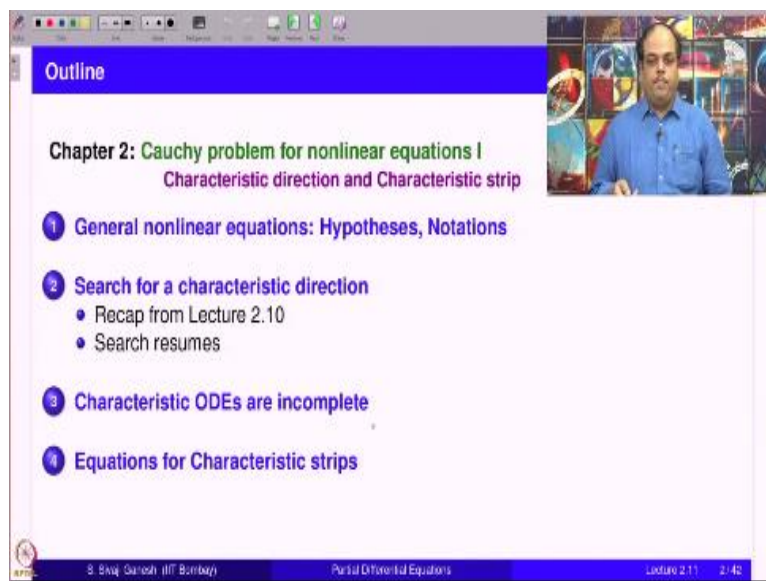


Partial Differential Equations
Prof. Sivaji Ganesh
Department of Mathematics
Indian Institute of Technology – Bombay

Lecture – 2.11
General Nonlinear Equations 2
Characteristic Direction and Characteristic Strip

Our journey of studying Cauchy problem for general nonlinear equations has started in the last lecture. That is lecture 2.10. Today we continue.

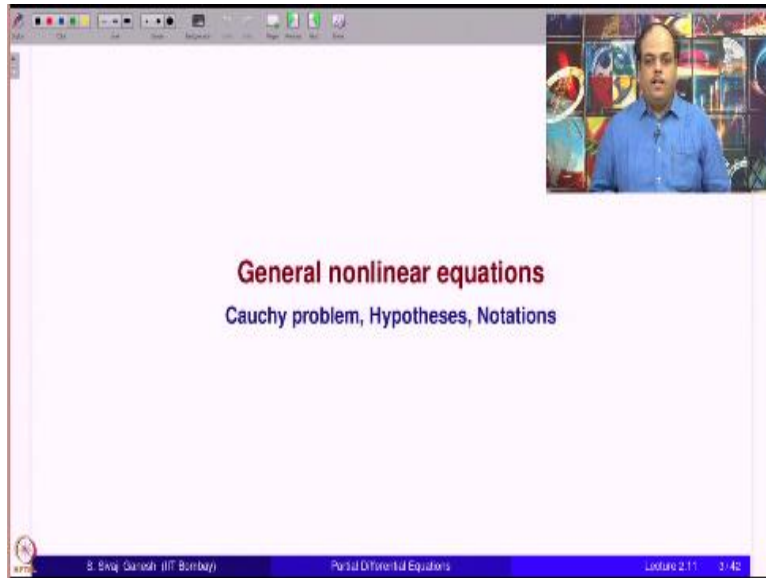
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The outline for this lecture is: First we recall once again the hypotheses and notations involving general nonlinear equations. And in the last lecture, we started our search for a characteristic direction we were not yet successful. Today the search resumes and we will find a characteristic direction for the general nonlinear equation. And then resulting characteristic ODEs are incomplete. We are going to see that.

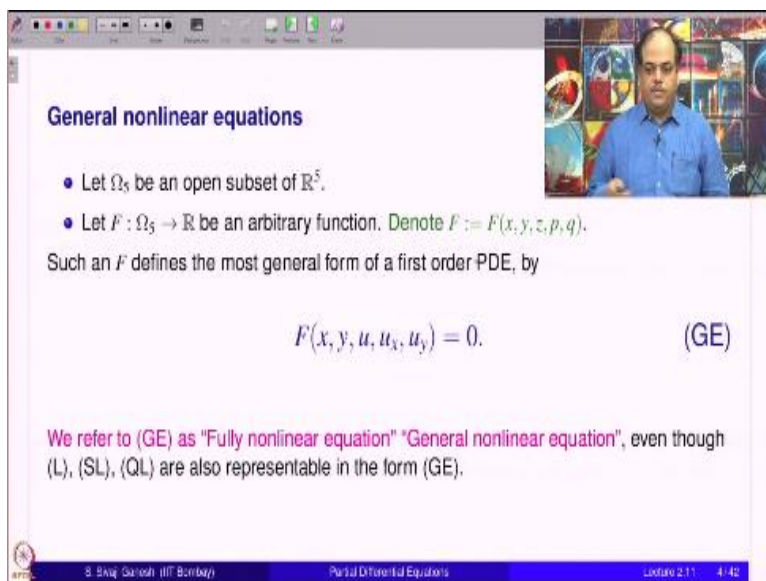
And therefore, we need to extend the system of characteristic ODEs to a new system, which is system of equations for characteristic strips.

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So, let us recall the notations and various hypotheses, which are involved in the Cauchy problem for general nonlinear equations.

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F is the function which is going to define as the general nonlinear equation. The arguments of F have been written as x, y, z, p, q defined on for a 5 tuples. So, lying in Ω_5 , which is an open subset of \mathbb{R}^5 . Such an F defines the most general form of a first order nonlinear general nonlinear PDE by F of x, y, u, u_x , u_y = 0. Though we know that this class GE contains in it linear equations, semi linear equations and Quasilinear equations, but we refer to this kind of form of general equation as fully nonlinear equation or general nonlinear equation.

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Hypotheses on the function F

The function $F : \Omega_5 \rightarrow \mathbb{R}$ is assumed to satisfy

- $F \in C^1(\Omega_5)$. Equivalently,

$$F, F_x, F_y, F_z, F_p, F_q \in C(\Omega_5).$$
- F_p and F_q satisfy

$$F_p^2(x, y, z, p, q) + F_q^2(x, y, z, p, q) \neq 0 \quad \forall (x, y, z, p, q) \in \Omega_5.$$

Notation: Ω_2 and Ω_3 denote the projections of Ω_5 to xy -plane and xyz -space respectively.

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So, hypotheses on the function F which defines the PDE are that F must be a C^1 function on its domain of definition, which is equal to saying that the function along with all first order partial derivatives of all the arguments which are x, y, z, p, q are continuous functions on Ω_5 . And we need to assume that F_p and F_q do not vanish simultaneously at any point in Ω_5 . Ω_2 and Ω_3 denotes the projections of Ω_5 to xy plane and xyz space respectively.

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Hypotheses on the Cauchy data

I is an interval in \mathbb{R} , and $f, g, h \in C^1(I)$ such that

$$(f'(s))^2 + (g'(s))^2 \neq 0 \text{ for all } s \in I.$$

Consider the space curve Γ described parametrically by

$$\Gamma : x = f(s), y = g(s), z = h(s), \quad s \in I.$$

Recall For a given space curve Γ , Cauchy problem for (GE)

$$F(x, y, u, u_x, u_y) = 0 \quad (\text{GE})$$

consists of finding a solution u to (GE) satisfying the Cauchy condition

$$u(f(s), g(s)) = h(s), \quad s \in I.$$

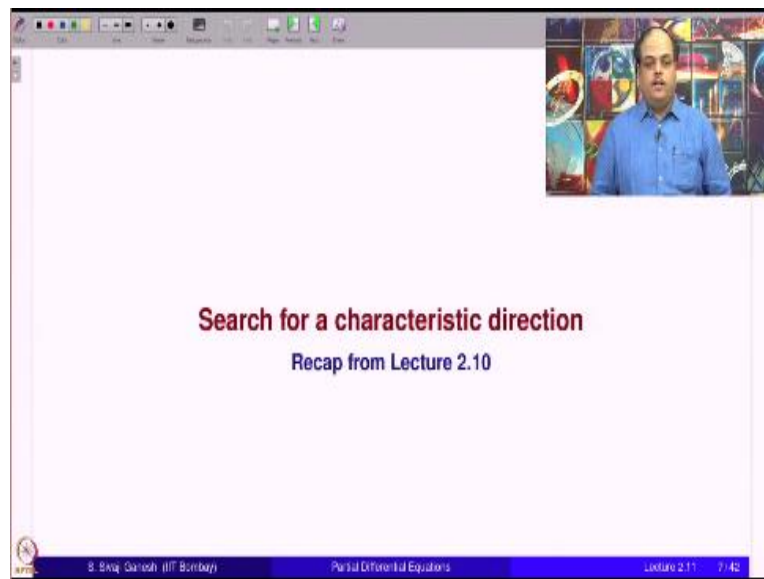
I' is a subinterval of I .

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The Cauchy data is prescribed by $\Gamma : x = f(s), y = g(s), z = h(s)$ for s belong to an interval I . And these functions are C^1 functions on the interval I and we assume the projection of Γ to xy plane which is denoted by Γ_2 is that Γ_2 is a regular curve, which means f' and g' do not vanish simultaneously at any point on the curve Γ_2 .

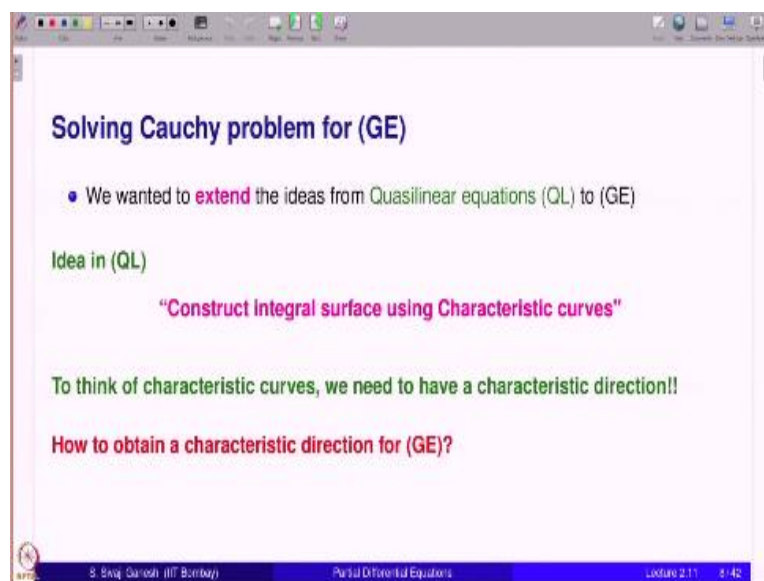
Recall that Cauchy problem means, we need to find a solution of the equation which satisfies this u of f s , g $s = h$ s . In other words f s , g $s = h$ s belong to the surface $z = u$ x y .

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Now, let us recap from the last lecture the search for a characteristic direction.

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So, we wanted to extend the ideas from Quasilinear equations to general equations. The idea in Quasilinear equations was construct integral surface using characteristic curves. To think of characteristic curves, we need to have a characteristic direction. So, how do you obtain a characteristic direction for GE? That is the question.

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Obtain Chara. direction for (GE)

Recall: For quasilinear equation (QL)

$$a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u), \quad (\text{QL})$$

we observed that at a point $P(x, y, z)$ on an integral surface $S: z = u(x, y)$,

- the normal direction at P is $(u_x(x, y), u_y(x, y), -1)$.
- (QL) says that $(a(x, y, z), b(x, y, z), c(x, y, z))$ is a direction in the tangent plane at P .
- This observation led to the definition of a characteristic system of ODE.

Remark There is no automatic choice of a characteristic direction suggested by (GE).

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Recall for Quasilinear equations, the equation gave us a characteristic direction. The equation QL is $a u_x + b u_y = c$. What we observed is if you take any point on an integral surface given by $z = u(x, y)$, the normal direction at P is $(u_x, u_y, -1)$. And the equation tells that (a, b, c) is a direction in the tangent plane at P . This observation led us to define the notions of characteristic system of ODE and characteristic curves.

Once again there is no automatic choice of such a characteristic direction for general equations.

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Family of possible tangent planes through a point for (QL)

(T2) for (QL)

- reads as

$$a(P_0)p + b(P_0)q = c(P_0).$$

- WLOG, assume $b(P_0) \neq 0$.
- Thus for each $p \in \mathbb{R}$, $q(p)$ is given by

$$q(p) = -\frac{a(P_0)}{b(P_0)}p + \frac{c(P_0)}{b(P_0)}.$$

(T1) for (QL) takes the form

$$z - z_0 = \frac{c(P_0)}{b(P_0)}(y - y_0) + p \left(x - x_0 - \frac{a(P_0)}{b(P_0)}(y - y_0) \right), \quad (T_p)$$

indexed by a parameter $p \in \mathbb{R}$.

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So, then we went on to consider this family of possible tangent planes to possible integral surfaces through a point for general nonlinear equation. Take a point P_0 in Ω . We get a 1 - parameter family of possible tangent planes given by T_1 and T_2 . What is T_1 ? T_1 is

simply equation of a plane passing through x_0, y_0, z_0 . But by restricting F P to satisfy T 2 will mean that this is going to be a tangent plane to possible integral surface.

It should satisfy these conditions T 1, T2. Note this you just see only the information coming from the equation. We are not pretending that we know the integral surface. That is not needed. Now, we observed that the T 2 for Quasilinear equation is this. And as one of them is nonzero we know that a or b or has to be nonzero by assumption. In the Quasilinear equations, we assume b is nonzero then we can solve q as a function of P.

In fact, it is a linear function of P that is what makes things much simpler and T 1 for QL becomes this after substituting for this. So, this is a family of possible tangent planes. We have explicitly got one equation.

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For (QL),

$$z - z_0 = \frac{c(P_0)}{b(P_0)}(y - y_0) + p \left(x - x_0 - \frac{a(P_0)}{b(P_0)}(y - y_0) \right), \quad (T_p)$$

represents a **1-parameter family of possible tangent planes at P_0** indexed by a parameter $p \in \mathbb{R}$.

For (GE),

$$z - z_0 = p(x - x_0) + q(y - y_0) \quad (T1)$$

$$F(x_0, y_0, z_0, p, q) = 0 \quad (T2)$$

represents a **1-parameter family of possible tangent planes at P_0** .

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So, this represents, the equation T p represents a 1 - parameter family of possible tangent planes indexed by a parameter. For general equations, we have to have both the equations. If you are able to eliminate p or q from here that is expressed q as a function of p, go back and substitute here then once again you have only one equation like this. In the case of Quasilinear it was very easy to express q in terms of P.

But in case of general nonlinear equations, it is not clear. So, this is still a 1- parameter family of possible tangent planes at the point P 0.

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Envelope of 1-parameter family of tangent planes for (QL)

turned out to be

$$\left\{ (x, y, z) : \frac{x - x_0}{a(P_0)} = \frac{y - y_0}{b(P_0)} = \frac{z - z_0}{c(P_0)} \right\}$$

which is nothing but the **characteristic direction at P_0** .

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Now, envelope of 1 - parameter family of tangent planes for QL we computed. It turned out to be this. What is this? This is nothing but the characteristic direction at P_0 . This is a line passing through the point x_0, y_0, z_0 and the direction abc is a characteristic direction.

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The plan for getting a chara. direction for (GE) is

- to obtain the **Envelope of 1-parameter family of possible tangent planes at P_0 for (GE)**

$$z - z_0 = p(x - x_0) + q(y - y_0) \quad (T1)$$

$$F(x_0, y_0, z_0, p, q) = 0 \quad (T2)$$

- and **choose a direction in the tangent plane for the envelope.**

This plan will be successful thanks to the Lemma on the next slide.

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So, therefore, the plan for getting a characteristic direction for GE is like this. Obtain an envelope of 1 - parameter family of possible tangent planes for GE which is this. So, get the envelope of this and choose a direction in the tangent plane for the envelope. This plan will be successful because we have a lemma that we proved in the last lecture, which is here.

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Lemma

Assumptions

- Let $S_\lambda : z = G(x, y, \lambda)$ be a 1-parameter family of surfaces.
- Let $\lambda = g(x, y)$ represent solutions of equation $G_\lambda(x, y, \lambda) = 0$.
- Let E denote the envelope of the family S_λ described by

$$z = G(x, y, g(x, y))$$

- Let C_λ be defined by equation

$$C_\lambda : \{(x, y, z) : z = G(x, y, \lambda), G_\lambda(x, y, \lambda) = 0\}.$$

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If you take a 1 - parameter family of surfaces given by $z = g(x, y, \lambda)$ and you find its envelope which is $z = G(x, y, g(x, y))$ and C_λ denotes the intersection of the envelope and the family.

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Lemma (contd.)

Conclusions

- 1 Assume that $C_\lambda \neq \emptyset$ for each λ . Then the envelope E intersects every member of the family S_λ along C_λ .
- 2 The envelope E and S_λ touch each other i.e., at each point of $E \cap S_\lambda$, which is nothing but C_λ , the envelope E and S_λ have a common tangent plane.

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Then look at the second conclusion. The envelope and S_λ touch each other. That means, wherever they intersect, which is along C_λ , the intersection is precisely C_λ . At every point on C_λ , they share the tangent plane. Therefore, if you can find a direction in the tangent plane for the envelope, we are done. That will also be a tangential direction for S_λ . That will also be S_λ .

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Search for a characteristic direction
Resumption

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Recall: For (GE),

$$z - z_0 = p(x - x_0) + q(y - y_0) \quad (T1)$$

$$F(x_0, y_0, z_0, p, q) = 0 \quad (T2)$$

represent a **1-parameter family of possible tangent planes at P_0** .

Let us find the envelope of the family of planes.

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So, let us resume our search of characteristic direction. So, this is T 1 and T 2. That represents 1 - parameter family of possible tangent planes at the point P 0. Let us find the envelope that is what we need to do.

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Computing envelope for (GE)

Assume that there exists a differentiable function $q = q(p)$ such that

$$F(x_0, y_0, z_0, p, q(p)) = 0. \quad (1)$$

Remark:

- It is always possible to find such a function locally.
- **Implicit function theorem** tells: Come with a particular solution, and **non-zero (invertible) derivative**; I guarantee existence of such a function.
- Let p_0, q_0 be such that $F(x_0, y_0, z_0, p_0, q_0) = 0$ and $F_q(x_0, y_0, z_0, p_0, q_0) \neq 0$.
- By implicit function theorem we can express q as a function of the variable p in an interval containing of p_0 .

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To do that, we need to assume that q can be expressed as a function of p in a differentiable way so that this equation is satisfied. In other words, we are solving F of $x_0, y_0, z_0, p, q = 0$ and q is expressed in terms of p . Question is, is it possible? It is always possible to find a such a function locally implicit function theorem tells you come with a particular solution and nonzero or invertible derivative, I guarantee the existence of such a function.

This is typically what implicit function theorem says. Now, we will go to the implicit function theorem with a particular solution with that means, you first find out a p_0, q_0 real number such that f of $x_0, y_0, z_0, p_0, q_0 = 0$. That means, we have the particular solution. Now, we need to see what is this nonzero derivative and F_q at this point x_0, y_0, z_0, p_0, q_0 is nonzero. Please note when we are applying implicit function theorem x_0, y_0, z_0 is fixed.

So it is only as a function of p and q that we are trying to solve this equation 1. So F_q is nonzero, if you assume then implicit function theorem guarantees existence of such a function, but locally. That means, in an interval containing p_0 you can express q as a function of p . That is the implicit function theorem.

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Computing envelope for (GE) (contd.)

Now the one-parameter family of planes given by (T1)-(T2) becomes

$$z - z_0 = p(x - x_0) + q(p)(y - y_0) \quad (T_p)$$

Differentiating the equations $F(x_0, y_0, z_0, p, q(p)) = 0$ and (T_p) w.r.t. p yields

$$F_p(x_0, y_0, z_0, p, q(p)) + F_q(x_0, y_0, z_0, p, q(p))q'(p) = 0, \quad (2a)$$

$$0 = (x - x_0) + q'(p)(y - y_0). \quad (2b)$$

Eliminating $q'(p)$ from (2), we get

$$\frac{x - x_0}{F_p(P_0, p, q(p))} = \frac{y - y_0}{F_q(P_0, p, q(p))}, \quad (3)$$

where $(P_0, p, q(p))$ is used to denote $(x_0, y_0, z_0, p, q(p))$.

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Therefore, the 1 - parameter family of planes becomes simply one equation now because I have solved q in terms of P and I put it here. Now only thing to do is differentiate this equation with respect to p and we get this equation. These 2 equations. We will differentiate this equation you get 2a, differentiate T p equation you get 2b. From 2a and 2b you can eliminate q prime p and we get this.

$x - x_0$ $y - y_0$ is proportional to F_p of F_q . Something we got. Now, what is this p_0 , p , q , p ? p not actually x_0 , y_0 , z_0 . F , F is a function of 5 variables. We need a 5 tuple here. For want of space here I have just made it a small notation p_0 . It stands for this.

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Computing envelope for (GE) (contd.)

Using

$$\frac{x - x_0}{F_p(P_0, p, q(p))} = \frac{y - y_0}{F_q(P_0, p, q(p))}, \quad (4)$$

and the equation of planes (T_p)

$$z - z_0 = p(x - x_0) + q(p)(y - y_0) \quad (T_p)$$

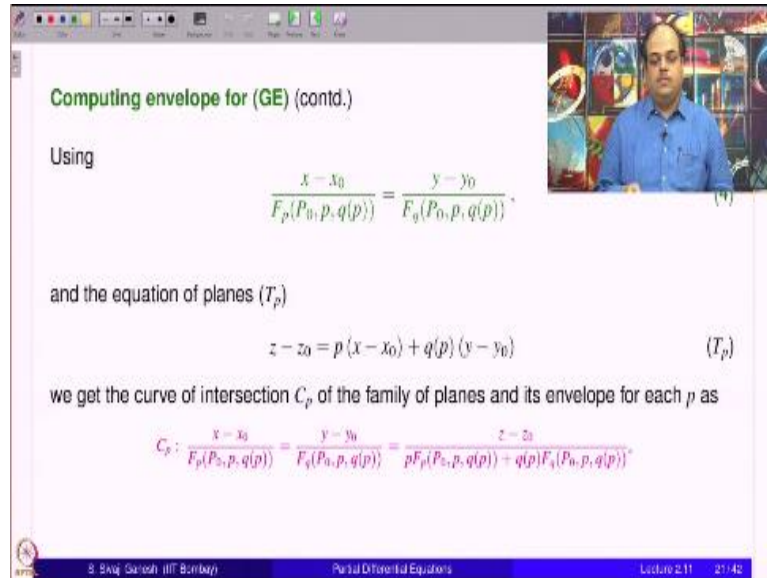
we get the curve of intersection C_p of the family of planes and its envelope for each p as

$$C_p: \frac{x - x_0}{F_p(P_0, p, q(p))} = \frac{y - y_0}{F_q(P_0, p, q(p))} = \frac{z - z_0}{pF_p(P_0, p, q(p)) + q(p)F_q(P_0, p, q(p))}$$

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Using this and the equation for tangent planes, which is TP here we get the curve of intersection and that is this. $x - x_0$ is proportional to F_p , $y - y_0$ is proportional to F_q . Therefore, $z - z_0$ is proportional to $p F_p$ plus $q F_q$. That is what we have written here.

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Computing envelope for (GE) (contd.)

Using

$$\frac{x - x_0}{F_p(P_0, p, q(p))} = \frac{y - y_0}{F_q(P_0, p, q(p))} \quad (1)$$

and the equation of planes (T_p)

$$z - z_0 = p(x - x_0) + q(p)(y - y_0) \quad (T_p)$$

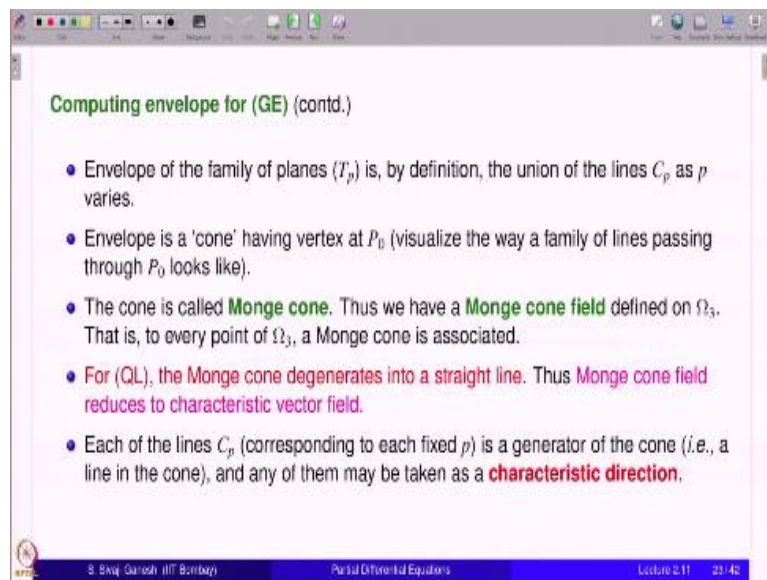
we get the curve of intersection C_p of the family of planes and its envelope for each p as

$$C_p: \frac{x - x_0}{F_p(P_0, p, q(p))} = \frac{y - y_0}{F_q(P_0, p, q(p))} = \frac{z - z_0}{pF_p(P_0, p, q(p)) + q(p)F_q(P_0, p, q(p))}$$

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So, this is a line right. C_p is a family of lines of course indexed by p . The same thing was actually one single line. It never depended on p . For a Quasilinear equation all of them pass through the point x_0, y_0, z_0 and having a direction $F_p, F_q, p F_p + q F_q$. The 3 tuple which is given in 5 is nonzero because we have assumed that $F_p^2 + F_q^2$ is not equal to 0 throughout the domain of F . That is the hypotheses.

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Computing envelope for (GE) (contd.)

- Envelope of the family of planes (T_p) is, by definition, the union of the lines C_p as p varies.
- Envelope is a 'cone' having vertex at P_0 (visualize the way a family of lines passing through P_0 looks like).
- The cone is called **Monge cone**. Thus we have a **Monge cone field** defined on Ω_3 . That is, to every point of Ω_3 , a Monge cone is associated.
- For (QL), the Monge cone degenerates into a straight line. Thus **Monge cone field reduces to characteristic vector field**.
- Each of the lines C_p (corresponding to each fixed p) is a generator of the cone (i.e., a line in the cone), and any of them may be taken as a **characteristic direction**.

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An envelope of the family of planes T_p by definition is a union of C_p . That means union of these lines. So, envelope is a cone having vertex at point p_0 . So, visualize the way a family

of lines passing through a point p_0 looks like. For example, you have this point p_0 . So, imagine lines in 3d. Like that. So, that is how. Looks like a cone. This is called Monge cone. Thus we have a Monge cone field defined in Ω_3 .

So, Ω_3 is in our \mathbb{R}^3 subset. At any point you have a cone sitting there. At another point, you have another cone like that. So every point you can attach a cone. So, this is called cone field. This is similar in spirit to a vector field. To any point if you attach one vector, associate one vector it is called vector field. If you recall it is called cone field. That is to every point of Ω_3 a Monge cone is associated. For Quasilinear equations, the Monge cone degenerates into a straight line.

Because we observe that the envelope is actually a straight line. It is independent of p . So, therefore, just one single straight line. So the Monge cone field reduces to a characteristic vector field in the case of Quasilinear equations. Now, each of the line C_p corresponding to each fixed p is a generator of the cone. Generator of the cone means it is a line in the cone. So, you have a for example, you have a cone like that. You have a line here.

So, at a point on the cone if you take tangent plane, that line is going to be there on the tangent plane. Therefore, what we wanted was to look at the envelope and take one direction in the tangent plane of the envelope. Now, here is very nice. Tangent plane of this cone at these points, one of the direction is clearly the line now C_p . So we can take that C_p to be characteristic direction.

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A Chara. Direction for (GE) is found

For every $P(x, y, z) \in \Omega_3$ and
for every (p, q) such that

$$F(P, p, q) = 0,$$

we have a choice for

a characteristic direction at P

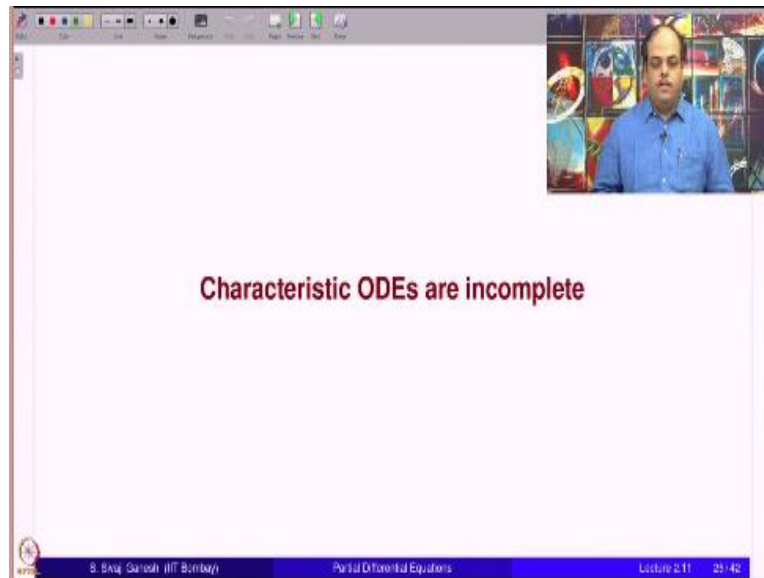
given by

$$(F_p(P, p, q), F_q(P, p, q), pF_p(P, p, q) + qF_q(P, p, q)).$$

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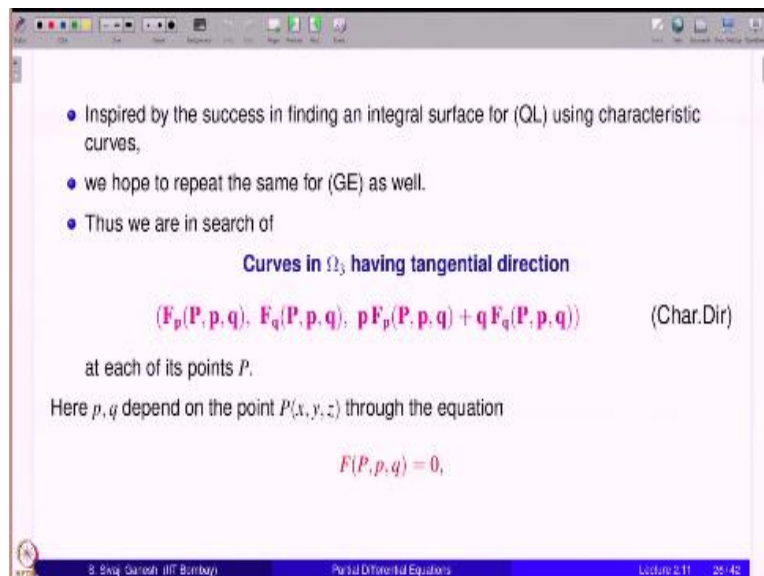
So, a characteristic direction for GE is found. For every p in Ω_3 and every p, q we do not we no longer require q of p here. After all q is a function of p , q is some value. It satisfies this equation. Still, we have found a characteristic direction at P . What is that? $F_p, F_q, p F_p + q F_q$

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Now, we have to look at the characteristic ODEs.

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We are going to see they are incomplete. We will see why. So, inspired by the success in finding an integral surface using characteristic curves, we hope to repeat the same for GE also. In QL we are successful. So, we hope same thing happens for GE as well. So, we are in search of curves in Ω_3 which have tangential direction equal to the characteristic direction that we just found which is this: $F_p, F_q, p F_p + q F_q$ at each of its points P .

Such curves we are looking for. Now, p, q of course, depends on the point x, y, z. They satisfy this equation. So it depends on x, y, z.

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Curve having Characteristic direction

- Let J be an interval in \mathbb{R} , and $t \in J$.

$$\gamma_{P_0} : (x(t), y(t), z(t)), t \in J$$

be a curve in parametric form such that

- It passes through $P_0(x_0, y_0, z_0)$ at $t = 0$.
- γ_{P_0} has the tangential direction (Char.Dir) at each of its points $P(x(t), y(t), z(t))$.

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Let us write a curve having a characteristic direction. So, let J be an interval, 0 belongs to J . γ_{P_0} be a curve given by $x(t), y(t), z(t)$, t varies in J such that it passes through the point P_0 at $t = 0$. And it has a tangential direction which is characteristic duration at each of its points.

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Curve having Characteristic direction

The following system of characteristic ODEs hold along γ_{P_0} :

$$\frac{dx}{dt} = F_p(x, y, z, p, q) \quad (6a)$$

$$\frac{dy}{dt} = F_q(x, y, z, p, q) \quad (6b)$$

$$\frac{dz}{dt} = pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q). \quad (6c)$$

along with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$.

Denote the above system by **(chara.ODE)**.

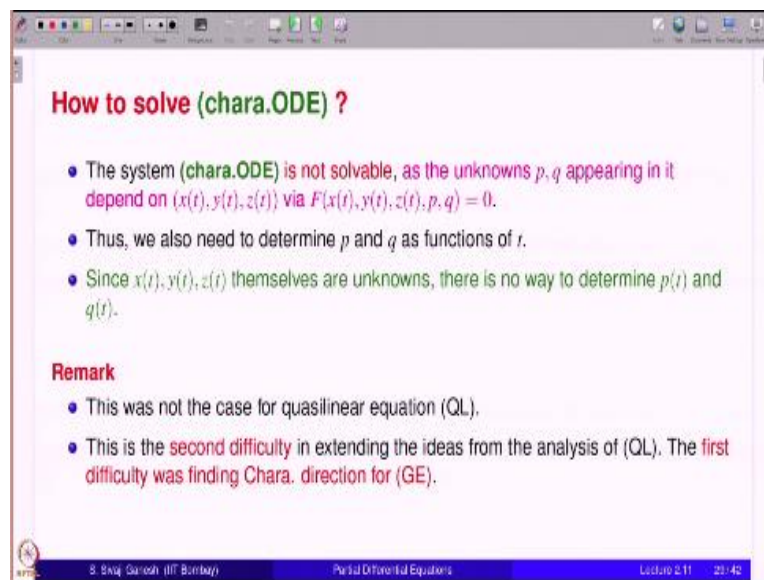
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This is what we would like to call a characteristic curve. Therefore, the following system of characteristic ODEs hold along γ_{P_0} . So, dx by dt , dy by dt , dz by dt are in F_p , F_q and $pF_p + qF_q$. And at $t = 0$, we want to be at the point x_0, y_0, z_0 . So, fine. Now, why is it

incomplete? I do not know what is p and q ? That is a problem. Let us denote the above system by chara.ODE exactly like we used for Quasilinear equations.

The solutions of this It is images will be characteristic curves. Only thing is that it does not determine characteristic curves. There is a problem, because there are p q s here.

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It is not solvable as unknowns p q appearing in it depend on x t , y t , z t via this equation F of x t , y t , z t , p , $q = 0$. p and q are also depending on t . Thus we also need to determine p and q as functions of t . Since x t , y t , z t themselves are unknowns, there is no hope of solving for p and q from this equation. And no this is not the case for Quasilinear equations, because characteristic ODE never involved a p and q .

And this is a second difficulty in extending the ideas from the Quasilinear case. What is the first one? Finding a characteristic direction. That was the first difficulty we have overcome. And second difficulty we will overcome by supplementing 2 more equations for one for p and one for q .

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How to find equations for $p(t)$ and $q(t)$

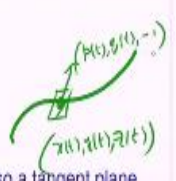
- We need to supplement (**chara.ODE**) with equations for p, q along the curve $(x(t), y(t), z(t))$ such that
 - $(p(t), q(t), -1)$ is normal direction to a possible integral surface passing through γ_{t_0} .

Thus we have to find **FIVE quantities**

$$x(t), y(t), z(t), p(t), q(t)$$

and not just the three quantities $x(t), y(t), z(t)$.

Geometrically speaking, we have to determine **not only a curve** but also a tangent plane to a possible integral surface that contains this curve.



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So, how to find the equations for p and q ? We need to supplement chara.ODE system with equations for p, q along the curve $x(t), y(t), z(t)$ and what should be your property? $p, q, -1$ should be the normal direction to a possible integral surface. They are not arbitrary functions. So, we have to determine 5 quantities, $x(t), y(t), z(t)$ and $p(t), q(t)$ and not just the 3 quantities because we are unable to determine the 3.

We would have been very happy if you had a got $x(t), y(t), z(t)$ we could have proceeded. But the equation for activities that involve p and q . Therefore, we need to find p and q also. So, geometrically speaking what we are saying is we have to determine not only a curve, but also a tangent plane to a possible integral surface that contains this curve. So, in other words we are trying to find a curve like that.

So, suppose this is the point $x(t), y(t), z(t)$. So, we want to find something like that $p(t), q(t)$, of course -1 , I write but then it is okay, -1 there is nothing to find. So, we need to find this.

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Definition of Characteristic strip

- Let J be an interval in \mathbb{R} , and

$$\gamma : x = x(t), y = y(t), z = z(t), t \in J$$
- be a curve having the tangential direction (Char.Dir) at each of its points,
- where $p(t), q(t)$ satisfy

$$F(x(t), y(t), z(t), p(t), q(t)) = 0.$$
- The 5-tuple $(x(t), y(t), z(t), p(t), q(t))$ is called a **characteristic strip**, and the point $(x(t), y(t), z(t))$ is called **support of the characteristic strip**.

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So, let J be an interval in \mathbb{R} and γ be a curve given by $x = x(t), y = y(t), z = z(t)$ be a curve having tangential direction as Char.Dir that is characteristic direction at each of its points, where $p(t), q(t)$ satisfy this equation F of $x(t), y(t), z(t), p(t), q(t)$ at $t = 0$. The 5-tuple $x(t), y(t), z(t), p(t), q(t)$ is called a characteristic strip. Maybe you may put like that, because we are saying tuple, but it is okay. Even without that it is fine.

All these 5 together is called a 5-tuple. That is called a characteristic strip and the point $x(t), y(t), z(t)$ is called support of the characteristic strip. Because that is a point at which you are putting a plane with the normal $p(t), q(t), -1$.

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Characteristic strip: An illustration

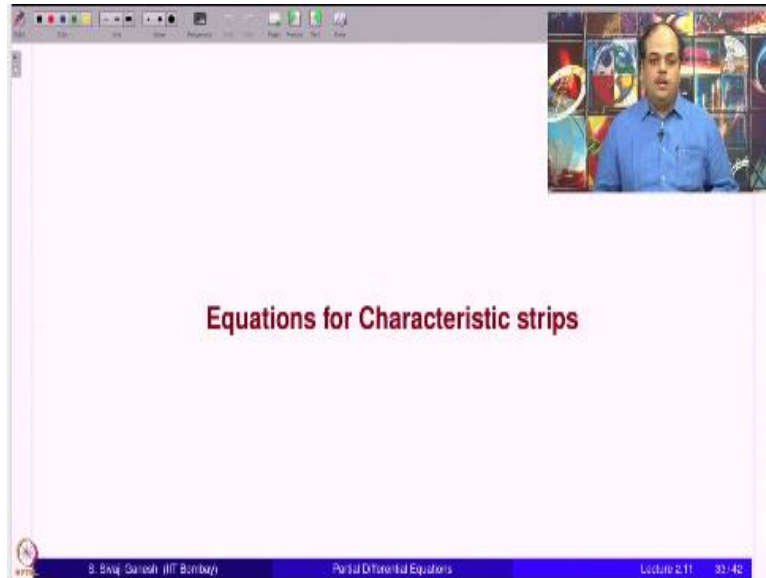
Strip can be thought of as a point $(x(t), y(t), z(t))$ along with an infinitesimal plane element passing through the point $(x(t), y(t), z(t))$ and having the normal direction $(p(t), q(t), -1)$.

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So, let us illustrate that pictorially. So, here there are 2 points I have considered indexed by $t = t_0$ here and $t = t_1$. So, $x(t_0), y(t_0), z(t_0)$ is a point. These are tangent. This is a plane with

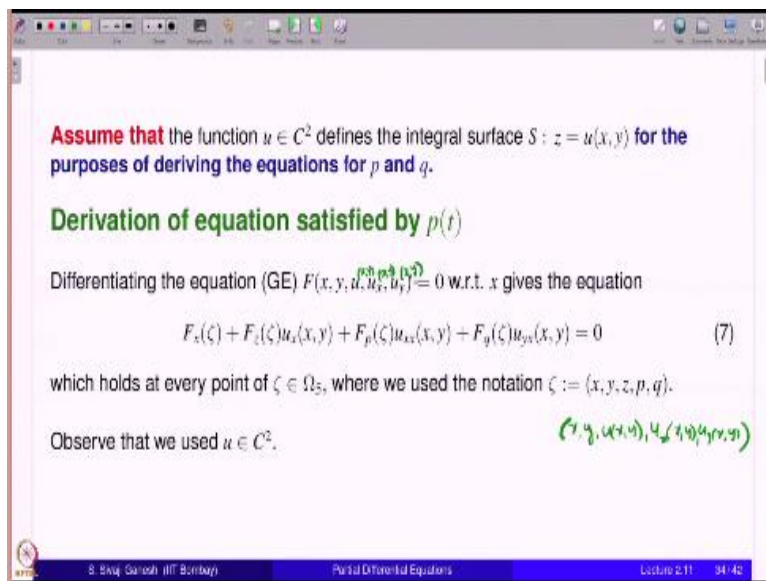
the normal $p, q - 1$. This is another point where the normal is $p, q - 1$. So, it can be, a strip can be thought of as a point along with an infinitely small plane element passing through that point x, y, z with the normal direction $p, q - 1$.

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Now, how do you get equations for p, q .

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So, here we assume some more nice properties about the solution u . See u is supposed to be a solution to first order PDE. We have not yet found. But we are assuming that it is C^2 . Is it okay? That is a question. It is okay because we are going to use this only to derive certain equations and for deriving the equation we assume what you want does not matter, but after getting the equations then you should show that solution exists.

There you should not suppose that C^2 etc. We will comment on this in the next lecture also. So, this is only to derive the equations for p, q that we are assuming u in C^2 . So, how do we derive the equation for p ? What are that at our hands? This equation F of $x, y, u, u_x, u_y = 0$. Differentiate that with respect to x . So, first is F_x . Here I am using the ζ to stand for x, y, z, p, q . So, F_x and then with respect to z you have u_x, y .

Therefore, you need to differentiate F_z and u with respect to x , then with here also p, F_p and then whichever is here with respect to x which is u_{xx} and here it is F_q and u_{xy} or $u_{yx} = 0$ at every point in Ω , where we use this notation $\zeta = x, y, z, p, q$. Yep, now actually this is ζ is not x, y, z, p, q . It should be x, y , we are to substitute $x, y, z, p, q = x, y, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$. That is what it is because we are going to differentiate here.

This u is a function of x, y , u_x is a function of x, y , u_y is also a function of x, y . So, that is why we get this by chain rule.

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Derivation of equation for $p(t)$

Let $(x(t), y(t), z(t))$ be a point lying on the integral surface $z = u(x, y)$.
 We require the following equation to hold:

$$p(t) = u_x(x(t), y(t)) \quad (8)$$

On differentiating the above equation w.r.t. t , we get

$$p'(t) = u_{xx}(x(t), y(t))x'(t) + u_{xy}(x(t), y(t))y'(t). \quad (9)$$

Since (chara.ODE) hold for $(x(t), y(t), z(t))$, denoting $\zeta(t) := (x(t), y(t), z(t), p(t), q(t))$, we get

$$p'(t) = u_{xz}(x(t), y(t))F_p(\zeta(t)) + u_{zz}(x(t), y(t))F_q(\zeta(t)). \quad (10)$$

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And x, y, z, t is a point lying on an integral surface $z = u(x, y)$. We require the following thing to hold namely p of t I want it to be like u_x . So p of $t = u_x$ of x, y, t . We are demanding this.. Because $p, q, t = 1$ should be such that F of x, y, z, t at p, q, t should be equal to 0. So p is supposed to play the role of u_x along the curve x, y, t . So, that is my p, t . We want this.

So on differentiating the above equation with respect to t and using chain rule we get p' prime t is equal to, t appears in both variables. So, differentiate this with respect to x , u_{xx} at the point x, y, t into derivative of x with respect to t that is x' prime t . And differentiate this with

respect to y that is u_y at the point (x, y, z) into derivative of y which is y' . Since characteristic ODE hold for x, y, z, p, q denoting ζ equal to this 5 tuple.

x, y, z, p, q we get p' is $= u_{xx} x' + u_{xy} y'$. So, we have got this equation for p' .

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Derivation of equation for $p'(t)$

Let $(x(t), y(t), z(t))$ be a point lying on the integral surface $z = u(x, y)$.
 We require the following equation to hold:

$$p(t) = u_x(x(t), y(t)) \quad (8)$$

On differentiating the above equation w.r.t. t , we get

$$p'(t) = u_{xx}(x(t), y(t))x'(t) + u_{xy}(x(t), y(t))y'(t). \quad (9)$$

Since (chara.ODE) hold for $(x(t), y(t), z(t))$, denoting $\zeta(t) := (x(t), y(t), z(t), p(t), q(t))$, we get

$$p'(t) = u_{xx}(x(t), y(t))F_p(\zeta(t)) + u_{xy}(x(t), y(t))F_q(\zeta(t)). \quad (10)$$

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And from the equation 7 that is the one we got after differentiating F of x, y . This one, this equation. And here we almost got an equation for p' . The only problem is there is u_{xx} and u_{xy} . We do not want that. Equation of p' should involve only F , it can involve x, y, z, p, q that no problem but not u_{xx} and u_{xy} .

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Derivation of equation for $p'(t)$

From the equation (7)

$$F_x(\zeta) + F_z(\zeta)u_x(x, y) + F_p(\zeta)u_{xx}(x, y) + F_q(\zeta)u_{xy}(x, y) = 0,$$

we have

$$u_{xx}(x(t), y(t))F_p(\zeta(t)) + u_{xy}(x(t), y(t))F_q(\zeta(t)) = -(F_x(\zeta(t)) + p(t)F_z(\zeta(t))).$$

From

$$p'(t) = u_{xx}(x(t), y(t))F_p(\zeta(t)) + u_{xy}(x(t), y(t))F_q(\zeta(t))$$

we get

$$p'(t) = -(F_x(\zeta(t)) + p(t)F_z(\zeta(t))).$$

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So, that needs to be removed now. So, we solve for that, which we do not want in terms of what we know F_x , p is what we want to find, F_z , which we know. Therefore, p' equal to this and now becomes $p' = -F_x + p F_z$. This is the equation for p' .

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Derivation of equation for $q(t)$

Let $(x(t), y(t), z(t))$ be a point lying on the integral surface $z = u(x, y)$.
 We require the following equation to hold:

$$q(t) = u_y(x(t), y(t)) \quad (12)$$

On differentiating the above equation w.r.t. t , we get

$$q'(t) = u_{xy}(x(t), y(t))x'(t) + u_{yy}(x(t), y(t))y'(t). \quad (13)$$

Since (chara.ODE) hold for $(x(t), y(t), z(t))$, denoting $\zeta(t) := (x(t), y(t), z(t), p(t), q(t))$, we get

$$q'(t) = u_{xy}(x(t), y(t))F_p(\zeta(t)) + u_{yy}(x(t), y(t))F_q(\zeta(t)). \quad (14)$$

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Similarly, we do for q . What do we do? We look at the same equation GE. Differentiate that with respect to y we get something. Then we propose the value for q . q is supposed to be u_y of x and y . Differentiate this with respect to t it will involve this u_{xy} and u_{yy} . Eliminate this using the previous equation, I think is equation 11 and then you get an equation for q' .

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Derivation of equation for $q(t)$

From the equation (11)

$$F_y(\zeta) + F_z(\zeta)u_y(x, y) + F_p(\zeta)u_{xy}(x, y) + F_q(\zeta)u_{yy}(x, y) = 0,$$

we have

$$u_{xy}(x(t), y(t))F_p(\zeta(t)) + u_{yy}(x(t), y(t))F_q(\zeta(t)) = -(F_y(\zeta(t)) + q(t)F_z(\zeta(t))).$$

From

$$q'(t) = u_{xy}(x(t), y(t))F_p(\zeta(t)) + u_{yy}(x(t), y(t))F_q(\zeta(t))$$

we get

$$q'(t) = -(F_y(\zeta(t)) + q(t)F_z(\zeta(t))).$$

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x is replaced by y . That is the only change. p is replaced by q . So, this is the equation for q' . So, we got the equation for Q' also.

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Chara.Strip.ODE

System of ODEs for the characteristic strip

$$\frac{dx}{dt} = F_p(x, y, z, p, q) \quad (15a)$$
$$\frac{dy}{dt} = F_q(x, y, z, p, q) \quad (15b)$$
$$\frac{dz}{dt} = pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q) \quad (15c)$$
$$\frac{dp}{dt} = -F_x(x, y, z, p, q) - pF_z(x, y, z, p, q) \quad (15d)$$
$$\frac{dq}{dt} = -F_y(x, y, z, p, q) - qF_z(x, y, z, p, q) \quad (15e)$$

The system of ODE (15) is denoted by **(Chara.Strip.ODE)**.

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So, this is the system where the first 3 we have a time or we have proposed using characteristic direction. We realised that p, q also depend on x, t, y, t, z, t and we said we have to find equation for p, t, q, t and we have got. We have appended Chara.ODE with these 2 equations. Now it is called Chara.Strip.ODE equations for the characteristic strip. Now, there should be no problem because x, y, z, p, q here also x, y, z, p, q .

Anything else F , we know F . Therefore, we know we have F_p, F_q . So, no problem. Of course, it is a nonlinear system of ordinary differential equations, how to solve this? We have to solve this to get the characteristic strip and that we will do in the next class.

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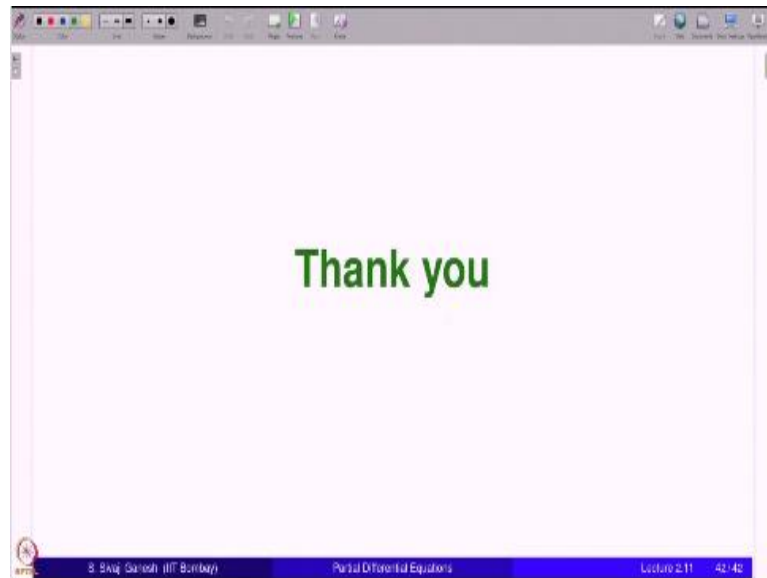
Summary

- 1 The (QL) defines a chara. direction right away. (GE) gives only possible tangent planes.
- 2 Using the idea of envelopes, we found chara. directions for (GE).
- 3 We observed that the system (chara.ODE) is incomplete.
- 4 The system (chara.ODE) is supplemented with two more equations, and obtained (Chara.Strip.ODE).

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So, let us summarize what happened so far. We saw that the Quasilinear equations has a characteristic direction right away given by the equation QL. GE gives only possible tangent planes. And using the idea of envelopes, we found characteristic directions for GE and we observed that the system Chara.ODE is incomplete. So, the system is supplemented with 2 more equations and we got characteristic strip equations, ordinary differential equations for characteristic strip.

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So, in the next lecture, we are going to take up the Chara.Strip.ODE system, which is system of 5 equations, nonlinear equations and we need to solve them and what is the strategy in causing these equations? We take the datum curve, take a point on the datum curve and pass a characteristic curve through that. So, now, if you take datum curve what is known is only x , y , z values on the datum curves namely F s, G s, H s which are given to us.

We will use them as initial conditions in the characteristic system of ODE in the positive case. But in the general nonlinear equations case, we have 2 more equations which is p t and q t equations for them. Therefore, we should know on the datum curve what should be the values of this p t, q t on the datum curve. So, that is what is called initial strip. So, initial data or datum curve is given that must be extended to a strip and that is called initial strip.

Using initial strip and using the initial conditions coming from the initial strip, we will solve

characteristic strip ODEs and get characteristic strip. We will determine characteristic strip and from there will come characteristic curves and from there we try to take the union and get

the integral surface. So, these steps will be implemented in the forthcoming lectures. Thank you