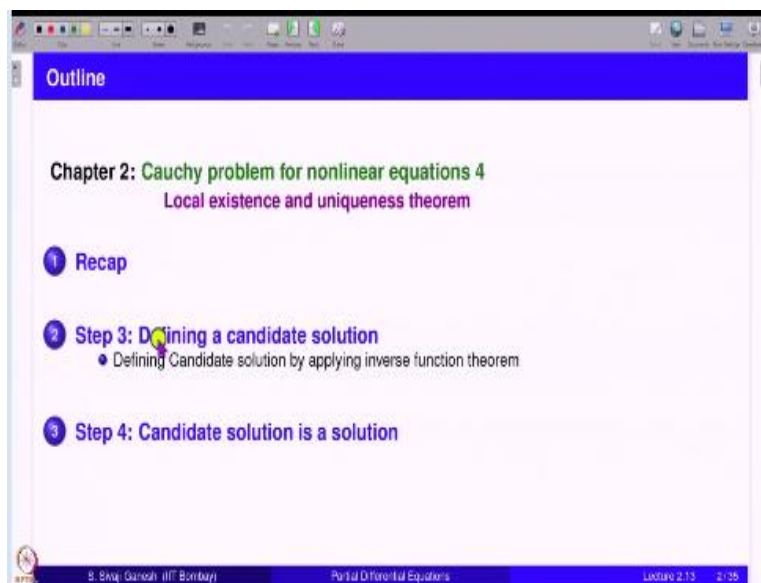


Partial Differential Equations
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Lecture – 2.13
General Nonlinear Equations 4
Local Existence and Uniqueness Theorem

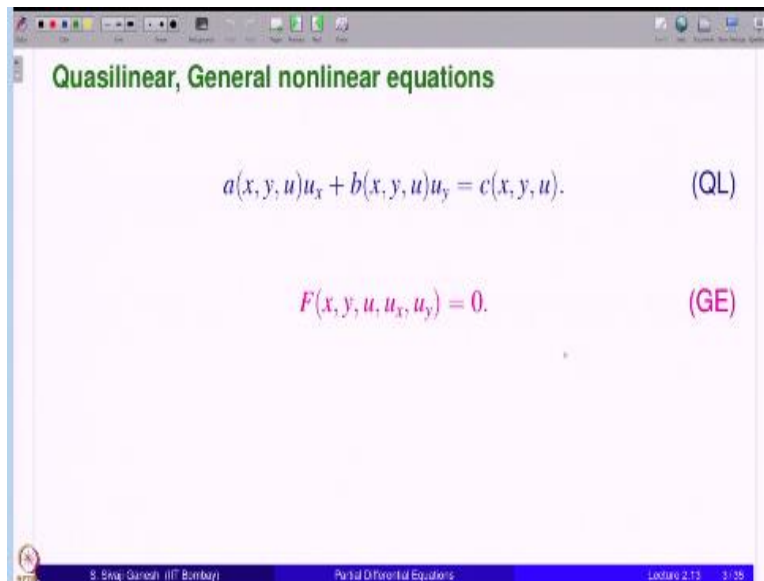
In this lecture, we are going to complete the proof of local existence and uniqueness theorem for Cauchy problems for the general nonlinear equations.

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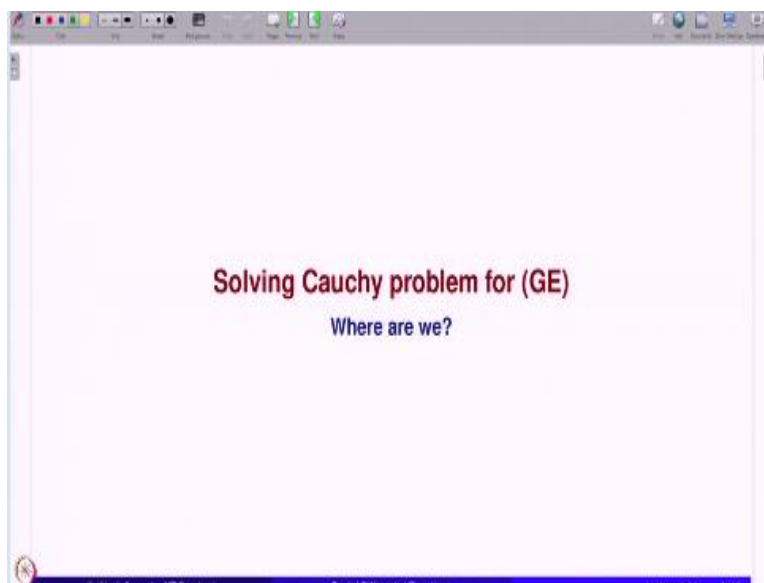
So, we start with a recap of what happened so far. And then we go to step 3 with namely defining a candidate solution. So, we will define a candidate solution by applying inverse function theorem and then we show that the candidate solution is indeed solution.

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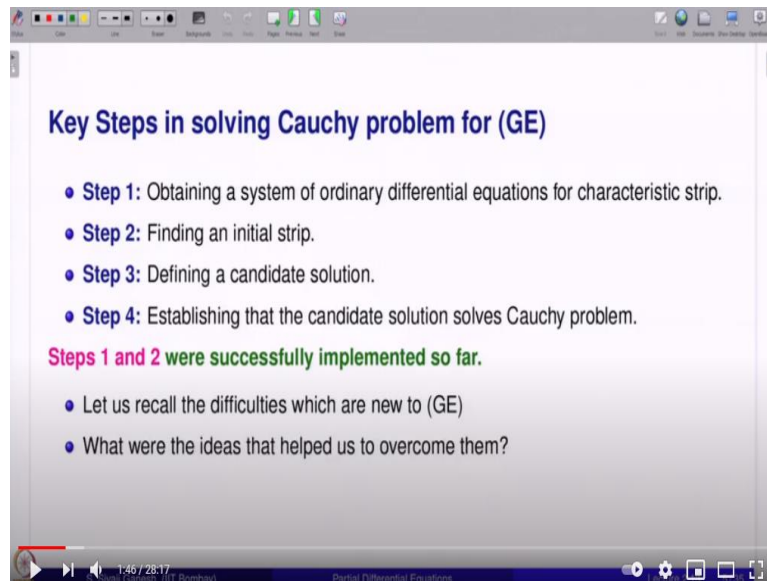


So, this is to recall the notations QL stands for Quasilinear equation and GE stands for general nonlinear equations or sometimes we will call it as fully nonlinear equations. Of course, GE contains L, SL as well as QL. But when it is presented in this form, it is called generally nonlinear equations.

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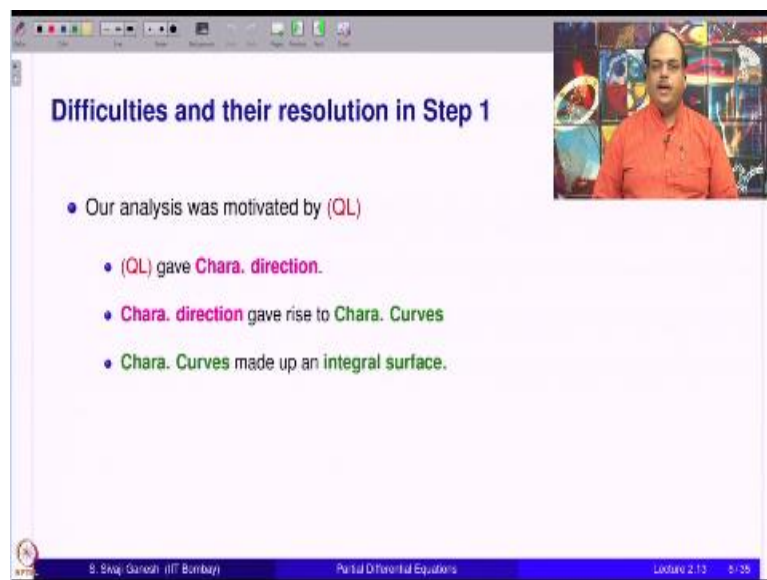
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Where are we in solving the Cauchy problem? The key steps involving the solution of the Cauchy problem were identified. Step 1 is to obtain a system of ODEs for the characteristic strip and step 2 is a finding an initial strip. Third one is to define a candidate solution and fourth one is establishing that the candidate solution is indeed a solution. So, steps 1 and 2 were successfully implemented so far.

Unless, we call the difficulties once again, which are new to GE when compared to QL. And what are the ideas that helped us to overcome them.

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So, our analysis was motivated by Quasilinear equations, QL. So, QL gave us characteristic direction. Characteristic direction gave rise to characteristic curves. Characteristic curves made up an integral surface.

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Step 1: Difficulty 1 for (GE):

Equation (GE) does NOT give away a Chara. direction.

Useful Idea

- We observed that for (QL), the **Chara. direction** is the **envelope of possible tangent planes**.
- We found that the same idea works for (GE) as well.

A **Chara. direction** at a point $P(x, y, z)$ is given by

$$(F_p(P, p, q), F_q(P, p, q), pF_p(P, p, q) + qF_q(P, p, q))$$

where p, q satisfy $F(x, y, z, p, q) = 0$.

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Now, equation GE does not give away a characteristic direction. So, useful idea, we observed that for Quasilinear equations, the characteristic direction is the envelope of possible tangent planes. In fact, the possible tangent planes envelope is a straight line which has the characteristic direction. So, we found that the same idea works for GE as well. So, characteristic direction at a point $P(x, y, z)$ is given by $(F_p, F_q, pF_p + qF_q)$.

The argument has to be $F_p, F_q, pF_p + qF_q$, P stands for (x, y, z) ; p and q , they are such that F of this is 0.

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Step 1: Difficulty 2 for (GE):

(Chara.ODE) are incomplete for (GE).

Curve having Characteristic direction

$$\frac{dx}{dt} = F_p(x, y, z, p, q)$$

$$\frac{dy}{dt} = F_q(x, y, z, p, q)$$

$$\frac{dz}{dt} = pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q)$$

along with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$.

(chara.ODE) is NOT solvable since p, q are unknown. Need to complete the system (chara.ODE).

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And characteristic ODE system is incomplete for general nonlinear equation, because when we try to find a curve which has a characteristic direction, it has to satisfy this set of ODEs, namely chara ODE, but here p and q appear which themselves are unknown. So, this is not

solvable and we need to add equations or supplement the system with another 2 equations; one for p and one for q.

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Step 1: Difficulty 2 for (GE):

System of ODEs for the characteristic strip was derived.

$$\begin{aligned} \frac{dx}{dt} &= F_p(x, y, z, p, q) \\ \frac{dy}{dt} &= F_q(x, y, z, p, q) \\ \frac{dz}{dt} &= pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q) \\ \frac{dp}{dt} &= -F_x(x, y, z, p, q) - pF_z(x, y, z, p, q) \\ \frac{dq}{dt} &= -F_y(x, y, z, p, q) - qF_z(x, y, z, p, q) \end{aligned}$$

The system of ODE (2) is denoted by **(Chara.Strip.ODE)**.

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It has been achieved and that was called chara strip ODE where the dp by dt, dq by dt equations have been added.

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Difficulties and their resolution in Step 2

Question: Can we pass Chara. curves through points of the datum curve?

Answer: We need to solve **(Chara.Strip.ODE)** with initial conditions given by points of the datum curve.

- initial conditions for x, y, z are given by datum curve.
- **NO initial conditions** are known for p, q .
- They were found in **Step 2**, which was called **Finding an initial strip**.

Next Step is to solve the initial value problem:

(Chara.Strip.ODE) + an initial strip

to define a **candidate solution**.

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Now, next question is that, can we pass characteristic curves through points of the datum curve? This is what we did in QL. So, for that we need to solve chara ODE with initial conditions given by points of the datum curve. Initially conditions for x y z are given by the datum curve; no initial conditions are known for p and q. So, they were found in step 2 which was called finding an initial strip.

Next step is to solve the initial value problem chara strip ODE plus an initial strip that will give you a candidate solution.

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Road ahead is smooth

From now onwards assume that

- 1 the system **(Chara.Strip.ODE)** is given and
- 2 an **Initial Strip**
 $(f(s), g(s), h(s), p(s), q(s))$
consisting of C^1 **functions** on an interval containing s_0 **is given.**

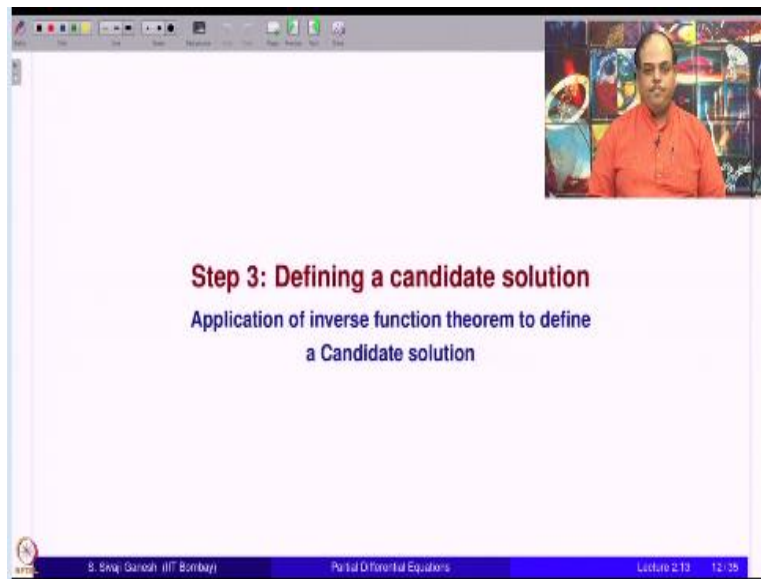
Let the functions $X(t, s), Y(t, s), Z(t, s), P(t, s), Q(t, s)$ solve the IVP
(Chara.Strip.ODE) and Initial Strip.

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It will define a candidate solution. We are going to see that. So, from now onwards, what we assume maybe in deriving these equations, we have assumed a certain higher smoothness conditions on f g h and F or maybe u but do not worry. Now onwards, we assume this that characteristics strip ODE is given; equations are given to you. You do not have to derive. And initial strip consisting of C^1 functions is found.

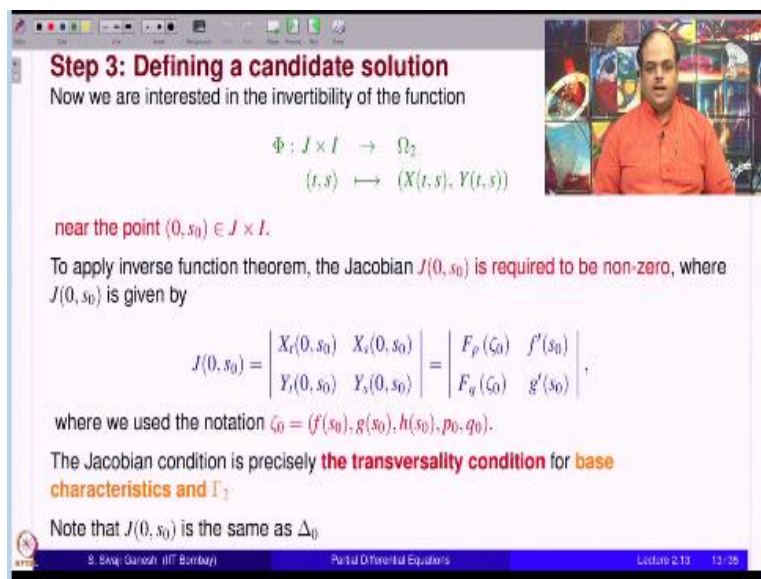
Of course, we cannot find perhaps throughout γ , but you look at a point $f(s_0), g(s_0), h(s_0)$ on γ . And hence, I want this to be defined nearby that point p_0 . And let this functions X Y Z P Q , which are functions of t and s ; solve the IVP. What is IVP? Chara strip ODE supplemented with this initial conditions coming from initial strip.

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Now; defining a candidate solution: this is where we need to apply inverse function theorem to certain functions and then get a candidate solution.

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So, we are interested in the invertibility of this function. What is this function? This is actually a base characteristic curves; the trace is base characteristic curves as a t varies passing through the point $f s g s$ which is on γ_2 . So, near this point $0, s_0$ which belongs to $J \times I$. So, now, to apply inverse function theorem, the Jacobian is required to be nonzero.

Of course, we need to check whether Φ is C^1 because x and y are solving ODEs therefore, the derivative with respect to t is continuous, no problem. With respect to X is also differentiable and C^1 because of differential dependence. So, we have this C^1 of Φ .

Now, we need that the Jacobian is nonzero. The Jacobian is this. As before, we do not once again analyze with g of t s because nothing is known for a nonzero t .

So, only at 0 , it is known. J of $0, s_0$, the Jacobian is this, which is $F_p F_q$ at the point C_0 and f dash s_0, g dash s_0 . Of course, the way we got p s and q s is such that p of s_0 is p_0 ; q of s_0 is q_0 . So, you could as well write here p s_0, q s_0 but remember, we started with $p_0 q_0$ a particular solution of system of equations, which define p s and q s later on. So, you can write p $s_0 q$ $s_0, p_0 q_0$ because both are actually the same.

Now, the Jacobian condition is precisely the transversality condition, f prime g prime corresponds to what, tangent to γ_2 , tangent to the γ_2 . What is γ_2 ? Projection of $\gamma_2 \times y$ coordinate ω_2 , these are tangent. This is the base characteristics; tangent to the base characteristic of this direction F_p and F_q . We have to assume these nonzero that means they are not parallel.

So, base characteristics cut γ_2 that is the transversality condition. This is same as the δ_0 that we saw in the step 2.

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Step 3: Defining a candidate solution

By **inverse function theorem**, there exist

- 1 an open subset E of $J \times I$ containing the point $(0, s_0)$, and an open subset F of Ω_2 containing the point $(X(0, s_0), Y(0, s_0))$ i.e., $(f(s_0), g(s_0))$,
- 2 a continuously differentiable function $\Psi : F \rightarrow E$ such that

$$\Psi \circ \Phi = I_E, \Phi \circ \Psi = I_F,$$

where I_E and I_F are the identity functions defined on E and F respectively.

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So, defining a candidate solution by inverse function theorem, there exists an open subset E of J cross I containing the point $0, s_0$. And F which is an open set of ω_2 which contains X of $0, s_0; Y$ of $0, s_0$, which is actually f s_0, g s_0 and a continuously differentiable function from F to E such that these 2 compositions are identity maps on respective spaces.

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Step 3: Defining a candidate solution

Denoting

$$\Psi(x, y) = (T(x, y), S(x, y)),$$

the equations

$$\Psi \circ \Phi = I_E, \quad \Phi \circ \Psi = I_F$$

yield

$$t = T(x, y), \quad s = S(x, y).$$

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There is a picture which depicts. So, this is the phi map event. Inverse function theorem told that there are neighbourhoods E and F and a map psi defined. These where we are using t s coordinates. Here we are using x y notation. We are restricted to, E is a diffeomorphism and so on. And denote psi of x y = T of x y, S of x y and the equation psi circle phi identity on E. Phi circle psi identity on F. They give t = T x y, s = S x y.

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Step 3: Defining a candidate solution

From last slide,

$$t = T(x, y), \quad s = S(x, y).$$

Recall that $Z(t, s)$ was expected to be the value of a solution at the point $(X(t, s), Y(t, s))$. This motivates the definition of a candidate solution for the Cauchy problem as

$$u : F \rightarrow \mathbb{R} \quad \text{given by} \quad u(x, y) = Z(T(x, y), S(x, y)).$$

As the function u is a composition of two C^1 functions, by Chain rule $u \in C^1(F)$.

Step 3 successfully completed!

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So, recall that $Z(t, s)$ was expected to be the value of the solution at the point $(X(t, s), Y(t, s))$. Therefore, this motivates us to define a candidate solution by using this Z . So, u define an F to \mathbb{R} , u of $x, y = Z$ of $T(x, y), S(x, y)$. As a function u is a composition of 2 C^1 functions. By chain rule, u itself is C^1 function on F . So, step 3 is successfully completed. We have defined a C^1 function as a candidate solution.

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Step 4: Candidate solution is a solution

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Candidate solution is a solution

The function

$$u : F \rightarrow \mathbb{R} \text{ given by } u(x, y) = Z(T(x, y), S(x, y))$$

is a solution to the Cauchy problem if

- 1 The following identity holds for every $(x, y) \in F$:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0,$$
- 2 and the Cauchy condition $u(f(s), g(s)) = h(s)$ is satisfied for $s \in I'$ where I' is a subinterval of I .

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In the next step, we are going to check that this is indeed a solution. So, this is a solution to the Cauchy problem if the following identity holds for every x, y in F , that is $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$. And the Cauchy condition $u(f(s), g(s)) = h(s)$ is satisfied.

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Proof of (2)

$u : F \rightarrow \mathbb{R}$ given by $u(x,y) = Z(T(x,y), S(x,y))$

Let $I' := \{s \in I : (f(s), g(s)) \in F\}$.

Observe that $T(f(s), g(s)) = 0$ and $S(f(s), g(s)) = s$ for $s \in I'$.

Thus for $s \in I'$, we have

$$\begin{aligned} u(f(s), g(s)) &= Z(T(f(s), g(s)), S(f(s), g(s))) \\ &= Z(0, s) \\ &= h(s). \end{aligned}$$

This shows that Cauchy condition is satisfied. □

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So, define I' to be those values of s in I for which $(f(s), g(s))$ belongs to F . So, this is the I' which came there. So, s belongs to I' . We will be; the datum curve will be on the corresponding integral surface defined by this s . So, observe that T of $(f(s), g(s))$ is 0 and the S of $(f(s), g(s))$ is s . Thus, for s in I' , we have u of $(f(s), g(s))$ equal to by definition u of $(f(s), g(s))$; of any 2 quantities is T of $(f(s), g(s))$, S of $(f(s), g(s))$ but that is nothing but Z of $0, s$ which by the definition of Z is h of s .

Therefore, Cauchy condition is satisfied that means, a piece of the datum curve lies on the integral surface corresponding defined by this function u . Now, we have to still check that u solves the PDE.

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Proof of (1)

Proving that for every $(x, y) \in F$,

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$$

holds is the same as showing that for every $(t, s) \in E$,

$$F(X(t, s), Y(t, s), Z(t, s), u_x(X(t, s), Y(t, s)), u_y(X(t, s), Y(t, s))) = 0.$$

We already know that for $(t, s) \in J \times I$,

$$F(X(t, s), Y(t, s), Z(t, s), P(t, s), Q(t, s)) = 0.$$

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So, proving that for every x, y, u satisfies the PDE namely $F(x, y, u, x_y, u_x, x_{yy}, u_{xy}, y_{xx}, y_{xy}) = 0$ holds is the same as showing that for every t, s in E , $F(X(t, s), Y(t, s), Z(t, s), u_x(X(t, s), Y(t, s)), u_y(X(t, s), Y(t, s))) = 0$ because x, y and t, s are in 1-to-1 correspondence via diffeomorphism. Therefore, showing this is same as showing this and we already know that for t, s in $J \times I$, $F(X(t, s), Y(t, s), Z(t, s), P(t, s), Q(t, s)) = 0$.

This, we have seen in step 2 in while defining initial strip, we did that. So, this is more we want to show this there is a difference between the 2. The difference being in the last 2 coordinates $u_x(X(t, s), Y(t, s))$ is here, $P(t, s)$ is here, $u_y(X(t, s), Y(t, s))$ is here, $Q(t, s)$ is here. Suppose, we show that this pair of functions of t, s is same as this pair, then we have shown this because it is already known. So, let us try to do that. Let us show that $P(t, s) = u_x(X(t, s), Y(t, s))$; $Q(t, s) = u_y(X(t, s), Y(t, s))$.

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Proof of (1) (contd.)

If we show that for $(t, s) \in E$,

$$P(t, s) = u_x(X(t, s), Y(t, s)), \quad Q(t, s) = u_y(X(t, s), Y(t, s)) \quad (***)$$

holds, then for $(t, s) \in E$,

$$\begin{aligned} F(X(t, s), Y(t, s), Z(t, s), u_x(X(t, s), Y(t, s)), u_y(X(t, s), Y(t, s))) \\ = F(X(t, s), Y(t, s), Z(t, s), P(t, s), Q(t, s)) \\ = 0. \end{aligned}$$

This completes the proof of (1). It remains to prove (***)

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So, this is a triple star is what we want to show, then this whole with this we already observed on the last slide that completes the proof. So, what remains to show is these 2 equalities.

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Proof of (1) (contd.)

For $(t, s) \in E$, how to prove

$$P(t, s) = u_x(X(t, s), Y(t, s)), \quad Q(t, s) = u_y(X(t, s), Y(t, s)) \quad (***)$$

We prove that the two pairs

$$(u_x(X(t, s), Y(t, s)), u_y(X(t, s), Y(t, s))) \quad \text{and} \quad (P(t, s), Q(t, s))$$

are solutions to a system of linear non-homogeneous equations having a unique solution. This proves (***)

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How are we going to show that? We prove that this pair and this pair are same by showing what. We will show that this pair is a solution to a system of non homogeneous linear equations. This is also a solution of non homogeneous linear equation, the same system has a unique solution because the coefficient matrix there will be invertible. Therefore, the solution must be the same.

We know that the system $Ax = b$ if you have $ax_1 = b$ and you also know you are at $Ax_2 = b$ is equal to b that would imply $x_1 = x_2$ if you know that A is invertible. Solution is unique. So, we are going to show this. We are going to derive the system of linear equations, which both the pairs satisfy and we will show that the coefficient matrix is invertible therefore, the solution is unique.

Therefore, if you knew $Ax_1 = b$ and $Ax_2 = b$, then it must be that x_1 should be equal to x_2 and then it will follow that this pair, this pair is same as this pair which is triple star. So, now we have to get those systems. How do we get that system?

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Proof of (1) (contd.)

Differentiating the equation $z(t, s) = u(X(t, s), Y(t, s))$ w.r.t. s and t gives

$$\begin{pmatrix} z_t(t, s) \\ z_s(t, s) \end{pmatrix} = \begin{pmatrix} X_t(t, s) & Y_t(t, s) \\ X_s(t, s) & Y_s(t, s) \end{pmatrix} \begin{pmatrix} u_x(X(t, s), Y(t, s)) \\ u_y(X(t, s), Y(t, s)) \end{pmatrix}.$$

The same system of equations is satisfied by the pair $(P(t, s), Q(t, s))$:

$$\begin{pmatrix} z_t(t, s) \\ z_s(t, s) \end{pmatrix} = \begin{pmatrix} X_t(t, s) & Y_t(t, s) \\ X_s(t, s) & Y_s(t, s) \end{pmatrix} \begin{pmatrix} P(t, s) \\ Q(t, s) \end{pmatrix}. \quad (5)$$

The second equation of (5) follows from (Chara.Strip.ODE).

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So, differentiating this equation $z(t, s) = u(X(t, s), Y(t, s))$ with respect to s and t will give us 2 equations, for convenience in the matrix form. What is z_s ? z_s that is the first equation, z_s means u_x into X_s . That is easy u_x into X_s plus u_y into Y_s so, Y_s into u_y . So, that is a first equation, this one, this into this. These are matrix, these are vector, these are vector. So, this equation is very clear. Now, we claim.

So, what is this equation? This is satisfied by $u_x X_t + u_y Y_t$, $u_y X_s + u_x Y_s$, this matrix acting on that will be z_s and z_t . Now, we are going to show P and Q also satisfy same system. See the coefficient matrix is same. In this case, I have written as the left hand side. So, this is same. This is same and if this is invertible, these 2 must be same. Is this invertible? That is the question.

It is invertible because change of variables, this is a Jacobian corresponding change of variables. Therefore, this will be always invertible. So, therefore, the moment we establish this system, it automatically follows that this pair is same as this pair. Now, how do I derive this 5 actually stands for this equation? How do I derive is missing in the latest compilation it has vanished.

The second equation follows from chara strip ODE because that is what it is $z_t = X_t p + Y_t q$. What is the X_t and Y_t ? They are F_p and F_q , so, it is p of p plus q of q , therefore, this follows. So, we are to show the first equation.

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Proof of (1) (contd.)

The first equation of (5) is $\forall (t,s) \in E$,

$$A(t,s) := z_s(t,s) - P(t,s)X_s(t,s) - Q(t,s)Y_s(t,s) = 0.$$

For each s , we show that $A(\cdot, s)$ solves the IVP

$$A_t + F_s A = 0, \quad A(0, s) = 0.$$

By uniqueness of solutions to IVP, we get $A(t,s) = 0$ for all $(t,s) \in E$.

Note that $A(0,s) = h'(s) - p(s)f'(s) - q(s)g'(s) = 0$ for all s , by definition of initial strip.

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The first equation is this we want to show. So, we want to show that this thing equal to 0. Let us call it by $A(t,s)$. We want to show that $A(t,s)$ is 0 for every (t,s) in E . How are we going to show this? Because these features derivative with respect to s that may be the difficulty. For each s , we show as a function of t , it solves an initial value problem $A_t + F_s A = 0$, $A(0,s) = 0$. We will show that it solves this initial value problem.

Now, by uniqueness of solutions to initial value problem $A(t,s)$ must be 0 for all t . Why? Because this initial value problem we have only 0; 0 solution is a solution, $A(0,s) = 0$, is a solution? Here, it satisfies this condition. Therefore, if this is a linear equation, if this is a continuous coefficients, this will be Lipschitz therefore, you have a unique solution. Therefore, $A(t,s)$ will be 0 for every t and this happens for every fixed s .

Therefore $A(t,s)$ is 0 for all (t,s) , fine. So, we have to derive this equation that is all remains to show. $A(0,s)$ is 0 for all s . This is coming from the definition of initial strip, $h' = p f' + q g'$.

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Proof of (1) (contd.)

Differentiating $A(t, s) := z_s(t, s) - P(t, s)X_s(t, s) - Q(t, s)Y_s(t, s)$ w.r.t. t (arguments are suppressed for functions below) gives

$$\begin{aligned}
 A_t &= z_{st} - P_t X_s - Q_t Y_s - P X_{st} - Q Y_{st} \\
 &= \frac{\partial}{\partial s} (z_t - P X_t - Q Y_t) + P_t X_t + Q_t Y_t - P_t X_t - Q_t Y_t \\
 &= P_s F_p + Q_s F_q + X_s (F_x + P F_z) + Y_s (F_y + Q F_z) \\
 &= \frac{\partial}{\partial s} (F) - F_z (z_t - P X_t - Q Y_t) \\
 &= -F_z A
 \end{aligned}$$

In this computation we used $\frac{\partial}{\partial s} (F) = 0$, a consequence of $F(X, Y, Z, P, Q) = 0$.

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Differentiating A w.r.t. s , we have to derive an equation satisfied by A w.r.t. s . So, the only thing you can do is differentiation. So, A is equal to $z_s - P X_s - Q Y_s$. Now, a small rearrangement is required it is an algebra. So, that we will get minus F_z into A . So, we are going to use chara strip ODE equations here and we end up getting this. That is same as minus $F_z A$, because this is 0. So, we use that τ by τ of F is 0, which is a consequence of F of $X Y Z P Q$ being 0.

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Proof of (1) (contd.)

- Note that the coefficient matrix appearing in the system of equations (4) and (5) is the same.
- Its determinant is given by

$$\Delta := \begin{vmatrix} X_t & Y_t \\ X_s & Y_s \end{vmatrix}$$
- It is the Jacobian corresponding to the change of coordinates (x, y) and (t, s) , thus non-zero.

Thus the linear system (4)-(5) has a unique solution. \square

Thus we have proved the following local existence and uniqueness theorem. Uniqueness proof proceeds as in (QL).

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So, note that the coefficient matrix appearing the system of equations 4 and 5 is the same, already observed. Its determinant is precisely this; it is the Jacobian corresponding to the change of coordinates x, y and t, s . Therefore, it is always nonzero. Thus, the linear system 4,5 has a unique solution. Thus, we have proved the following local existence and uniqueness

theorem. Uniqueness proof proceeds as in the case of Quasilinear case, so I am not going to do it here.

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Theorem

Assumptions on F :

- 1 Let Ω_5 be an open and connected subset of \mathbb{R}^5 .
- 2 Let $F \in C^2(\Omega_5)$, F_p and F_q satisfy

$$F_p^2(x, y, z, p, q) + F_q^2(x, y, z, p, q) \neq 0 \quad \forall (x, y, z, p, q) \in \Omega_5.$$

Assumptions on Cauchy data Γ :

- 3 Let Γ be a continuously differentiable parametrized curve: that is,

$$\Gamma: x = f(s), y = g(s), z = h(s) \quad s \in I,$$
 where $I \subseteq \mathbb{R}$ is an interval, and $f, g, h \in C^1(I)$.

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So, what is the theorem? Assumptions on F , Ω_5 is an open set connectedness is not required, but open set. Let F be a C^2 function that we cannot dispense with. F_p and F_q satisfy that both of them cannot vanish simultaneously at any point Ω_5 . Assumptions on Cauchy data, f, g, h are C^1 functions. So, no need of C^2 functions, just C^1 .

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Theorem (contd.)

Assumptions on Initial strip and transversality condition:

- 1 Assume that the system

$$F(f(s), g(s), h(s), p(s), q(s)) = 0 \quad (\text{IS-1})$$

$$p(s)f'(s) + q(s)g'(s) = h'(s). \quad (\text{IS-2})$$
 admits a solution $(p(s), q(s))$ where $p(s)$ and $q(s)$ are continuously differentiable functions on the interval I such that the transversality condition holds at $s = s_0$:

$$\Delta = \begin{vmatrix} f'(s_0) & F_p(f(s_0), g(s_0), h(s_0), p_0, q_0) \\ g'(s_0) & F_q(f(s_0), g(s_0), h(s_0), p_0, q_0) \end{vmatrix} \neq 0.$$

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And assumption on initial strip and transversality condition: So, assume that the system $f, g, h, p, q, s = 0$; $p, f, g, h, p, q, s = 0$; $p, f, g, h, p, q, s = 0$; $p, f, g, h, p, q, s = 0$ admits a solution p, s, q, s where p, s, q, s are continuously differentiable functions on the interval I . Actually we have, it is enough to work with the existence of these kind of functions for s in a small interval containing some point s

0. We got finally the conclusion of inverse function theorem or implicit function theorem or local.

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Theorem (contd.)

Conclusions:

- The general nonlinear PDE $F(x, y, u, u_x, u_y) = 0$ admits a solution defined on an open subset D of Ω_2 .
- The point $(f(s_0), g(s_0)) \in F$, and satisfies $u(f(s), g(s)) = h(s)$ for those $s \in I$ for which $(f(s), g(s)) \in F$.
- Further, the solution is locally unique.

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So, I assume that transversality condition. Conclusions are the general nonlinear PDE admits a solution defined an open subset of omega 2 for the Cauchy problem that is missing here Cauchy problem. Cauchy problem for general nonlinear PDE admits a solution defined an open set of omega 2 which is F actually, it is not D. It is actually F, we have found. D is equal to F in the proof we have. D is equal to F and the point f s 0 g s 0 in the F satisfies u f s g s = h s s for those s for which this is in F. Further, the solution is locally unique.

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Example

Consider the Cauchy problem for the nonlinear equation

$$u_x^2 + u_y^2 = 1,$$

where Cauchy data is given by

$$u(x, y) = 0 \text{ for } (x, y) \text{ such that } x^2 + y^2 = 1.$$

Let us parametrize the given Cauchy data as

$$\Gamma: x = \cos s, y = \sin s, z = 0, s \in [0, 2\pi).$$

The system of ODEs for characteristic strips for the given equation is

$$\frac{dx}{dt} = 2p, \frac{dy}{dt} = 2q, \frac{dz}{dt} = 2, \frac{dp}{dt} = 0, \frac{dq}{dt} = 0.$$

Handwritten notes on the slide:

$$F(x, y, z, p, q) = p^2 + q^2 - 1$$

$$F_p = 2p$$

$$F_q = 2q$$

$$pF_p + qF_q = 2p^2 + 2q^2 = 2$$

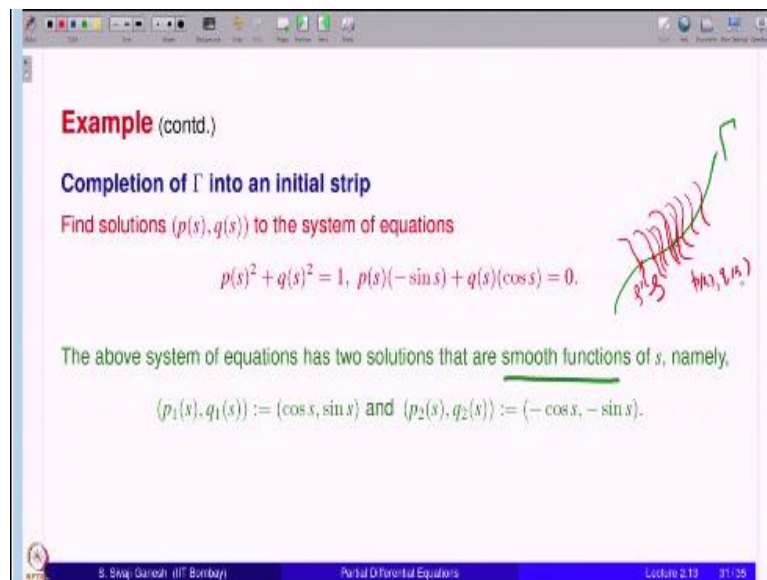
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So, let us solve an example of where we are going to solve a Cauchy problem for a nonlinear equation, the simplest nonlinear equation $u_x^2 + u_y^2 = 1$. Cauchy data is given

by u is 0 on the circle $x^2 + y^2 = 1$. So, first thing as always is to parameterize the Cauchy data, $x = \cos s$, $y = \sin s$, $z = 0$, s in the interval 0 to 2π system of ODEs. So, what is F of x, y, z, p, q , $p^2 + q^2 - 1$.

So, in this example, so, we should always write this, what is this, function. From here, we can compute very easily F_x, F_y, F_z are 0; F_p is $2p$; F_q is $2q$. So, dx/dt is F_p therefore, $2p$; dy/dt is F_q therefore, $2q$; dz/dt is $p F_p + q F_q$. What is $p F_p + q F_q$? Let us write down once again. F_p is $2p$; F_q is $2q$; $p F_p + q F_q$ is equal to $2p^2 + 2q^2$, but the equation says $p^2 + q^2 = 1$. Therefore, this is 2. And dp/dt is 0; dq/dt is 0 because our F does not involve x, y, z at all.

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So, we have to now complete into an initial strip the datum curve. We have to find p, q, s satisfying 2 conditions. One is the equation $p^2 + q^2 = 1$. Second is $p f' + q g' = h'$ that will give us this. Now, we have to find solutions. Here, we see clearly 2 solutions of course, you may say infinitely many solutions, because you can mix both of them because it is an algebraic or whatever transcendental equation.

At some s , you can be here; at some other s , you can be here. You can keep on oscillating, but that is not good, what we need; is not smooth function? It is very important, because what we are trying to solve just recall whether Quasilinear or fully nonlinear, we take γ ; we take a point on this and we pass a characteristic curve through that and repeat for everything. Whenever you choose a point on γ , it will correspond to some s .

And then you are simply passing a characteristic curve through that. Another s dash you take, you pass another characteristic curve. Why should these characteristic curves together teach or view a surface that would happen if things underlying are smooth functions that is why f g h, we assume C 1 functions. So, here we need to assume p s q s also smooth functions, otherwise we do not expect.

So, that is why there are 2 choices for smooth functions. So, p 1 s q 1 s is cos s sin s, other one is minus cos s sin s. So, smooth is very important.

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Example (contd.)

By taking the initial strip as $(\cos s, \sin s, 0, \cos s, \sin s)$, the solution of the (Chara.Strip.ODE) satisfying the initial conditions

$$x(0) = \cos s, y(0) = \sin s, z(0) = 0, p(0) = \cos s, q(0) = \sin s$$

is given by

$$X(t, s) = (2t + 1) \cos s, Y(t, s) = (2t + 1) \sin s, Z(t, s) = 2t,$$

$$P(t, s) = \cos s, Q(t, s) = \sin s.$$

From the first three equations, we get

$$X^2 + Y^2 = (Z + 1)^2$$

Thus, the function Z is given by

$$Z(t, s) = -1 \pm \sqrt{X^2(t, s) + Y^2(t, s)}.$$

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So, there are only 2 if you insist on smoothness. Now, if you take the initial strip where p q is taken to be cos s and sin s, the solution of chara strip ODE will be this. I am not going to the computation because a very simple ODEs that you can solve. So, these are the solutions X t s Y t s, Z t s P t s Q t s. Now, from the first 3 equations, we get this relation X square + Y square = Z + 1 whole square and Z t s is given by this formula on simplification again that.

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Example (contd.)

We define a pair of functions by

$$u_{1,2}(x,y) = -1 \pm \sqrt{x^2 + y^2}.$$

Thus the solution to Cauchy problem is given by

$$u_1(x,y) = -1 + \sqrt{x^2 + y^2}.$$

The function $u_2(x,y) = -1 - \sqrt{x^2 + y^2}$ does not satisfy the Cauchy data.

If we proceeded by taking the initial strip as $(\cos s, \sin s, 0, -\cos s, -\sin s)$, then we would have got the solution as $u(x,y) = 1 - \sqrt{x^2 + y^2}$.

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So, we define a pair of functions u_1 and u_2 ; one with plus sign, one with minus sign, then the solution to Cauchy problem is given by this and not the minus 1. Why? It does not satisfy the Cauchy data, you can check that. So, if we proceeded by taking the initial strip as $\cos s$ and $\sin s$, then we would have got this as a solution and not this. This will not satisfy the Cauchy data, this will satisfy.

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Summary

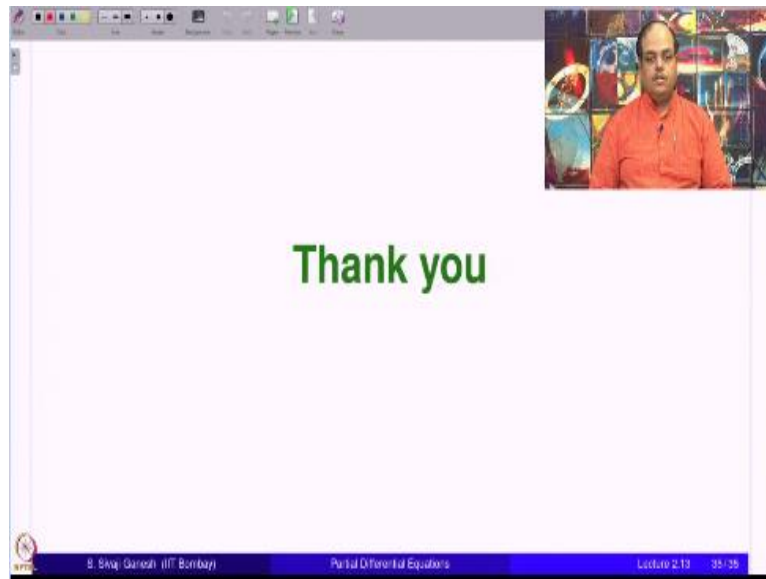
- 1 A candidate solution was defined.
- 2 Verified that it is indeed a solution to the Cauchy problem.
- 3 A closer look at the proof of Existence and Uniqueness theorem reveals
 - Proofs of existence and uniqueness theorems for both (QL) and (GE) are strikingly similar, once Characteristic strips have been obtained.
 - The extension of ideas from (QL) to (GE) were clearly brought out in our presentation.

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So, summary is for this lecture is that the candidate solution was defined, we verified that is indeed a solution to the Cauchy problem. A closer look at the proof of existence and uniqueness theorem reveals proofs of the existence and uniqueness theorem for both QL and GE are strikingly similar, actually the same, but for obvious modifications. Once; characteristics strips have been obtained, because we need that $X(t,s), Y(t,s), Z(t,s)$. And we always work with $X(t,s), Y(t,s)$ to get inversion. Inverse function theorem is applied only for X

t s Y t s whether it is QL or GE and the extension of ideas were clearly brought out in our presentation.

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So, this completes the analysis of Cauchy problem for general nonlinear equations. In the forthcoming lectures, we take up some problems based on this. Thank you.