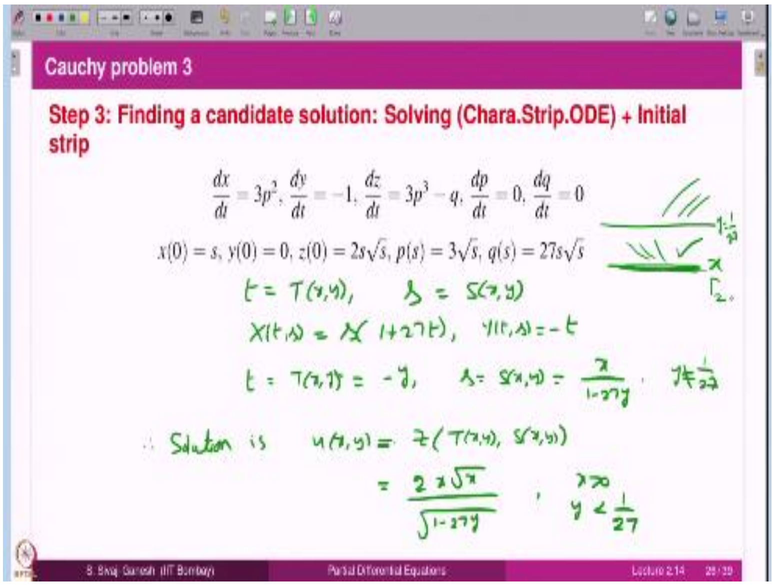
(Refer Slide Time: 28:34)



We need to solve a system of equations. Now, here p and q are the simplest equation. So, that gives us P t s is a constant and that has to be 3 root s. Similarly, Q t s has to be constant that has to be 27 s root s and Y t s is also easy, y of t s is – t + constant; when t = 0, it should be 0 therefore, this is – t and we have now compute x and z. So, dx by dt is 3 p square that is 3 into 3 root s whole square that is 27s. So, therefore, that implies X t s is 27 s t + constant.

So, when t = 0, I want it to be s therefore, that must be s, 27 s t + s. Now let us go to dz by dt that is 3p cube – q actually that is 2 p cube because p cube – q, we have chosen it to be 0. So, that is actually 2p cube so, which is nothing 2 into 3 root s cube, 27, 54 s root s. Therefore z of t s 54s root s into t, because this is constant; does not depend on t. When t = 0, I want it to be 2s root s so I add 2 s root s because it is 0 at t = 0, so that if you take 2s root s common, 1 + 27 t is what we have.

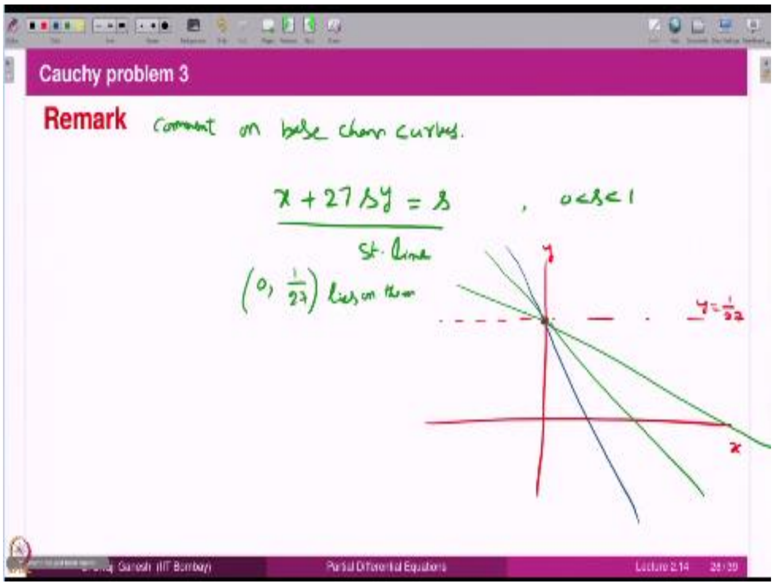
(Refer Slide Time: 30:51)



So, next step is to find out t as a function of x y and s as a function of x y. So, X t s is equal to s into 1 + 27 t and Y t s is equal to – t. So, that will give us t is equal to T x y that is very simple equal to – y and s = s of x y = x by 1 – 27 y. Obviously, y should not be equal to 1 by 27, so, something is going to happen when y is 1 by 27. Therefore, solution to the Cauchy problem is given by u of x y = Z of T x y, S x y and that will be equal to 2x root x divided by 1 – 27 y square root.

Of course, x should be positive and y should be less than 1 by 27 because the initial data is given on the x axis and y = 1 by 27 is here. Solution is not meaningful at y = 1 by 27. So, you have to choose either this domain or this domain. We choose this domain because this is where it is in touch with the Cauchy data. We want to solve solution nearby these gamma 2.

(Refer Slide Time: 32:48)

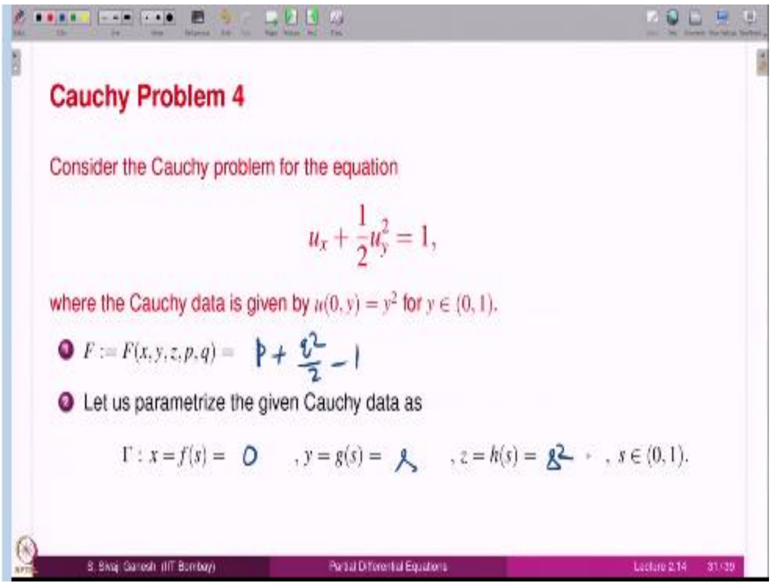


Now, let us give a comment on base characteristic curve. Base characteristic curves, they are given by this equation for each s in 0 to 1. We are considered s in 0 to 1. So, for each fixed s, it is a straight line. This straight land is like or reference it is x axis, y axis and this is a line y = 1 by 27. This point, something is happening. The base characteristic curves are straight line. They always pass through this point 0, 1 by 27 lies on them that means all of them are meeting up at this point. Slopes are 1 by 27 s; – 1 by 27 s yes.

As you are coming closer, they are becoming steeper and steeper like that. So, all of them are meeting at this point and u is not defined at y is equal to 1 by 27. So, this example tells us that intersecting base characteristic curves pose a problem or stop or prevent a solution from being global. That is a problem. Global with respect to domain and solution of course, is global with respect to datum curve in this example.

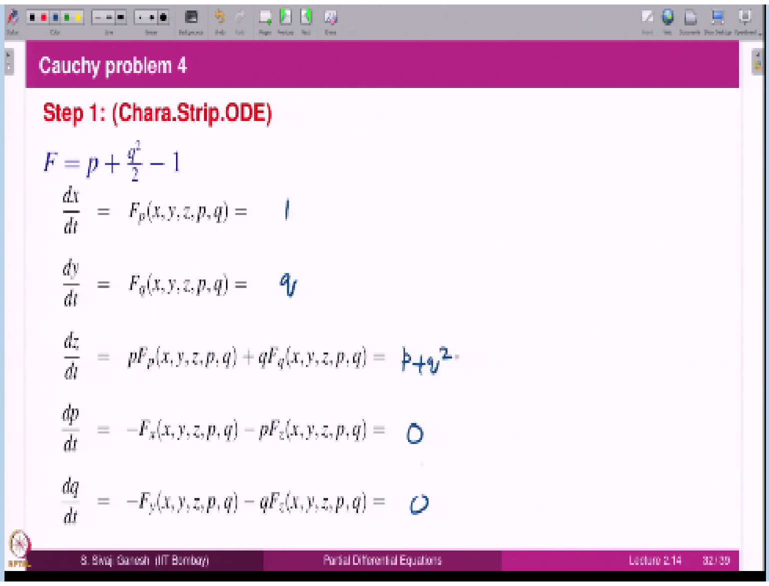
Though existence and uniqueness theorem asserts local existence near with respect to datum curve that kind of solution, it asserts, we observed this example that solutions are actually global with respect to the datum curve. Solution is actually global with respect to datum curve, but it is not global with respect to the domain, because these base characteristic are meeting up. We will also see an example using Burgers' equation in the next lecture that this is indeed one of the main problems even there.

(Refer Slide Time: 35:28)



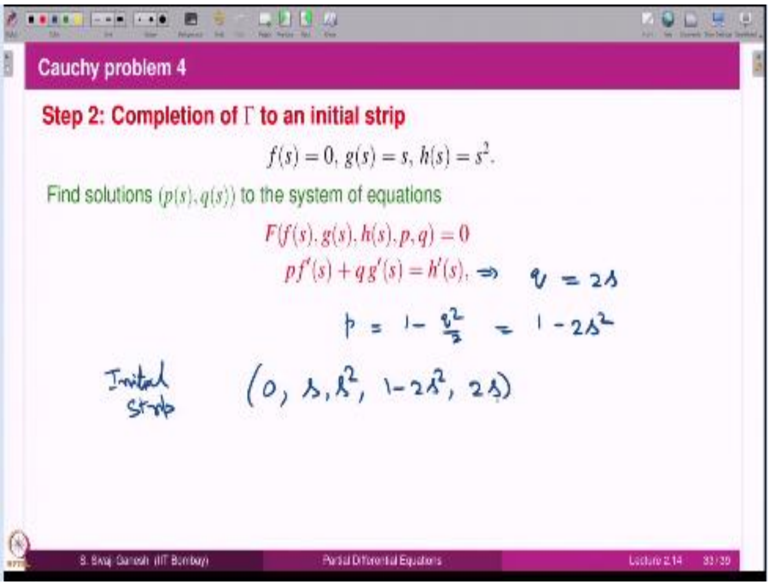
And the last Cauchy problem for today is this equation u x + half y square = 1 and this is a datum curve and F as usual p + q square by 2 – 1.

(Refer Slide Time: 35:43)



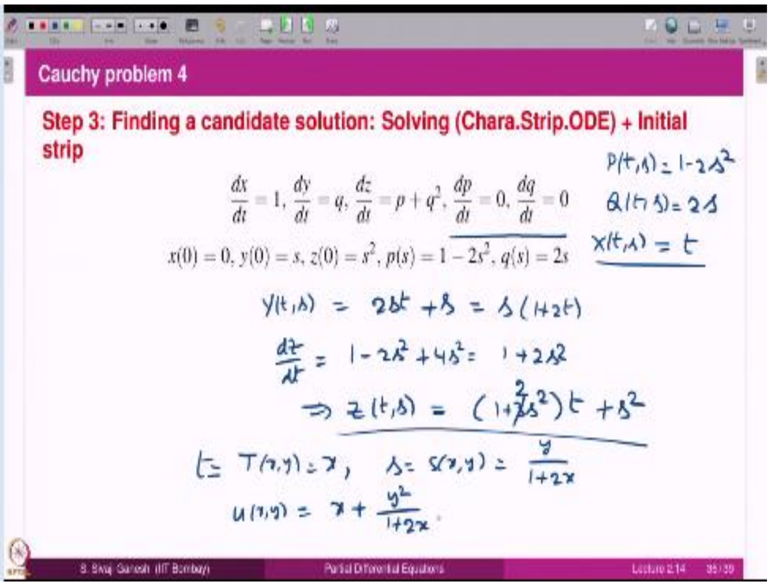
And once again, it does not depend on x and z. So, these are 0. F p is 1; F q is q; p F p + q F q is p + q square.

(Refer Slide Time: 35:58)



Now, this is p into f prime is 0, so, it is not there. q and g prime is 1 and that is equal to h s. So, q is determined. Now, use the equation and get the p. p is equal to 1 – q square by 2 by the equation that is equal 1 – 2s whole square, 4s square by 2 that is 2s square. So, this is p. So, we got p s and q s, only one initial strip f is 0; g is s; h is s square; p is 1 – 2s square and q is 2s.

(Refer Slide Time: 36:45)

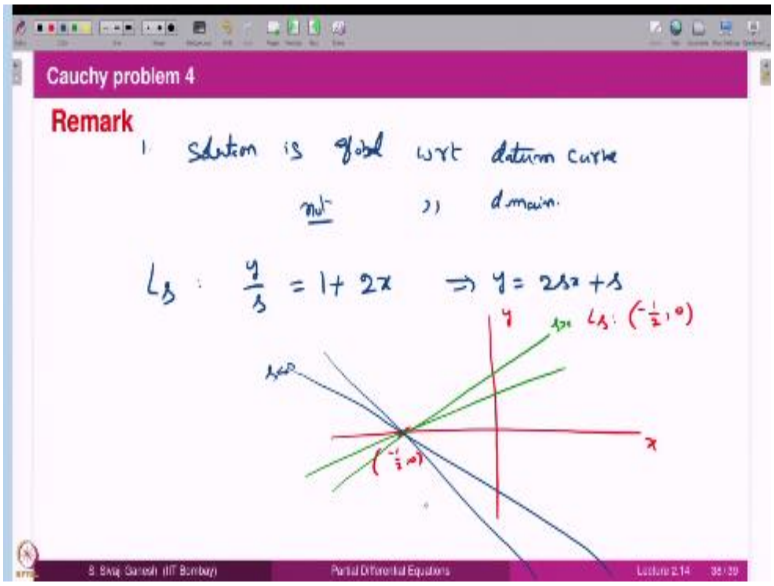


Now, as before P t s is 1 – 2s square; it is a constant. Q t s is 2s. What else is easy, X t s is easy. It is t plus some constant and that constant has to be 0 because it stratifies 0 condition; X t s is t. Now, what is Y T s? Y t s is going to; q is a constant ; 2s so, 2s into t but a t = 0. It should be s. So, it is this and dz by dt, we need to write down slightly. dz by dt is p + q square; p is 1 – 2s square; q square is 4s square.

So, that is equal to 1 + 3s square therefore, that gives Z t s is equal to 1 + 3s square into t; when t = 0 it should be a square; sorry this is not 3; it is 2; 1 + 2s square. So, there is not 3; it is 2; 1 + 2s square. So, t is here; s is here; where is s; t is here. From here, we can get for s; s into 1 + 2 t. So, what we get is t is equal to T of x y is equal to x and s is equal to S of x y is equal to y by 1 + 2x.

So, when you substitute in the formula for z, the solution we obtained will be u x y. After simplification, it is x + y square by 1 + 2x.

(Refer Slide Time: 39:00)



So, once again the solution is global with respect to datum curve but, not global with respect to domain as we see, formula has a denominator 1 by 1 + 2x. So, there is a problem at the point when the denominator is 0. So, it turns out that at that point, base characteristics meet up. So, equations for base characteristics is y by s = 1 + 2x, that implies y is equal to 2 s x + s, all of them pass through minus half 0.

When x is minus half by 0, so, whatever maybe the s pass L s goes through this point. So, that is this point minus half, 0, passing through this, like that. This is for s positive. These are family correspond to positive s s and for the other one, for negative s s is going to be like that and so on. So, all of them pass through. So, base characteristic meet up at this point. Therefore, we see a trouble in the formula for the solution 1 by 1 + 2x.

So, the solution surface will be singular at this point and that is clear by the formula. So, basic characteristics meet, we expect troubles. So, this completes the tutorial on Cauchy problems for nonlinear equations. Please solve as many problems as you can and in each of the problem, analyze if you can find out some expression for the base characteristic curves and see whether they meet or not meet etcetera. Consider all these considerations. Thank you.