Partial Differential Equations
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Lecture – 2.13
General Nonlinear Equations 4
Local Existence and Uniqueness Theorem

In this lecture, we are going to complete the proof of local existence and uniqueness theorem for Cauchy problems for the general nonlinear equations.

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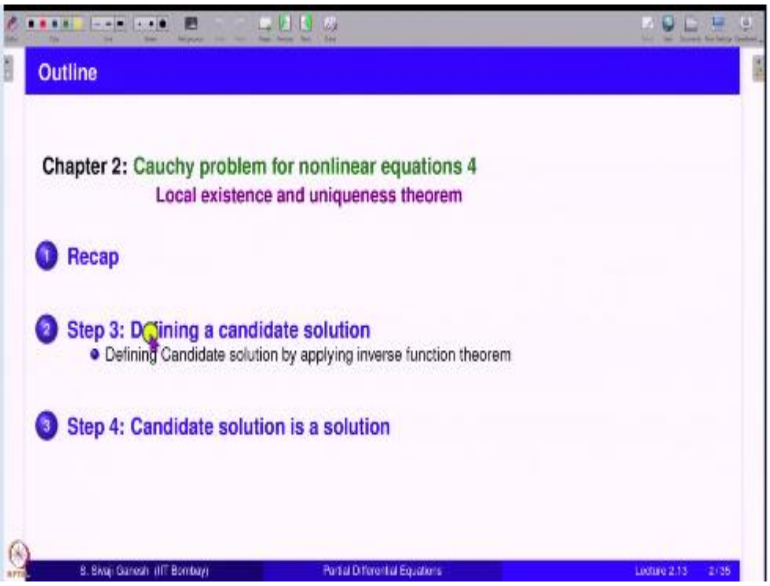

So, we start with a recap of what happened so far. And then we go to step 3 with namely defining a candidate solution. So, we will define a candidate solution by applying inverse function theorem and then we show that the candidate solution is indeed solution.

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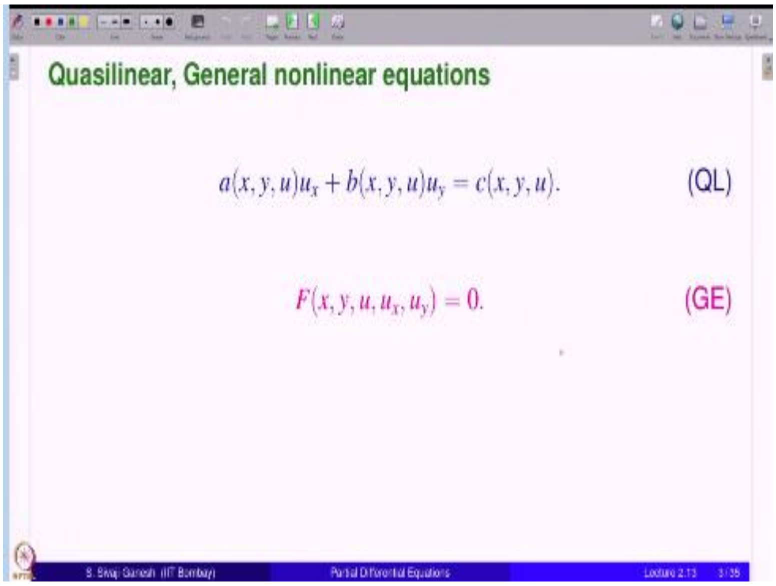

So, this is to recall the notations QL stands for Quasilinear equation and GE stands for general nonlinear equations or sometimes we will call it as fully nonlinear equations. Of course, GE contains L, SL as well as QL. But when it is presented in this form, it is called generally nonlinear equations.

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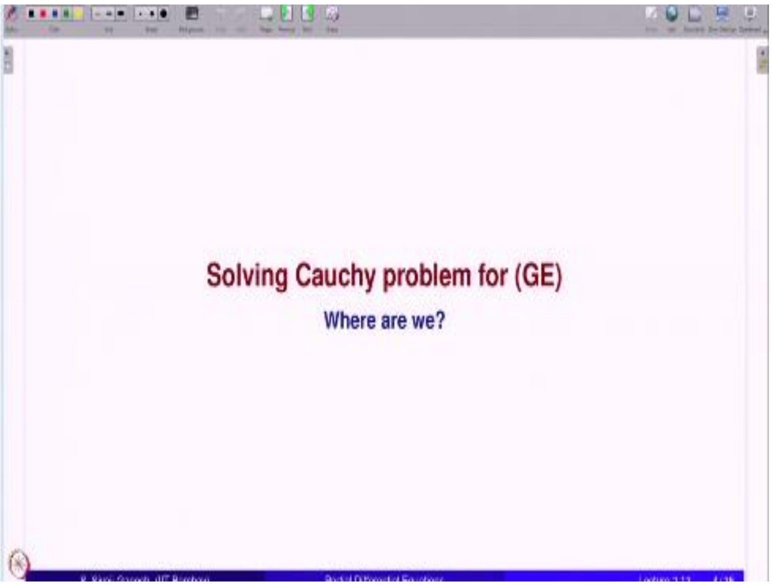

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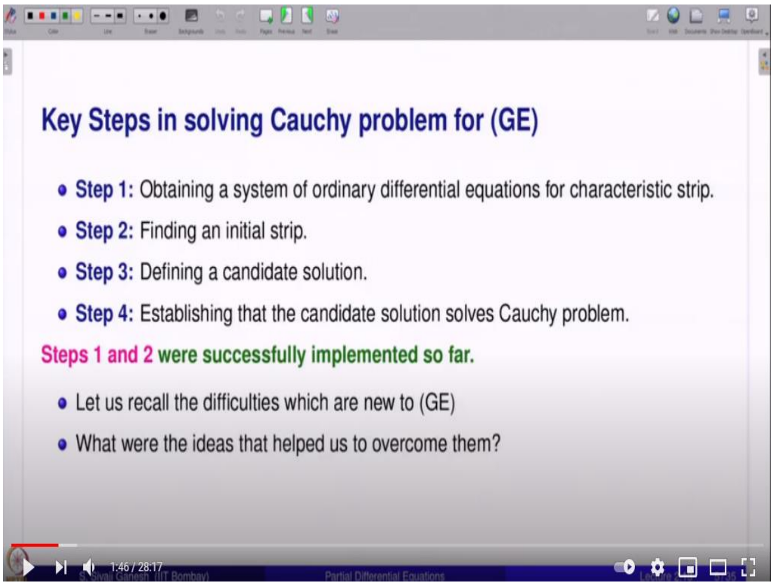

Where are we in solving the Cauchy problem? The key steps involving the solution of the Cauchy problem were identified. Step 1 is to obtain a system of ODEs for the characteristic strip and step 2 is a finding an initial strip. Third one is to define a candidate solution and fourth one is establishing that the candidate solution is indeed a solution. So, steps 1 and 2 were successfully implemented so far.

Unless, we call the difficulties once again, which are new to GE when compared to QL. And what are the ideas that helped us to overcome them.

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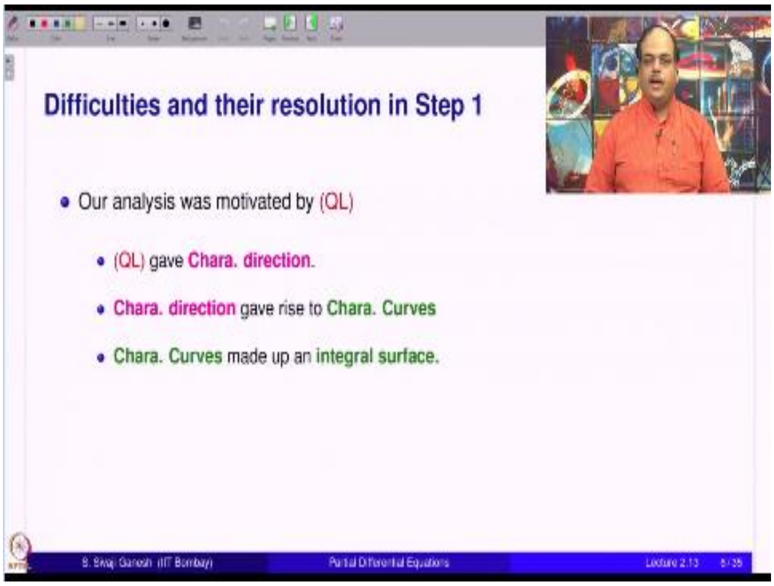

So, our analysis was motivated by Quasilinear equations, QL. So, QL gave us characteristic direction. Characteristic direction gave rise to characteristic curves. Characteristic curves made up an integral surface.

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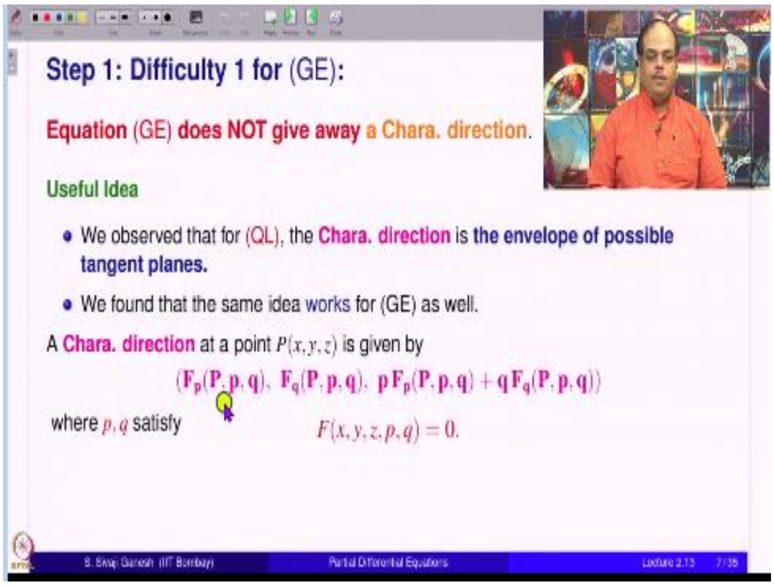

Now, equation GE does not give away a characteristic direction. So, useful idea, we observed that for Quasilinear equations, the characteristic direction is the envelope of possible tangent planes. In fact, the possible tangent planes envelope is a straight line which has the characteristic direction. So, we found that the same idea works for GE as well. So, characteristic direction at a point P x y z is given by F p F q p F p + q F q.

The argument has to be F p P p q, P stands for x y z; p and q, they are such that F of this is 0.

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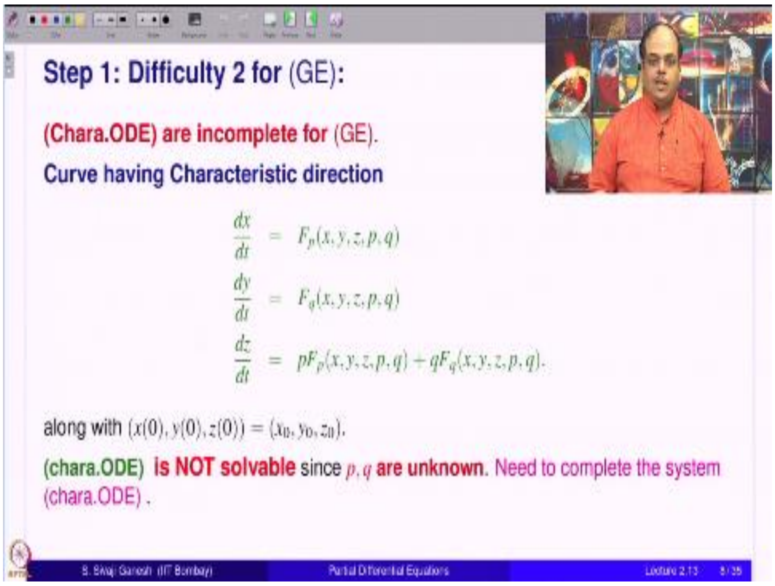

And characteristic ODE system is incomplete for general nonlinear equation, because when we try to find a curve which has a characteristic direction, it has to satisfy this set of ODEs, namely chara ODE, but here p and q appear which themselves are unknown. So, this is not solvable and we need to add equations or supplement the system with another 2 equations; one for p and one for q.

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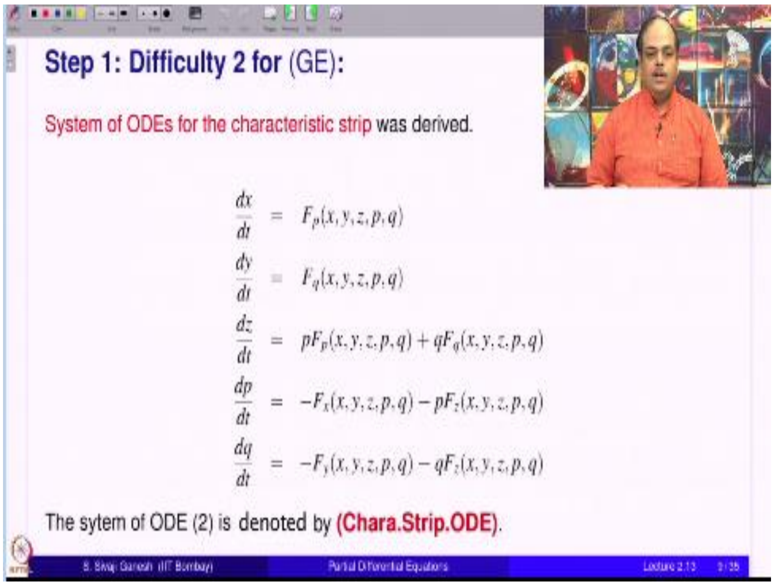

It has been achieved and that was called chara strip ODE where the dp by dt, dq by dt equations have been added.

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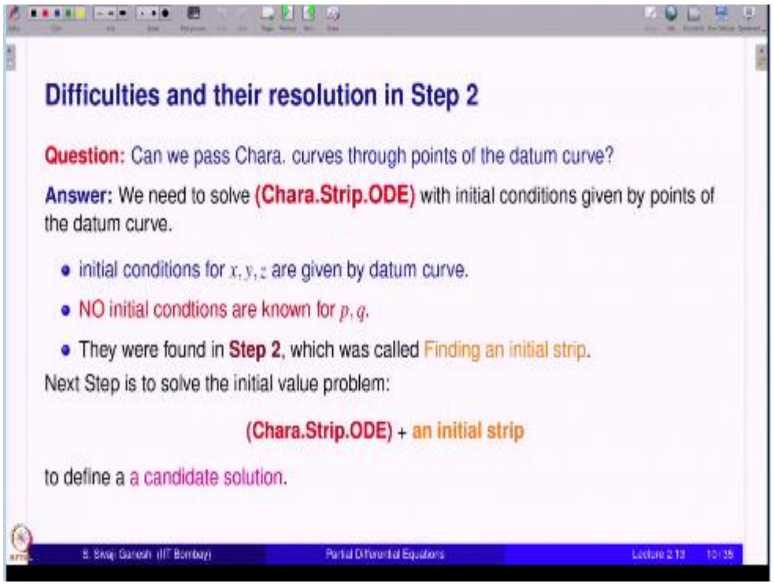

Now, next question is that, can we pass characteristic curves through points of the datum curve? This is what we did in QL. So, for that we need to solve chara ODE with initial conditions given by points of the datum curve. Initially conditions for x y z are given by the datum curve; no initial conditions are known for p and q. So, they were found in step 2 which was called finding an initial strip.

Next step is to solve the initial value problem chara strip ODE plus an initial strip that will give you a candidate solution.

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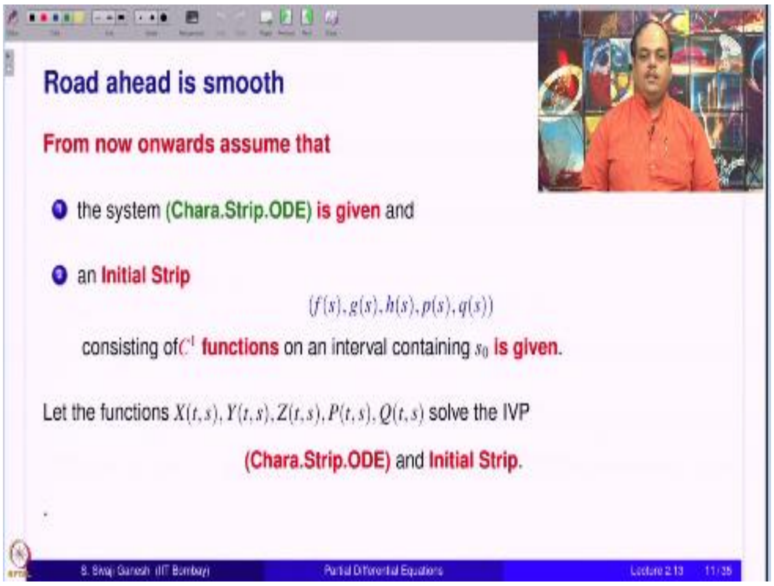

It will define a candidate solution. We are going to see that. So, from now onwards, what we assume maybe in deriving these equations, we have assumed a certain higher smoothness conditions on f g h and F or maybe u but do not worry. Now onwards, we assume this that characteristics strip ODE is given; equations are given to you. You do not have to derive. And initial strip consisting of C 1 functions is found.

Of course, we cannot find perhaps throughout gamma, but you look at a point f s 0, g s 0, h s 0 on gamma. And hence, I want this to be defined nearby that point p 0. And let this functions X Y Z P Q, which are functions of t and s; solve the IVP. What is IVP? Chara strip ODE supplemented with this initial conditions coming from initial strip.

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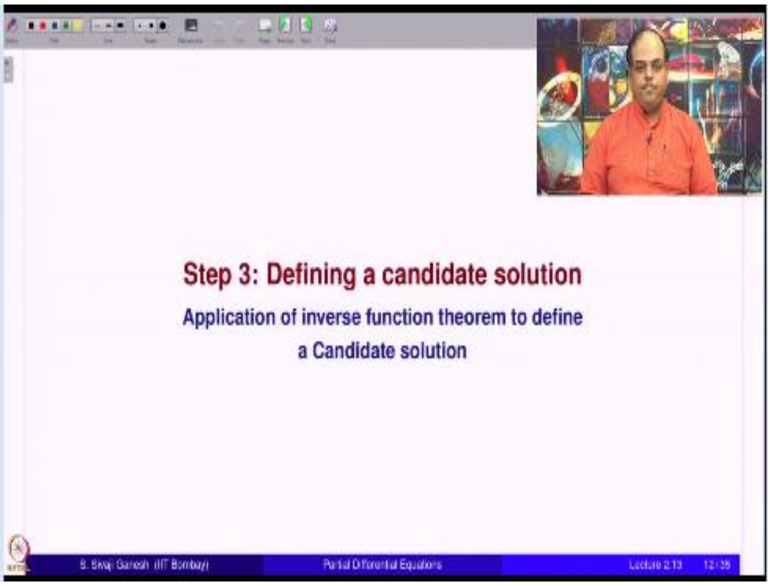

Now; defining a candidate solution: this is where we need to apply inverse function theorem to certain functions and then get a candidate solution.

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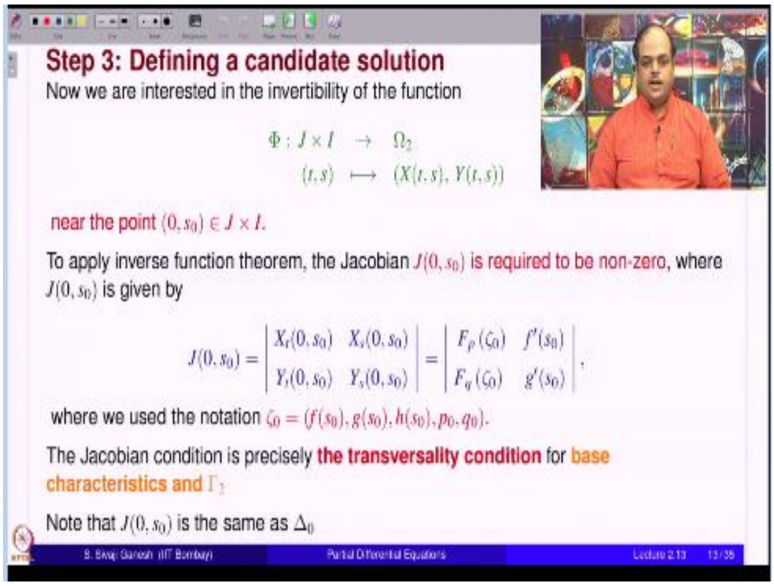

So, we are interested in the invertibility of this function. What is this function? This is actually a base characteristic curves; the trace is base characteristic curves as a t various passing through the point f s g s which is on gamma 2. So, near this point 0, s 0 which belongs to J cross I. So, now, to apply inverse function theorem, the Jacobian is required to be nonzero.

Of course, we need to check whether phi is 1 that is C 1 because x and y are solving ODEs therefore, the derivative with respect to t is continuous, no problem. With respect to X is also differentiable and C 1 because other differential dependence. So, we have this C 1 of phi.

Now, we need that the Jacobian is nonzero. The Jacobian is this. As before, we do not once again analyze with g of t s because nothing is known for a nonzero t.

So, only at 0, it is known. J of 0, s 0, the Jacobian is this, which is F p F q at the point C 0 and f dash s 0, g dash s 0. Of course, the way we got p s and q s is such that p of s 0 is p 0; q of s 0 is q 0. So, you could as well write here p s 0, q s 0 but remember, we started with p 0 q 0 a particular solution of system of equations, which define p s and q s later on. So, you can write p s 0 q s 0, p 0 q 0 because both are actually the same.

Now, the Jacobian condition is precisely the transversality condition, f prime g prime corresponds to what, tangent to gamma 2, tangent to the gamma 2. What is gamma 2? Projection of gamma 2 x y coordinate omega 2, these are tangent. This is the base characteristics; tangent to the base characteristic of this direction F p and F q. We have to assume these nonzero that means they are not parallel.

So, base characteristics cut gamma 2 that is the transversality condition. This is same as the delta 0 that we saw in the step 2.

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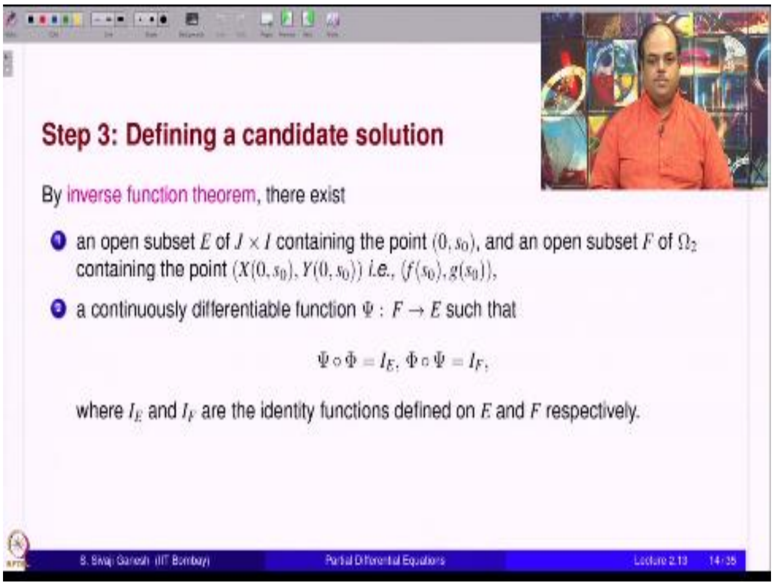

So, defining a candidate solution by inverse function theorem, there exists an open subset E of J cross I containing the point 0, s 0. And F which is an open set of omega 2 which contains X of 0, s 0; Y of 0, s 0, which is actually f s 0, g s 0 and a continuously differentiable function from F to E such that these 2 compositions are identity maps on respective spaces.

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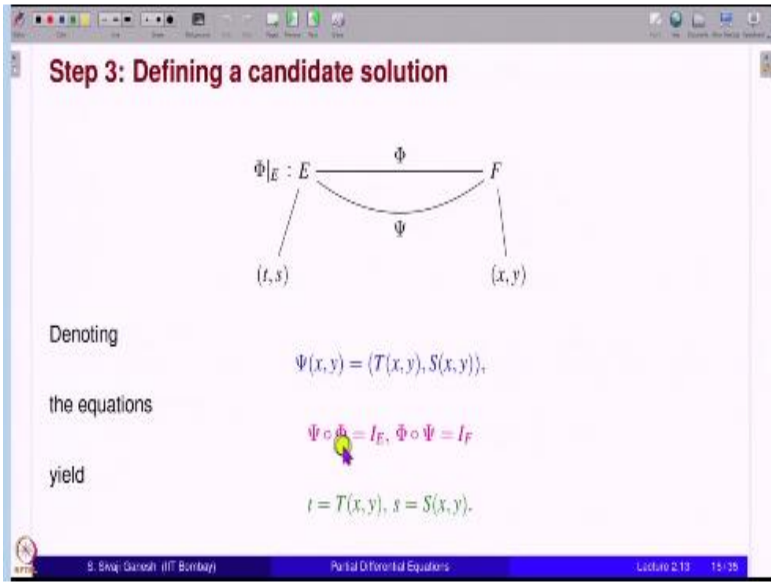

There is a picture which depicts. So, this is the phi map event. Inverse function theorem told that there are neighbourhoods E and F and a map psi defined. These where we are using t s coordinates. Here we are using x y notation. We are restricted to, E is a diffeomorphism and so on. And denote psi of x y = T of x y, S of x y and the equation psi circle phi identity on E. Phi circle psi identity on F. They give t = T x y, s = S x y.

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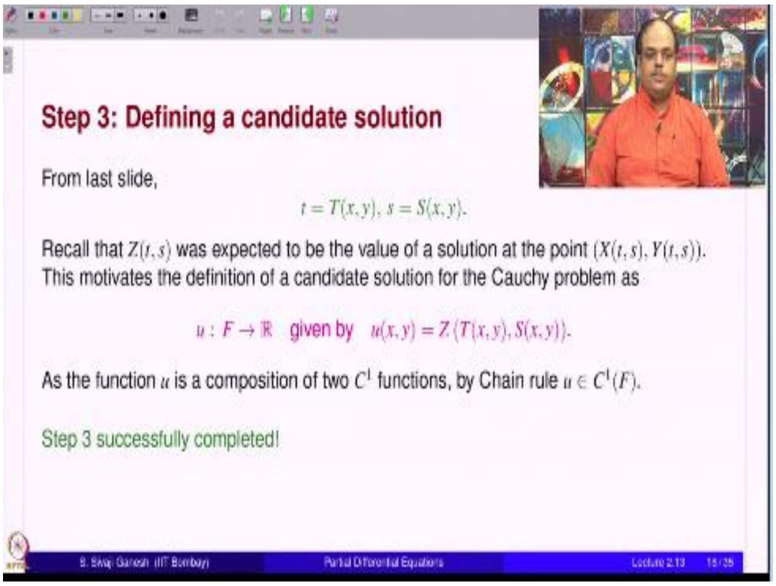

So, recall that Z t s was expected to be the value of the solution at the point X t s, Y t s. Therefore, this motivates us to define a candidate solution by using this Z. So, u define an F to R, u of x y = Z of T x y, S of x y. As a function u is a composition of 2 C 1 functions. By chain rule, u itself is C 1 function on F. So, step 3 is successfully completed. We have defined a C 1 function as a candidate solution.

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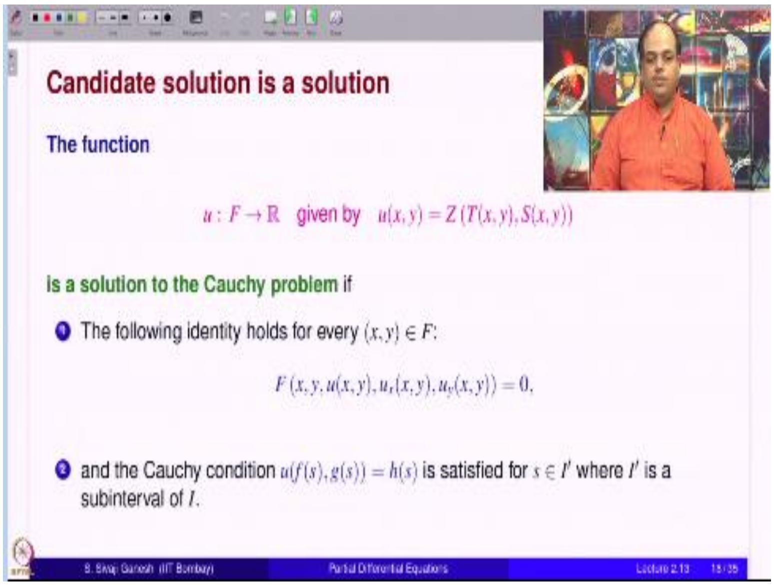

In the next step, we are going to check that this is indeed a solution. So, this is a solution to the Cauchy problem if the following identity holds for every x y in F, that is F of x y, u x y, u x x y, u y x y= 0. And the Cauchy condition u of f s g s = h s is satisfied.

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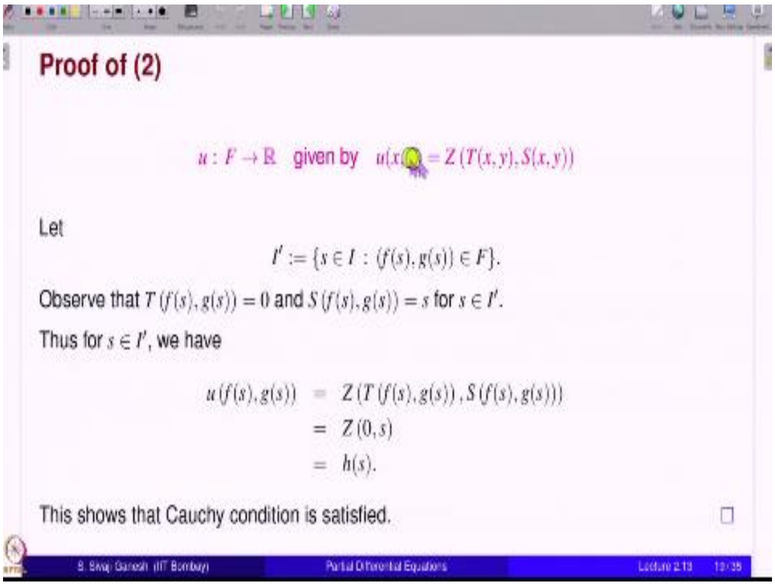

So, define I dash to be those values of s in I for which f s g s belongs to F. So, this is the I dash which came there. So, s belongs I dash. We will be; the datum curve will be on the corresponding integral surface defined by this s. So, observe that T of f s g s is 0 and the S of f s g s is s. Thus, for s in I dash, we have u of f s g s equal to by definition u of f s g s u; of any 2 quantities is T of f s g s, S of f s g s but that is nothing but Z of 0, s which by the definition of Z is h of s.

Therefore, Cauchy condition is satisfied that means, a piece of the datum curve lies on the integral surface corresponding defined by this function u. Now, we have to still check that u solves the PDE.

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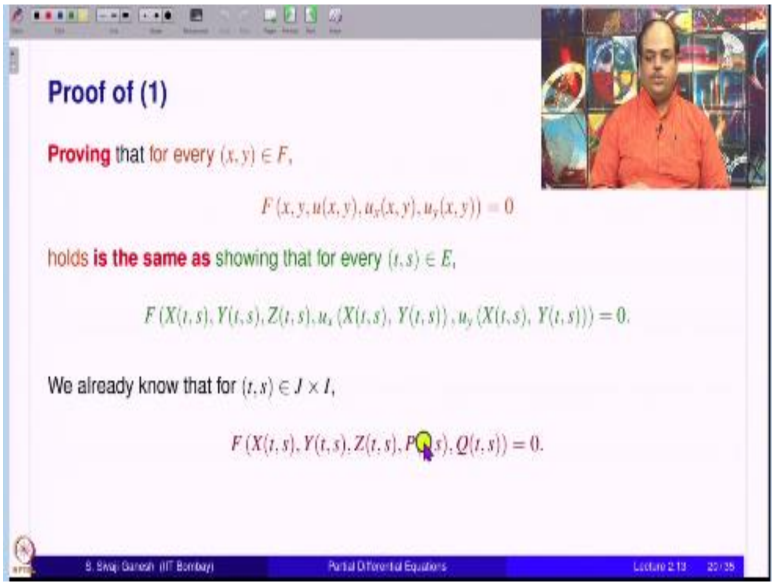

So, proving that for every x y, u satisfies the PDE namely F of x y u x y u x x y u y x y = 0 holds is the same as showing that for every t s in E, F X t s Y t s Z t s u x of X t s Y t s u y of X t s Y t s is 0 because x y and t s are in 1-to-1 correspondence via diffeomorphism. Therefore, showing this is same as showing this and we already know that for t s in J cross I, F of X t s Y t s Z t s P t s Q t s is 0.

This, we have seen in step 2 in while defining initial strip, we did that. So, this is more we want to show this there is a difference between the 2. The difference being in the last 2 coordinates u x X t s Y t s is here, P t s is here, u y X t s Y t s is here, Q t s is here. Suppose, we show that this pair of functions of t s is same as this pair, then we have shown this because it is already known. So, let us try to do that. Let us show that P t s is u x X t s Y t s; Q t s is u y X t s Y t s.

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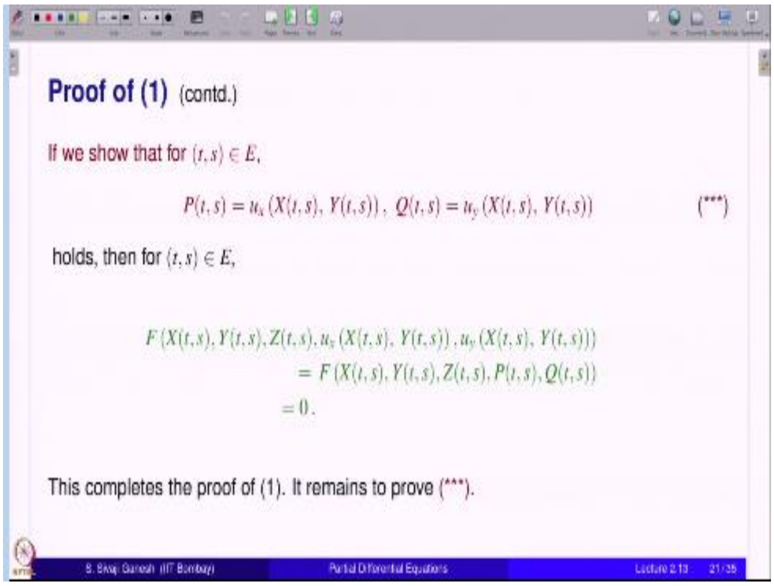

So, this is a triple star is what we want to show, then this whole with this we already observed on the last slide that completes the proof. So, what remains to show is these 2 equalities.

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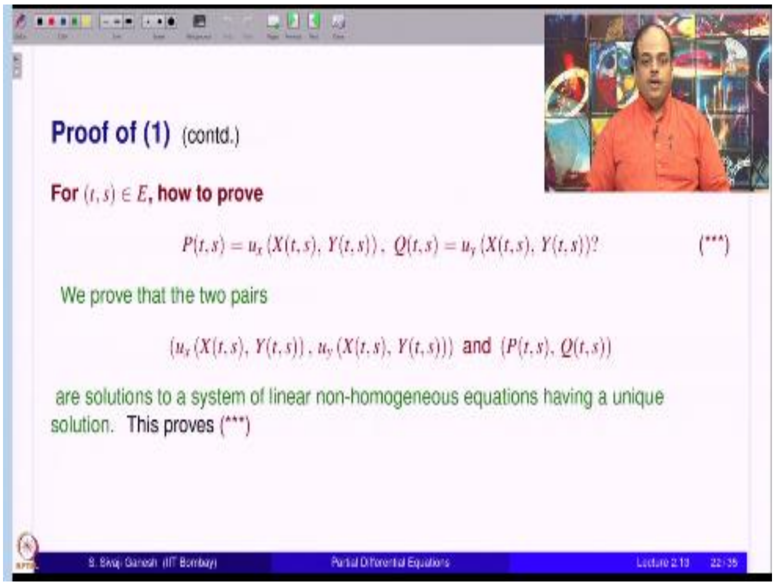

How are we going to show that? We prove that this pair and this pair are same by showing what. We will show that this pair is a solution to a system of non homogeneous linear equations. This is also a solution of non homogeneous linear equation, the same system has a unique solution because the coefficient matrix there will be invertible. Therefore, the solution must be the same.

We know that the system A x = b if you have a x 1 = b and you also know you are at A x 2 is equal to b that would imply x 1 = x 2 if you know that A is invertible. Solution is unique. So, we are going to show this. We are going to derive the system of linear equations, which both the pairs satisfy and we will show that the coefficient matrix is invertible therefore, the solution is unique.

Therefore, if you knew A x 1 = b and A x 2 = b, then it must be that x 1 should be equal to x 2 and then it will follow that this pair, this pair is same as this pair which is triple star. So, now we have to get those systems. How do we get that system?

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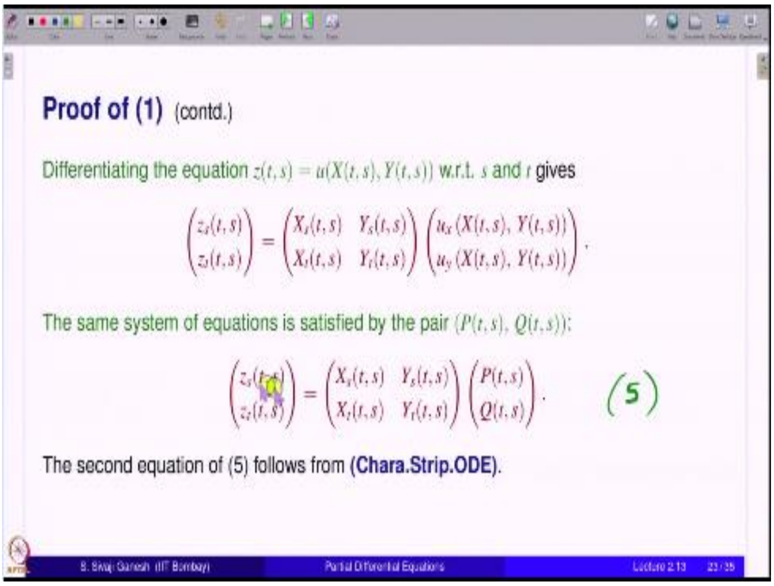

So, differentiating this equation z t s equal u of X t s Y t s with respect to s and t will give us 2 equations, for convenience in the matrix form. What is z s? z s that is the first equation, z s means u x into X s. That is easy u x into X s plus u y into Y s so, Y s into u y. So, that is a first equation, this one, this into this. These are matrix, these are vector, these are vector. So, this equation is very clear. Now, we claim.

So, what is this equation? This is satisfied by u x X t s Y t s, u y X t s Y t s, this matrix acting on that will be z s t s z t t s. Now, we are going to show P t s Q t s also satisfy same system. See the coefficient matrix is same. In this case, I have written as the left hand side. So, this is same. This is same and if this is invertible, these 2 must be same. Is this invertible? That is the question.

It is invertible because change of variables, this is a Jacobian corresponding change of variables. Therefore, this will be always invertible. So, therefore, the moment we establish this system, it automatically follows that this pair is same as this pair. Now, how do I derive this 5 actually stands for this equation? How do I derive is missing in the latest compilation it has vanished.

The second equation follows from chara strip ODE because that is what it is z t = X t p plus Y t q. What is the X t and Y t? They are F p and F q, so, it is p of p plus q of q, therefore, this follows. So, we are to show the first equation.

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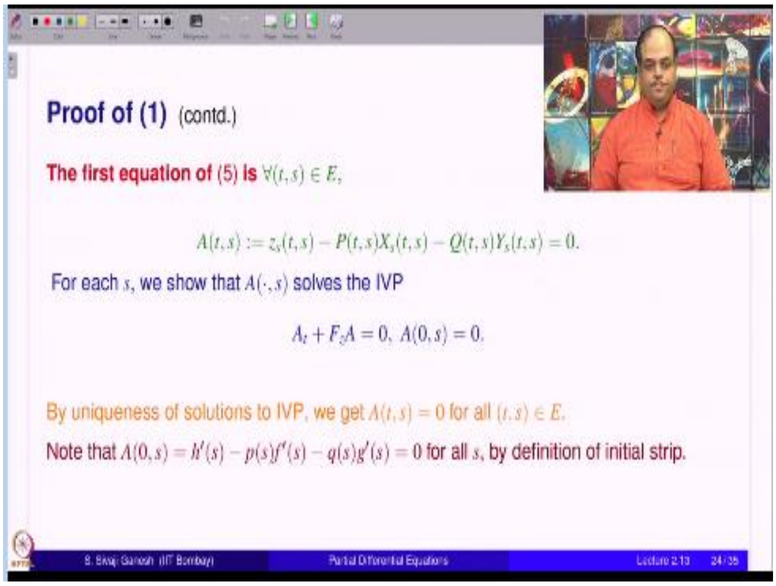

The first equation is this we want to show. So, we want to show that this thing equal to 0. Let us call it by A t s. We want to show that A t s is 0 for every t s in E. How are we going to show this? Because these features derivative with respect to s that may be the difficulty. For each s, we show as a function of t, it solves an initial value problem A t plus F z = 0, A of 0 s = 0. We will show that it solves this initial value problem.

Now, by uniqueness of solutions to initial value problem A t s must be 0 for all t. Why? Because this initial value problem we have only 0; 0 solution is a solution, A of t = 0, is a solution? Here, it satisfies this condition. Therefore, if this is a linear equation, if this is a continuous coefficients, this will be Lipschitz therefore, you have a unique solution. Therefore, A t s will be 0 for every t and this happens for every fixed s.

Therefore A t s is 0 for all t s, fine. So, we have to derive this equation that is all remains to show. A of 0 s is 0 for all s. This is coming from the definition of initial strip, h prime equal to p f prime plus q of q g prime.

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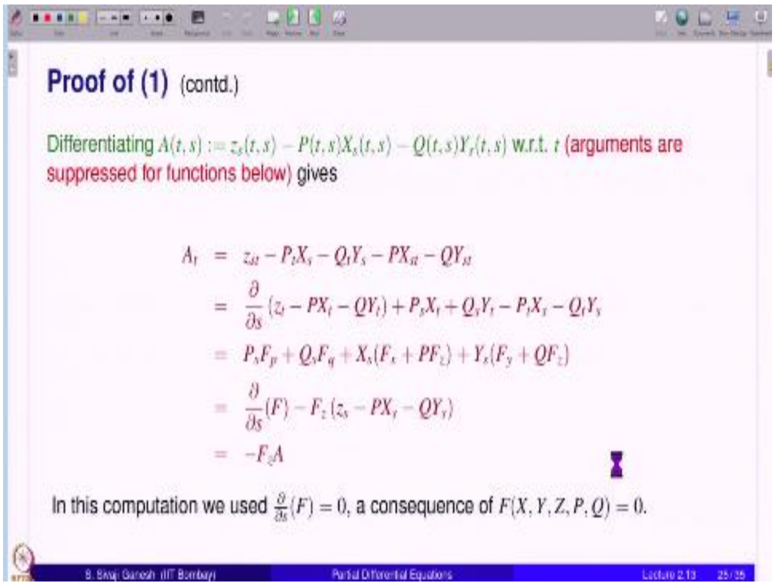

Differentiating A t s, we have to derive an equation satisfied by A t s. So, the only thing you can do is differentiation. So, A t equal to z s t at, P t X s, Q t Y s, PX s t QY s t. Now, a small rearrangement is required it is an algebra. So, that we will get minus F z into A. So, we are going to use chara strip ODE equations here and we end up getting this. That is same as minus F z A, because this is 0. So, we use that tau by tau of F is 0, which is a consequence of F of X Y Z P Q being 0.

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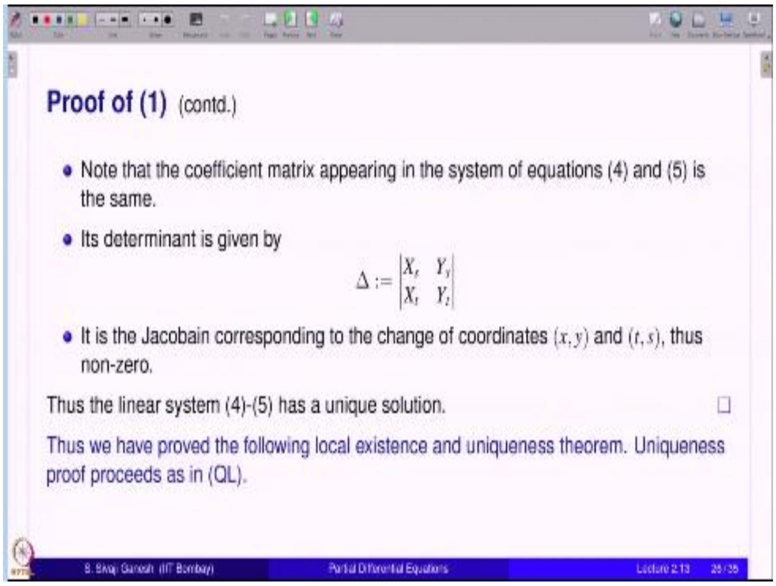

So, note that the coefficient matrix appearing the system of equations 4 and 5 is the same, we already observed. Its determinant is precisely this; it is the Jacobian corresponding to the change of coordinates x, y and t s. Therefore, it is always nonzero. Thus, the linear system 4,5 has a unique solution. Thus, we have proved the following local existence and uniqueness theorem. Uniqueness proof proceeds as in the case of Quasilinear case, so I am not going to do it here.

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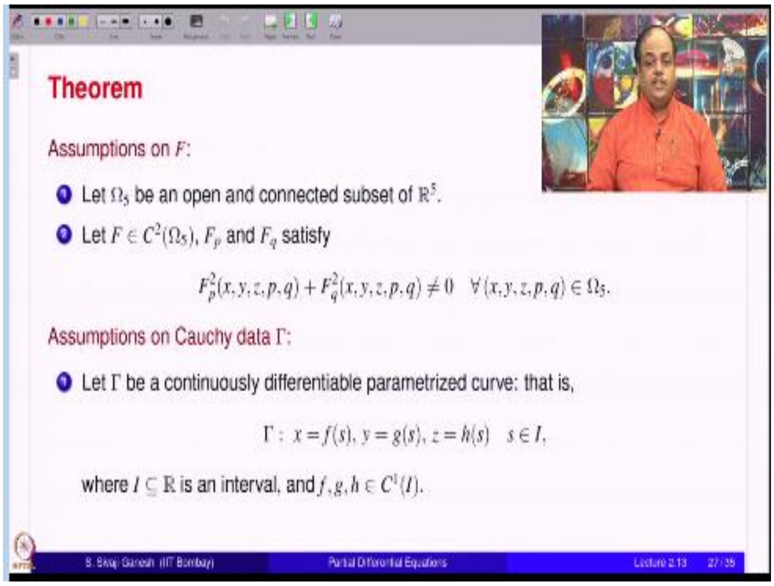

So, what is the theorem? Assumptions on F, omega 5 is an open set connectedness is not required, but open set. Let F be a C 2 function that we cannot dispense with. F p and F q satisfy that both of them cannot vanish simultaneously at any point omega 5. Assumptions on Cauchy data, f g h are C 1 functions. So, no need of C 2 functions, just C 1.

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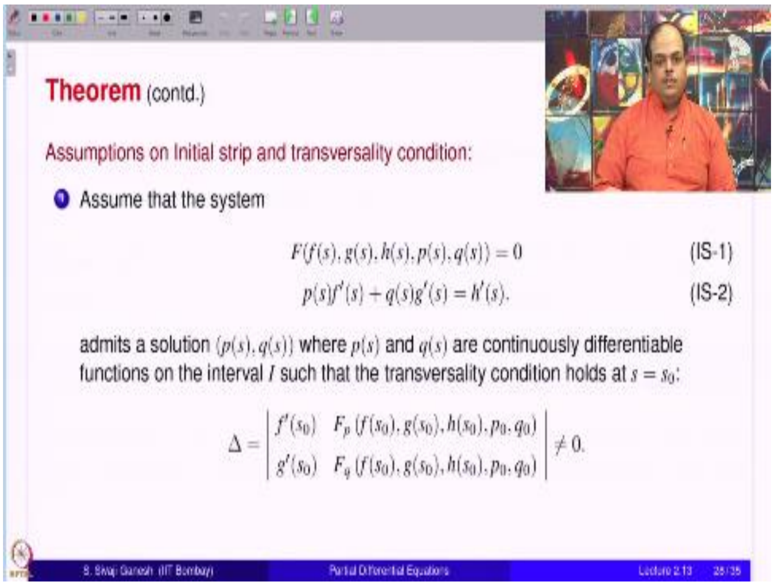

And assumption on initial strip and transversality condition: So, assume that the system f s g s h s p s q s = 0; p s f dash plus q g dash = h dash admits a solution p s, q s where p s, q s are continuously differentiable functions on the interval I. Actually we have, it is enough to work with the existence of these kind of functions for s in a small interval containing some point s

0. We got finally the conclusion of inverse function theorem or implicit function theorem or local.

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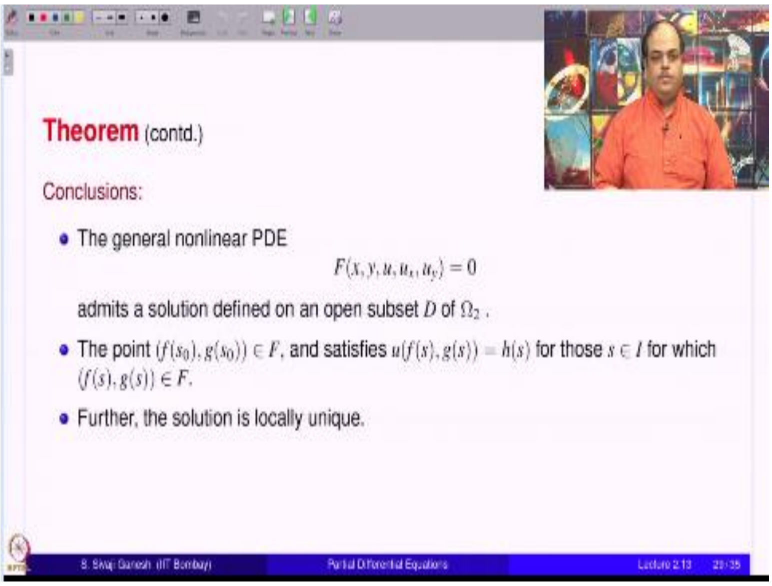

So, I assume that transversality condition. Conclusions are the general nonlinear PDE admits a solution defined an open subset of omega 2 for the Cauchy problem that is missing here Cauchy problem. Cauchy problem for general nonlinear PDE admits a solution defined an open set of omega 2 which is F actually, it is not D. It is actually F, we have found. D is equal to F in the proof we have. D is equal to F and the point f s 0 g s 0 in the F satisfies u f s g s = h s s for those s for which this is in F. Further, the solution is locally unique.

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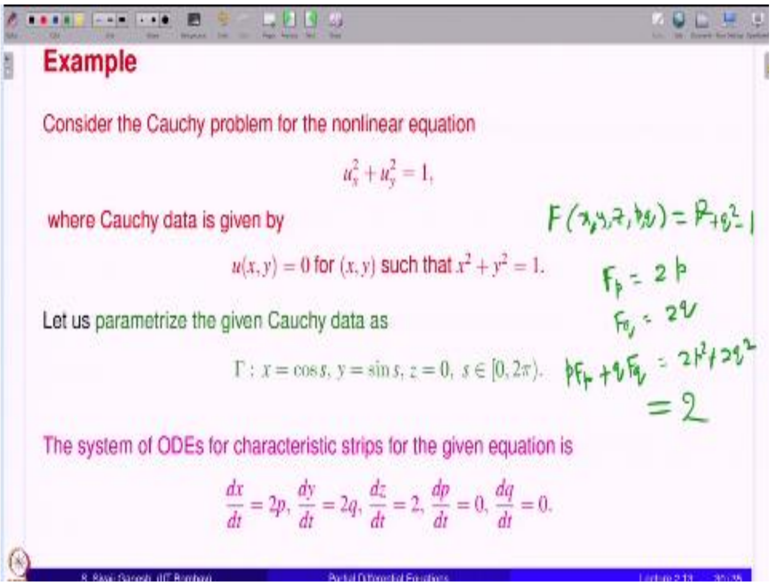

So, let us solve an example of where we are going to solve a Cauchy problem for a nonlinear equation, the simplest nonlinear equation u x square + u y square = 1. Cauchy data is given by u is 0 on the circle x square + y square = 1. So, first thing as always is to parameterize the Cauchy data, x = cos s, y = sig s, z = 0, s in the interval 0 to 2 phi system of ODEs. So, what is F of x y z p q, p square + q square – 1.

So, in this example, so, we should always write this, what is this, function. From here, we can compute very easily F x F y F z are 0; F p is 2p; F q is 2q. So, dx by dt is F p therefore, 2p; dy by dt is F q therefore, 2q; dz by dt is p F p + q F q. What is p F p + q F q? Let us write down once again. F p is 2 p; F q is 2q; p F p + q F q is equal to 2p square + 2q square, but the equation says p square + q square = 1. Therefore, this is 2. And dp by dt is 0; dq by dt is 0 because our F does not involve x y z at all.

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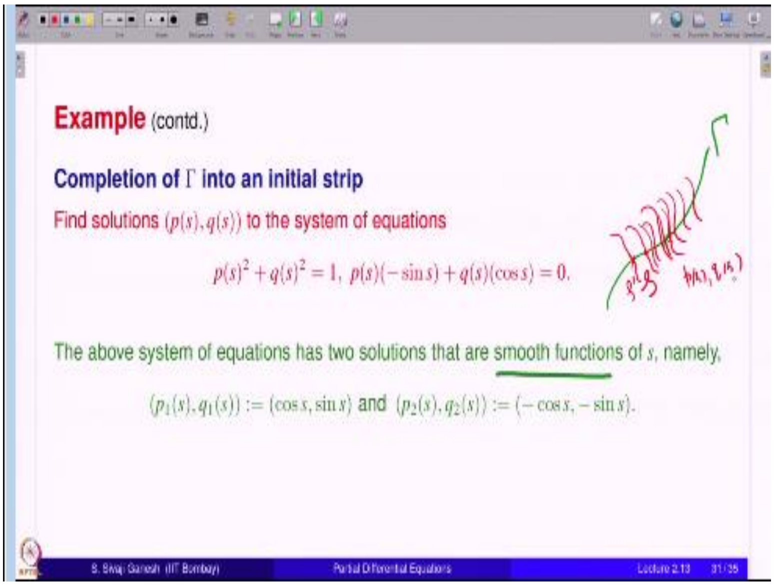

So, we have to now complete into an initial strip the datum curve. We have to find p s q s satisfying 2 conditions. One is the equation p s square + q s squared = 1. Second is p f prime + q g prime = h prime that will give us this. Now, we have to find solutions. Here, we see clearly 2 solutions of course, you may say infinitely many solutions, because you can mix both of them because it is an algebraic or whatever transcendental equation.

At some s, you can be here; at some other s, you can be here. You can keep on oscillating, but that is not good, what we need; is not smooth function? It is very important, because what we are trying to solve just recall whether Quasilinear or fully nonlinear, we take gamma; we take a point on this and we pass a characteristic curve through that and repeat for everything. Whenever you choose a point on gamma, it will correspond to some s.

And then you are simply passing a characteristic curve through that. Another s dash you take, you pass another characteristic curve. Why should these characteristic curves together teach or view a surface that would happen if things underlying are smooth functions that is why f g h, we assume C 1 functions. So, here we need to assume p s q s also smooth functions, otherwise we do not expect.

So, that is why there are 2 choices for smooth functions. So, p 1 s q 1 s is cos s sin s, other one is minus cos s sin s. So, smooth is very important.

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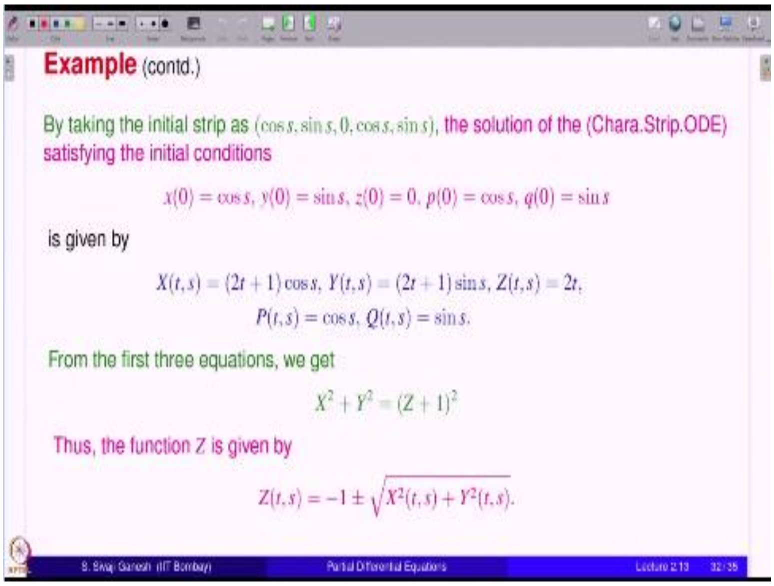

So, there are only 2 if you insist on smoothness. Now, if you take the initial strip where p q is taken to be cos s and sin s, the solution of chara strip ODE will be this. I am not going to the computation because a very simple ODEs that you can solve. So, these are the solutions X t s Y t s, Z t s P t s Q t s. Now, from the first 3 equations, we get this relation X square + Y square = Z + 1 whole square and Z t s is given by this formula on simplification again that.

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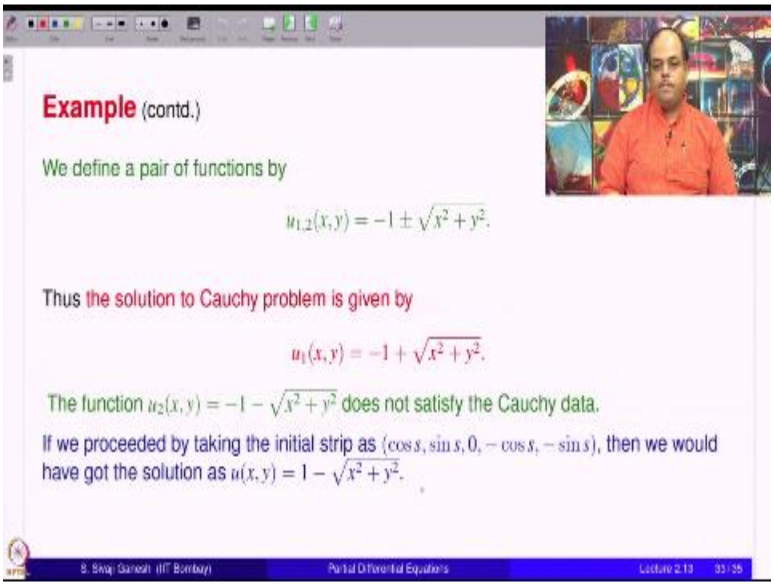

So, we define a pair of functions u 1 and u 2; one with plus sign, one with minus sign, then the solution to Cauchy problem is given by this and not the minus 1. Why? It does not satisfy the Cauchy data, you can check that. So, if we proceeded by taking the initial strip as minus cos s and minus sin s, then we would have got this as a solution and not this. This will not satisfy the Cauchy data, this will satisfy.

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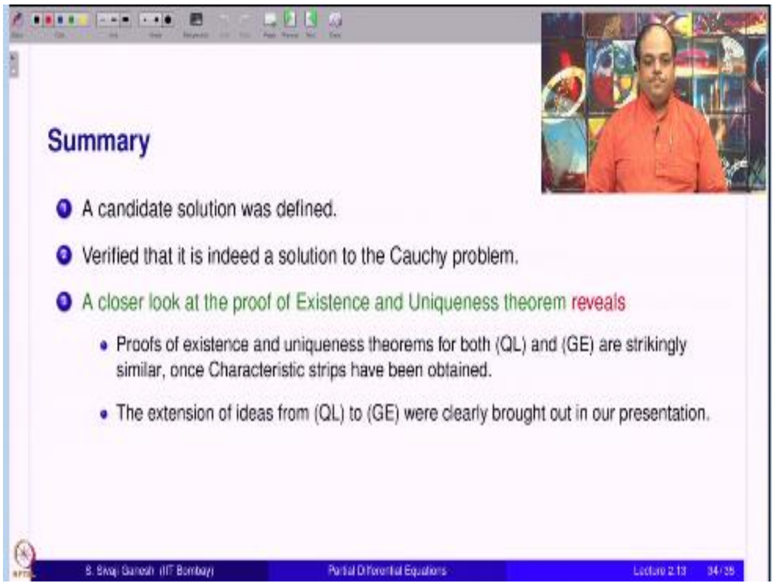

So, summary is for this lecture is that the candidate solution was defined, we verified that is indeed a solution to the Cauchy problem. A closer look at the proof of existence and uniqueness theorem reveals proofs of the existence and uniqueness theorem for both QL and GE are strikingly similar, actually the same, but for obvious modifications. Once; characteristics strips have been obtained, because we need that X t s, Y t s, Z t s. And we always work with X t s, Y t s to get inversion. Inverse function theorem is applied only for X t s Y t s whether it is QL or GE and the extension of ideas were clearly brought out in our presentation.

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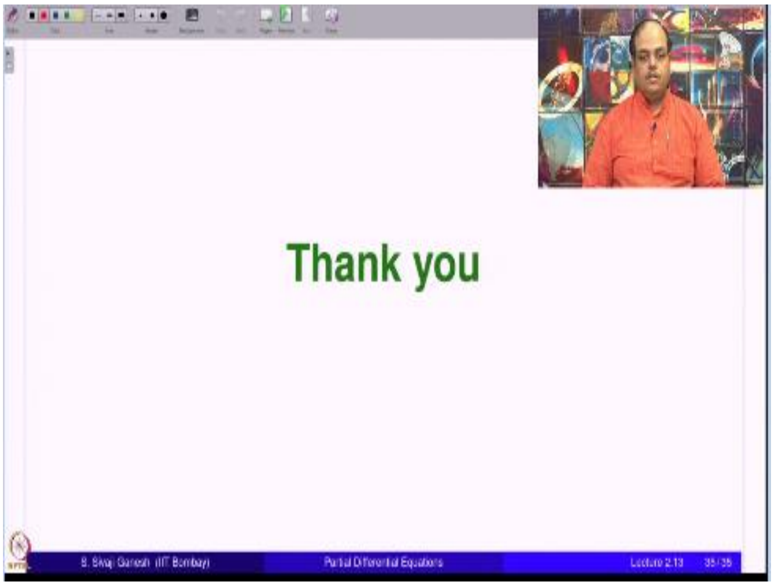

So, this completes the analysis of Cauchy problem for general nonlinear equations. In the forthcoming lectures, we take up some problems based on this. Thank you.