Partial Differential Equations
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Lecture – 2.11
General Nonlinear Equations 2
Characteristic Direction and Characteristic Strip

Our journey of studying Cauchy problem for general nonlinear equations has started in the last lecture. That is lecture 2.10. Today we continue.

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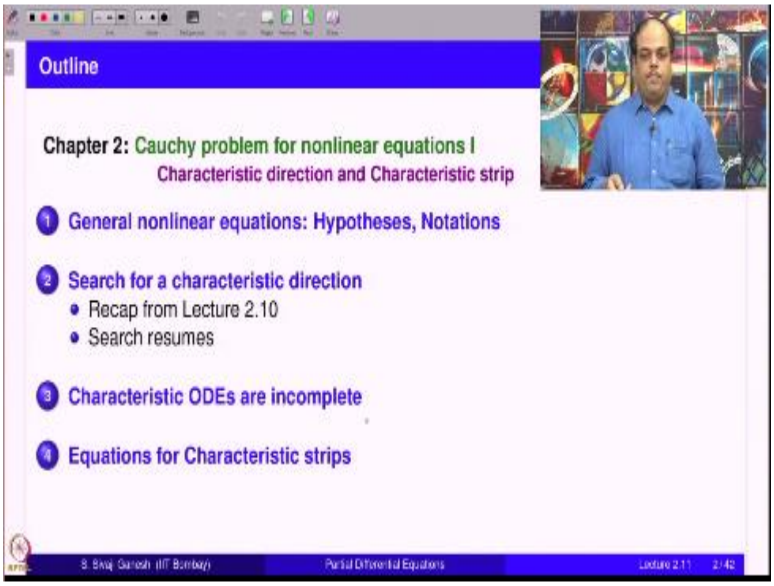

The outline for this lecture is: First we recall once again the hypotheses and notations involving general nonlinear equations. And in the last lecture, we started our search for a characteristic direction we were not yet successful. Today the search resumes and we will find a characteristic direction for the general nonlinear equation. And then resulting characteristic ODEs are incomplete. We are going to see that.

And therefore, we need to extend the system of characteristic ODEs to a new system, which is system of equations for characteristic strips.

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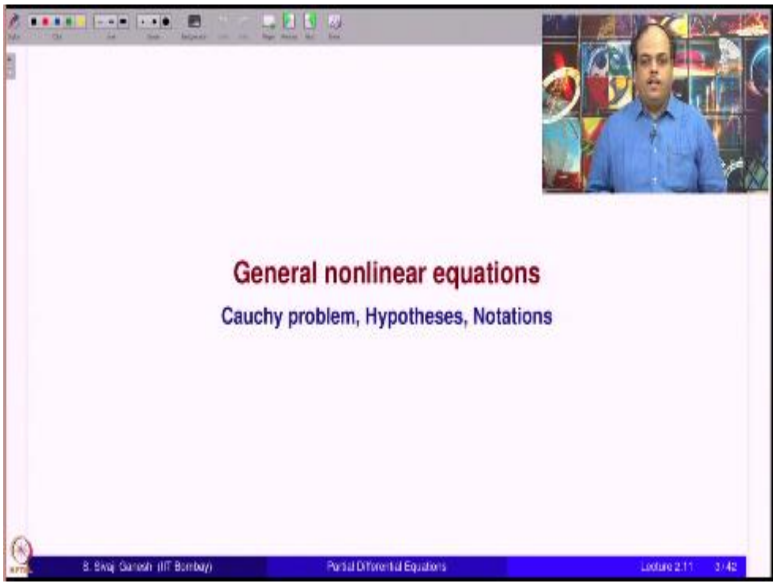

So, let us recall the notations and various hypotheses, which are involved in the Cauchy problem for general nonlinear equations.

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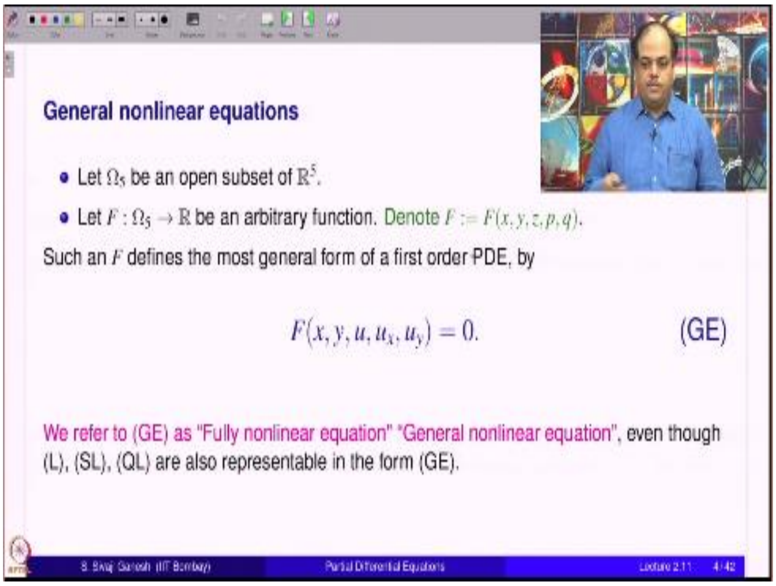

F is the function which is going to define as the general nonlinear equation. The arguments of F have been written as x, y, z, p, q defined on for a 5 tuples. So, lying in omega 5, which is an open subset of R 5. Such an F defines the most general form of a first order nonlinear general nonlinear PDE by F of x, y, u, u x, u y = 0. Though we know that this class GE contains in it linear equations, semi linear equations and Quasilinear equations, but we refer to this kind of form of general equation as fully nonlinear equation or general nonlinear equation.

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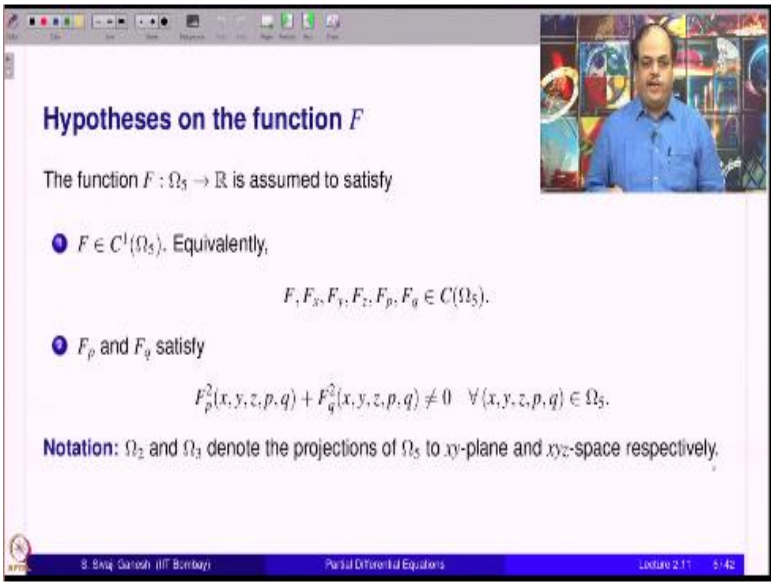

So, hypotheses on the function F which defines the PDE are that F must be a C 1 function on its domain of definition, which is equalent to saying that the function along with all first order partial derivatives of all the arguments which are x, y, z, p, q are continuous functions on omega 5. And we need to assume that F p and F q do not vanish simultaneously at any point in omega 5. Omega 2 and omega 3 denotes the projections of omega 5 to xy plane and xyz space respectively.

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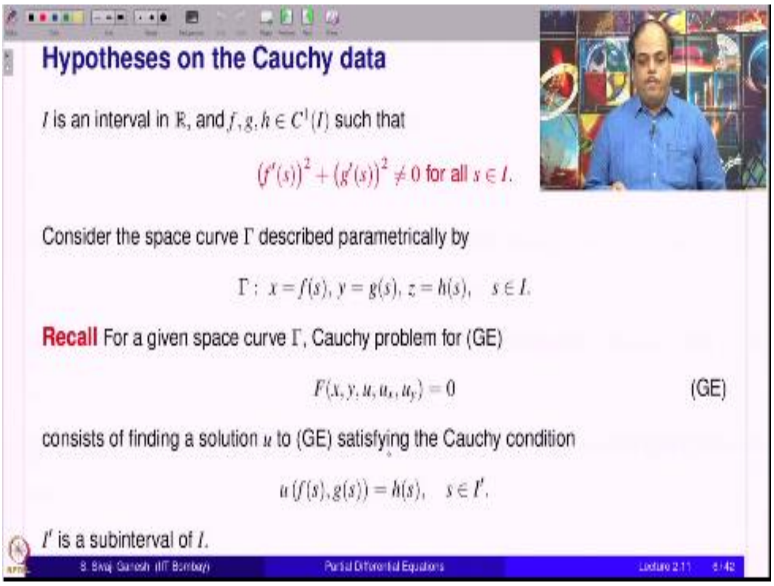

The Cauchy data is prescribed by gamma x = f s, y = g s, z = h s for s belong to an interval I. And these functions are C 1 functions on the interval I and we assume the projection of gamma to xy plane which is denoted by gamma 2 is that gamma 2 is a regular curve, which means f primes and g prime do not vanish simultaneously at any point on the curve gamma 2.

Recall that Cauchy problem means, we need to find a solution of the equation which satisfies this u of f s, g s = h s. In other words f s, g s = h s belong to the surface z = u x y.

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Now, let us recap from the last lecture the search for a characteristic direction.

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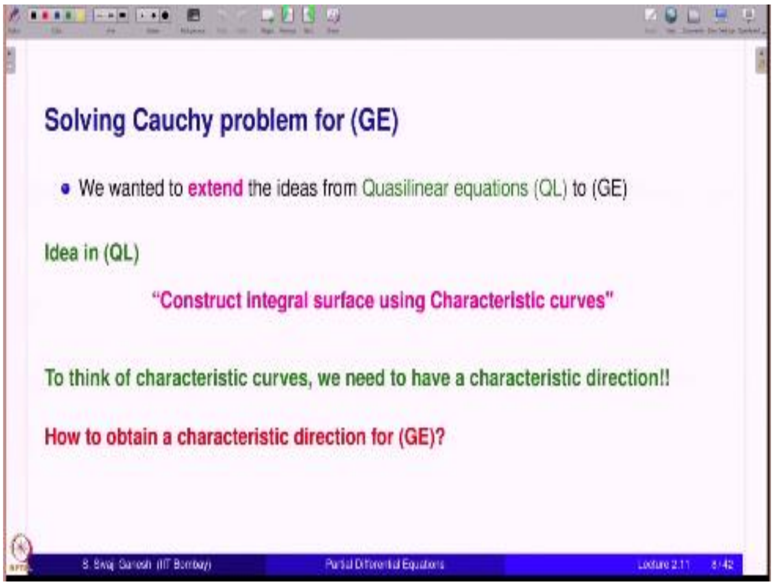

So, we wanted to extend the ideas from Quasilinear equations to general equations. The idea in Quasilinear equations was construct integral surface using characteristic curves. To think of characteristic curves, we need to have a characteristic direction. So, how do you obtain a characteristic direction for GE? That is the question.

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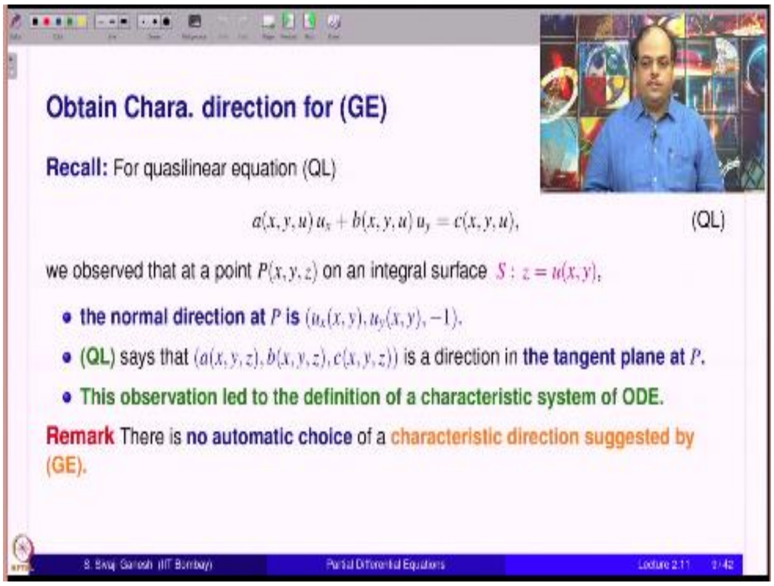

Recall for Quasilinear equations, the equation gave us a characteristic direction. The equation QL is a u x + b u y = c. What we observed is if you take any point on an integral surface given by z = u x y, the normal direction at P is u x, u y – 1. And the equation tells that abc is a direction in the tangent plane at P. This observation led us to define the notions of characteristic system of ODE and characteristic curves.

Once again there is no automatic choice of such a characteristic direction for general equations.

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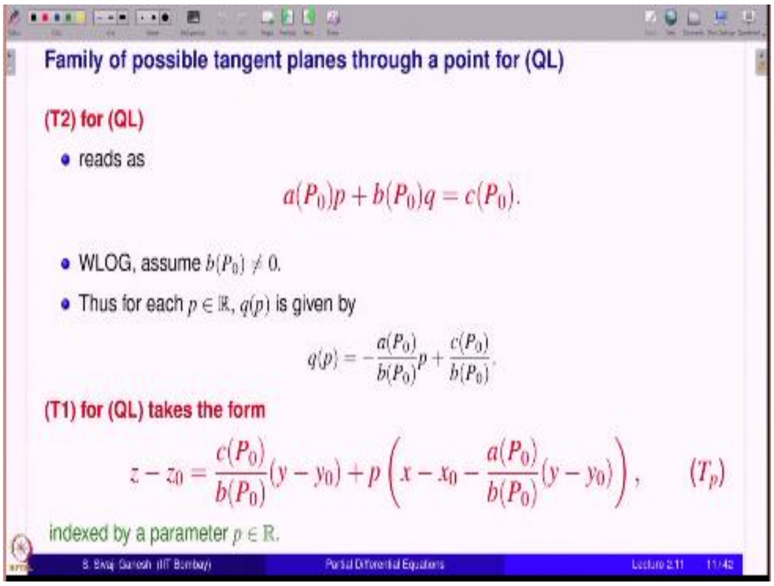

So, then we went on to consider this family of possible tangent planes to possible integral surfaces through a point for general nonlinear equation. Take a point P 0 in omega 3. We get a 1 - parameter family of possible tangent planes given by T 1 and T 2. What is T 1? T 1 is simply equation of a plane passing through x 0, y 0, z 0. But by restricting F P to satisfy T 2 will mean that this is going to be a tangent plane to possible integral surface.

It should satisfy these conditions T 1, T2. Note this you just see only the information coming from the equation. We are not pretending that we know the integral surface. That is not needed. Now, we observed that the T 2 for Quasilinear equation is this. And as one of them is nonzero we know that a or b or has to be nonzero by assumption. In the Quasilinear equations, we assume b is nonzero then we can solve q as a function of P.

In fact, it is a linear function of P that is what makes things much simpler and T 1 for QL becomes this after substituting for this. So, this is a family of possible tangent planes. We have explicitly got one equation.

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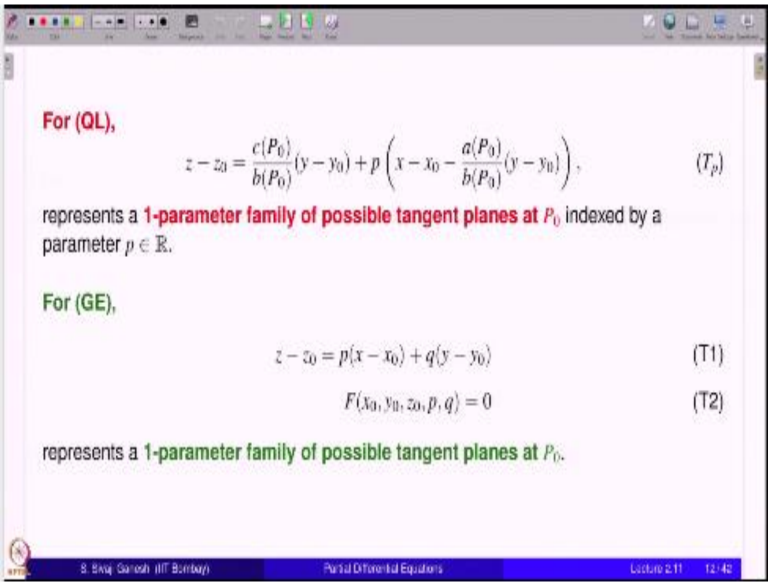

So, this represents, the equation T p represents a 1 - parameter family of possible tangent planes indexed by a parameter. For general equations, we have to have both the equations. If you are able to eliminate p or q from here that is expressed q as a function of p, go back and substitute here then once again you have only one equation like this. In the case of Quasilinear it was very easy to express q in terms of P.

But in case of general nonlinear equations, it is not clear. So, this is still a 1- parameter family of possible tangent planes at the point P 0.

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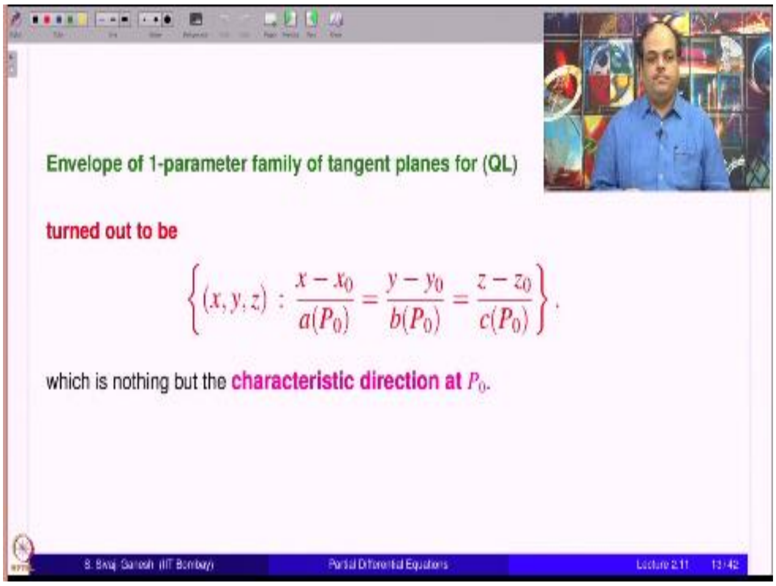

Now, envelope of 1 - parameter family of tangent planes for QL we computed. It turned out to be this. What is this? This is nothing but the characteristic direction at P 0. This is a line passing through the point x 0, y 0, z 0 and the direction abc is a characteristic direction.

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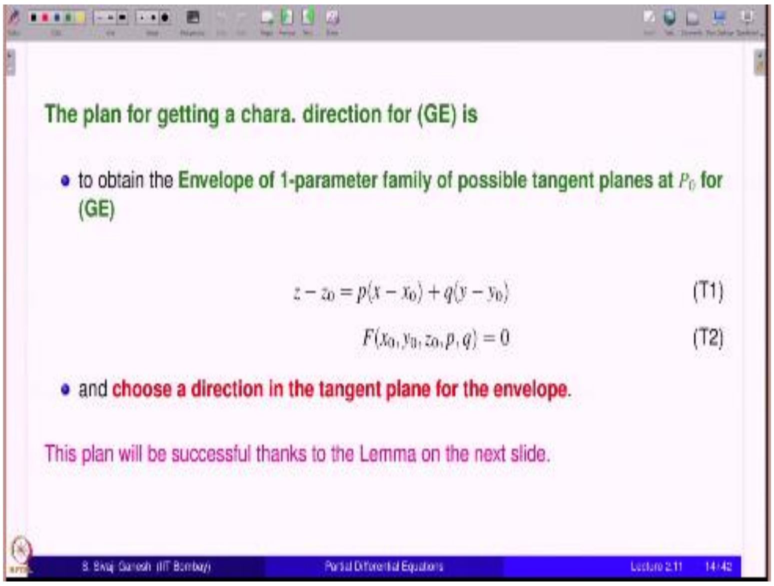

So, therefore, the plan for getting a characteristic direction for GE is like this. Obtain an envelope of 1 - parameter family of possible tangent planes for GE which is this. So, get the envelope of this and choose a direction in the tangent plane for the envelope. This plan will be successful because we have a lemma that we proved in the last lecture, which is here.

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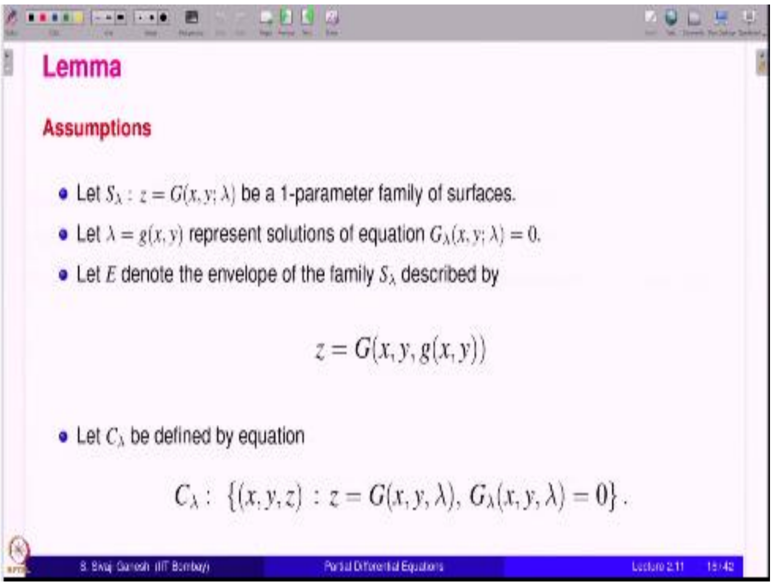

If you take a 1 - parameter family of surfaces given by z = g x, y, lambda and you find its envelope which is this z = G x, y, g x y and C lambda denotes the intersection of the envelope and the family.

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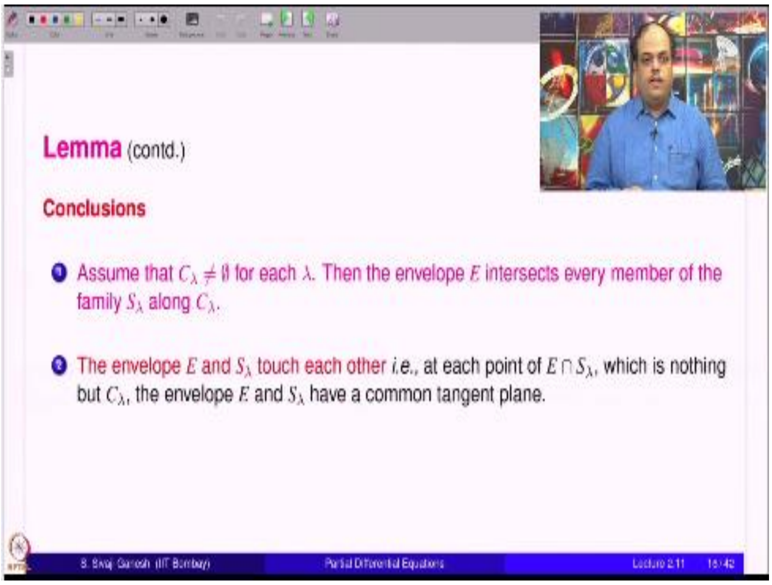

Then look at the second conclusion. The envelope and S lambda touch each other. That means, wherever they intersect, which is along C lambda, the intersection is precisely C lambda. At every point on C lambda, they share the tangent plane. Therefore, if you can find a direction in the tangent plane for the envelope, we are done. That will also be a tangential direction for S lambda. That will also be S lambda.

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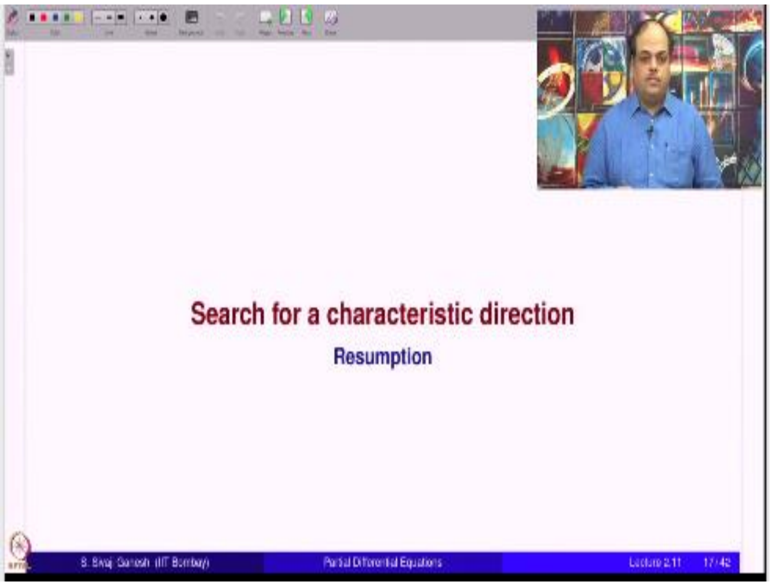

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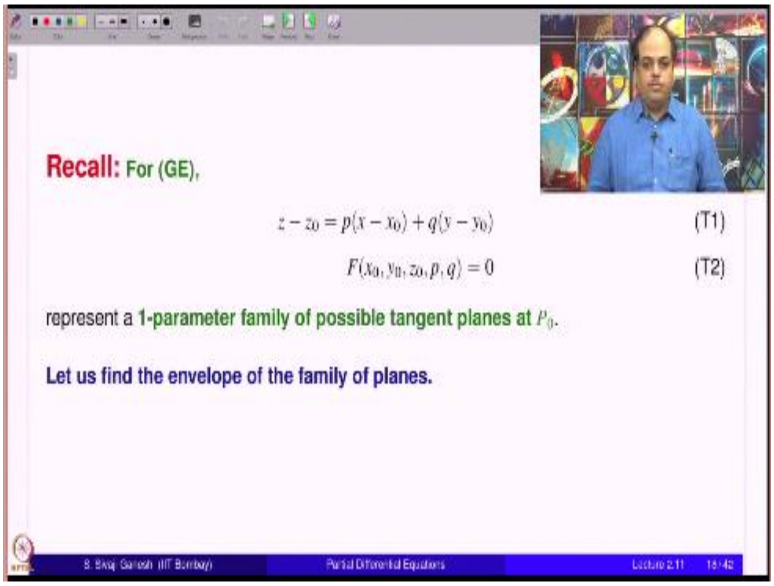

So, let us resume our search of characteristic direction. So, this is T 1 and T 2. That represents 1 - parameter family of possible tangent planes at the point P 0. Let us find the envelope that is what we need to do.

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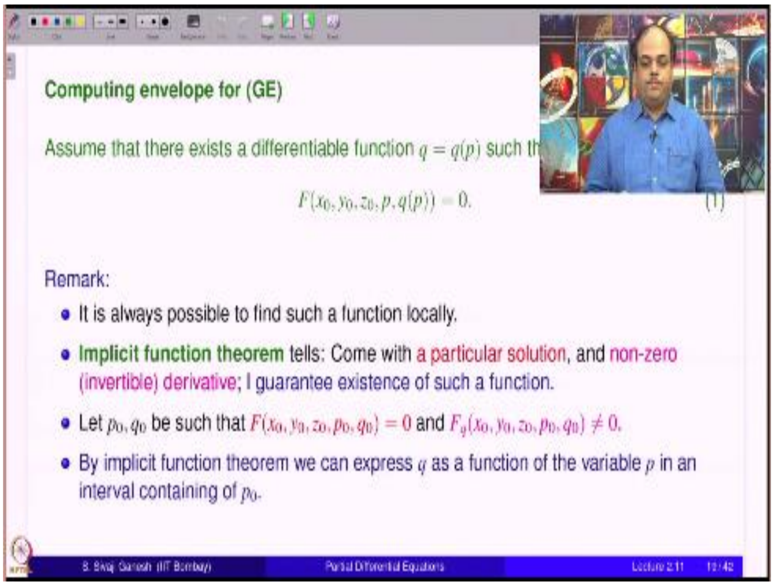

To do that, we need to assume that q can be expressed as a function of p in a differentiable way so that this equation is satisfied. In other words, we are solving F of x 0, y 0, z 0, p. q = 0 and q is expressed in terms of p. Question is, is it possible? It is always possible to find a such a function locally implicit function theorem tells you come with a particular solution and nonzero or invertible derivative, I guarantee the existence of such a function.

This is typically what implicit function theorem says. Now, we will go to the implicit function theorem with a particular solution with that means, you first find out a p 0, q 0 real number such that f of x 0, y 0, z 0, p 0, q = 0. That means, we have the particular solution. Now, we need to see what is this nonzero derivative and F q at this point x 0, y 0, z 0, p 0, q 0 is nonzero. Please note when we are applying implicit function theorem x 0, y 0, z 0 is fixed.

So it is only as a function of p and q that we are trying to solve this equation 1. So F q is nonzero, if you assume then implicit function theorem guarantees existence of such a function, but locally. That means, in an interval containing p 0 you can express q as a function of p. That is the implicit function theorem.

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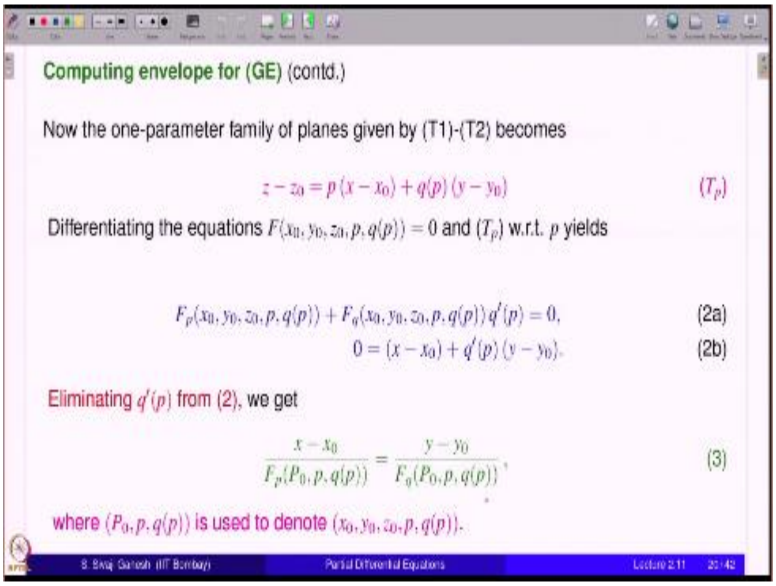

Therefore, the 1 - parameter family of planes becomes simply one equation now because I have solved q in terms of P and I put it here. Now only thing to do is differentiate this equation with respect to p and we get this equation. These 2 equations. We will differentiate this equation you get 2a, differentiate T p equation you get 2b. From 2a and 2b you can eliminate q prime p and we get this.

x - x 0 y – y 0 is proportional to F p of F q. Something we got. Now, what is this p 0, p, q, p? p not actually x 0, y 0, z 0. F, F is a function of 5 variables. We need a 5 tuple here. For want of space here I have just made it a small notation p 0. It stands for this.

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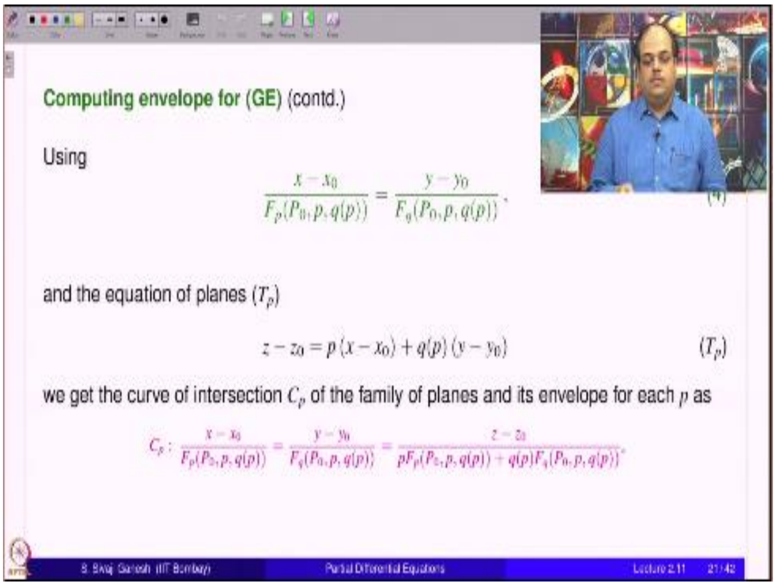

Using this and the equation for tangent planes, which is TP here we get the curve of intersection and that is this. x - x 0 is proportional to F p, y - y 0 is proportional to F q. Therefore, z – z 0 is proportional to p F p plus q F q. That is what we have written here.

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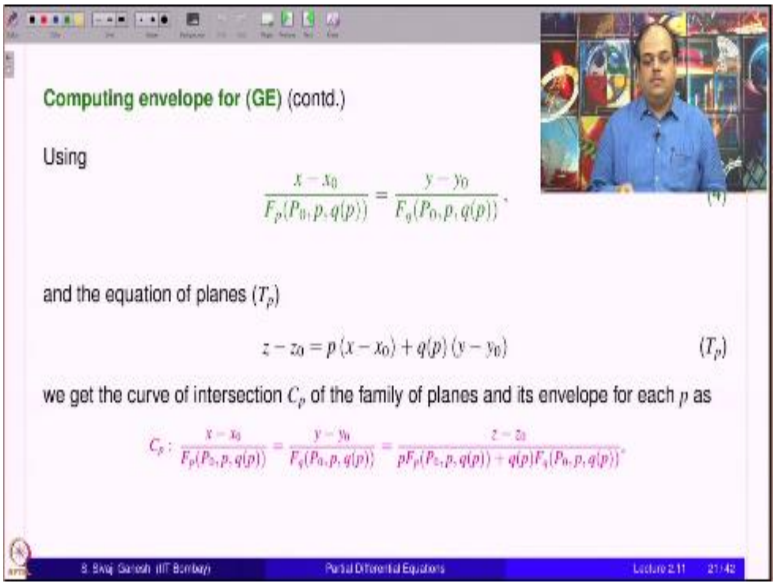

So, this is a line right. C p is a family of lines of course indexed by p. The same thing was actually one single line. It never depended on p. For a Quasilinear equation all of them past the point x 0, y 0, z 0 and having a direction F p, F q, p F p + q F q. The 3 tuple which is given in 5 is nonzero because we have assumed that F p square + F q square is not equal to 0 throughout the domain of F. That is the hypotheses.

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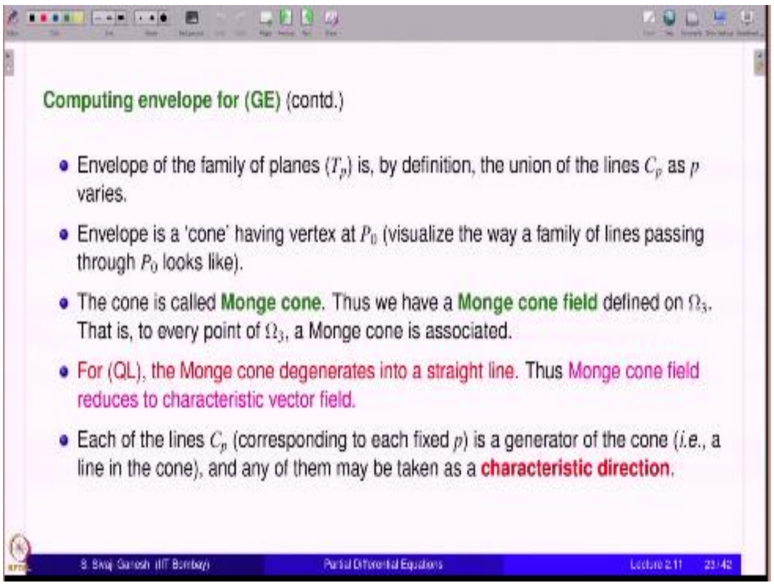

An envelope of the family of planes T p by definition is a union of C p. That means union of these lines. So, envelope is a cone having vertex at point p 0. So, visualize the way a family of lines passing through a point p 0 looks like. For example, you have this point p 0. So, imagine lines in 3d. Like that. So, that is how. Looks like a cone. This is called Monge cone. Thus we have a Monge cone field defined in omega 3.

So, omega 3 is in our R 3 subset. At any point you have a cone sitting there. At another point, you have another cone like that. So every point you can attach a cone. So, this is called cone field. This is similar in spirit to a vector field. To any point if you attach one vector, associate one vector it is called vector field. If you recall it is called cone field. That is to every point of omega 3 a Monge cone is associated. For Quasilinear equations, the Monge cone degenerates into a straight line.

Because we observe that the envelope is actually a straight line. It is independent of p. So, therefore, just one single straight line. So the Monge cone field reduces to a characteristic vector field in the case of Quasilinear equations. Now, each of the line C p corresponding to each fixed p is a generator of the cone. Generator of the cone means it is a line in the cone. So, you have a for example, you have a cone like that. You have a line here.

So, at a point on the cone if you take tangent plane, that line is going to be there on the tangent plane. Therefore, what we wanted was to look at the envelope and take one direction in the tangent plane of the envelope. Now, here is very nice. Tangent plane of this cone at these points, one of the direction is clearly the line now C p. So we can take that C p to be characteristic direction.

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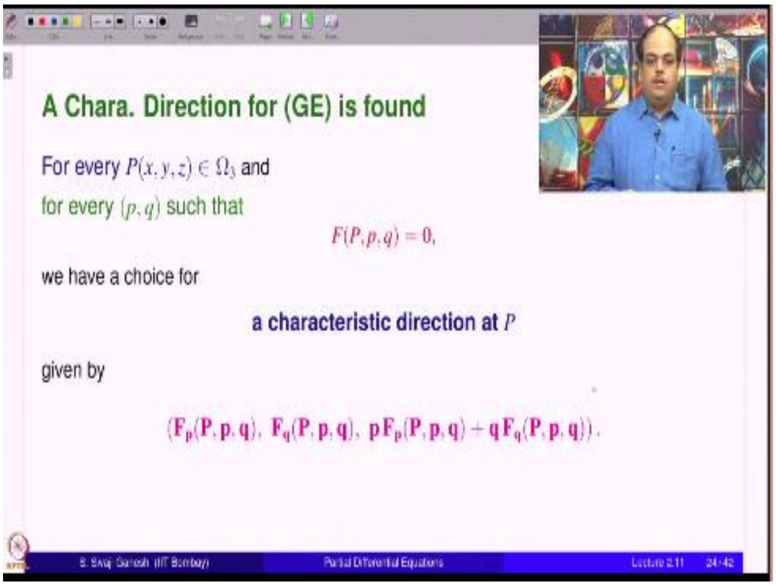

So, a characteristic direction for GE is found. For every p in omega 3 and every p q we do not we no longer require q of p here. After all q is a function of p, q is some value. It satisfies this equation. Still, we have found a characteristic direction at P. What is that? F p, F q, p F p + q F q.

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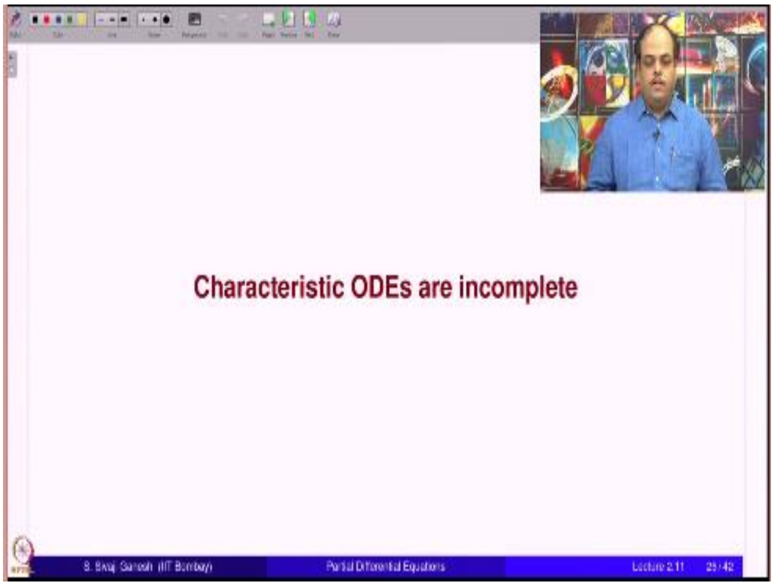

Now, we have to look at the characteristic ODEs.

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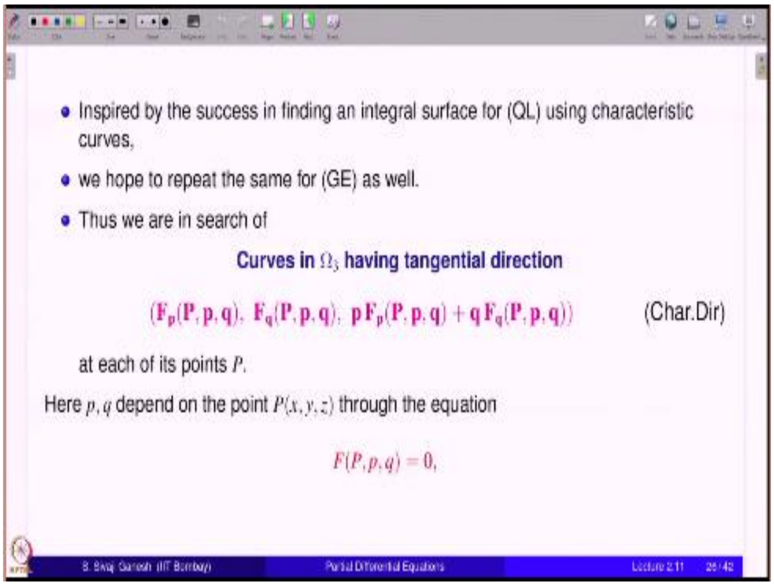

We are going to see they are incomplete. We will see why. So, inspired by the success in finding an integral surface using characteristic curves, we hope to repeat the same for GE also. In QL we are successful. So, we hope same thing happens for GE as well. So, we are in search of curves in omega 3 which have tangential direction equal to the characteristic direction that we just found which is this: F p, F q, p F p + q F q at each of its points P.

Such curves we are looking for. Now, p, q of course, depends on the point x, y, z. They satisfy this equation. So it depends on x, y, z.

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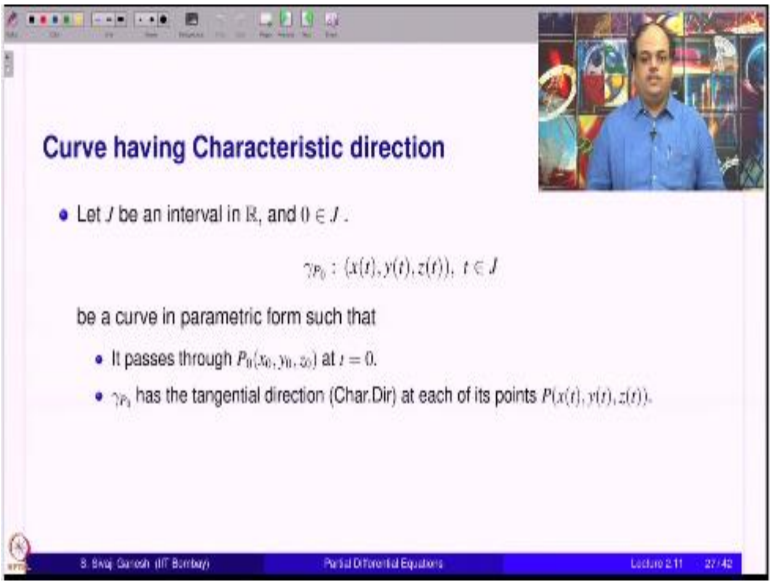

Let us write a curve having a characteristic direction. So, let J be an interval, 0 belongs to J. Gamma P 0 be a curve given by x t, y t, z t, t varies in J such that it passes through the point P 0 at t = 0. And it has a tangential direction which is characteristic duration at each of its points.

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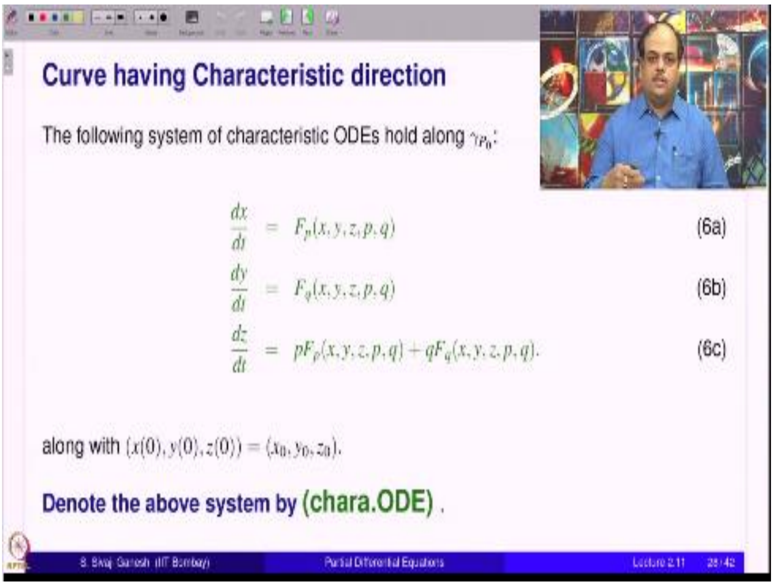

This is what we would like to call a characteristic curve. Therefore, the following system of characteristic ODEs hold along gamma p 0. So, dx by dt, dy by dt, dz by dt are in F p, F q and p F p + q F q. And at t = 0, we want to be at the point x0, y0, z0. So, fine. Now, why is it incomplete? I do not know what is p and q? That is a problem. Let us denote the above system by chara.ODE exactly like we used for Quasilinear equations.

The solutions of this It is images will be characteristic curves. Only thing is that it does not determine characteristic curves. There is a problem, because there are pq s here.

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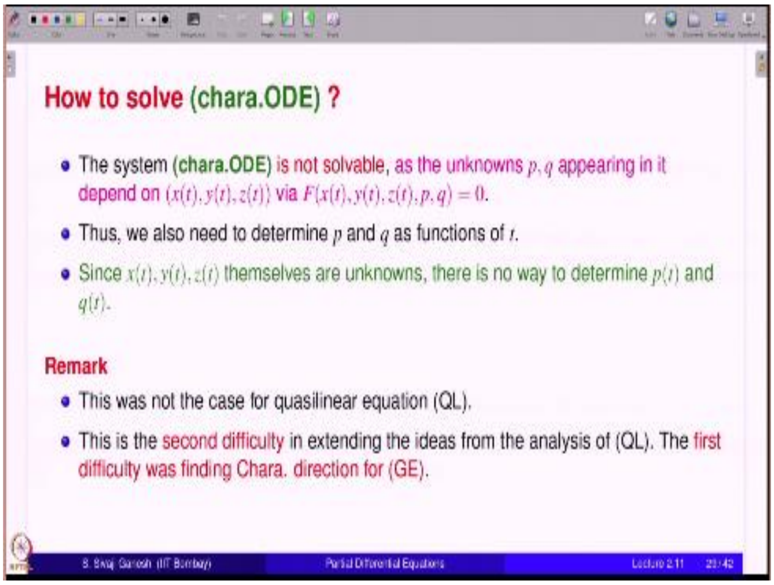

It is not solvable as unknowns pq appearing in it depend on x t, y t, z t via this equation F of x t, y t, z t, p, q = 0. p and q are also depending on t. Thus we also need to determine p and q as functions of t. Since x t, y t, z t themselves are unknowns, there is no hope of solving for p and q from this equation. And no this is not the case for Quasilinear equations, because characteristic ODE never involved a p and q.

And this is a second difficulty in extending the ideas from the Quasilinear case. What is the first one? Finding a characteristic direction. That was the first difficulty we have overcome. And second difficulty we will overcome by supplementing 2 more equations for one for p and one for q.

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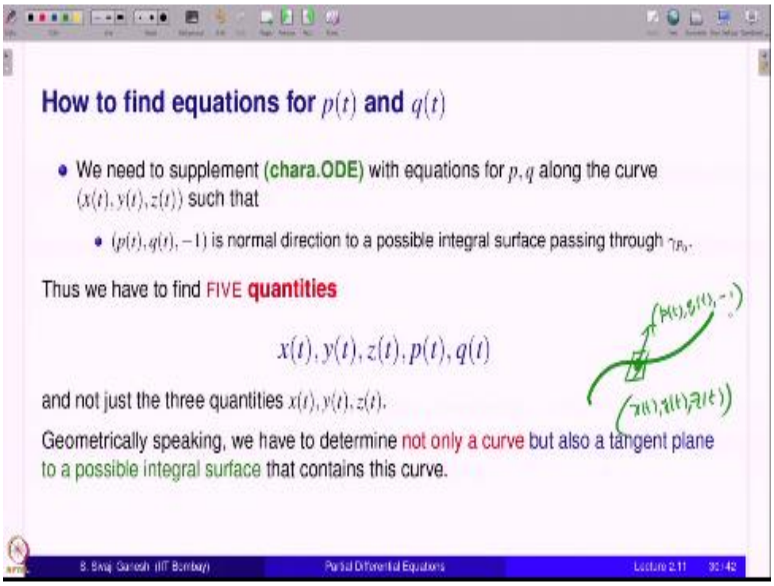

So, how to find the equations for p t and q t? We need to supplement chara.ODE system with equations for p, q along the curve x t, y t, z t and what should be your property? P t, q t - 1 should be the normal direction to a possible integral surface. They are not arbitrary functions. So, we have to determine 5 quantities, x t, y t, z t and p t, q t and not just the 3 quantities because we are unable to determine the 3.

We would have been very happy if you had a got x t, y t, z t we could have proceeded. But the equation for activities that involve p and q. Therefore, we need to find p and q also. So, geometrically speaking what we are saying is we have to determine not only a curve, but also a tangent plane to a possible integral surface that contains this curve. So, in other words we are trying to find a curve like that.

So, suppose this is the point x t, y t, z t. So, we want to find something like that p t, q t, of course – 1, I write but then it is okay, - 1there is nothing to find. So, we need to find this.

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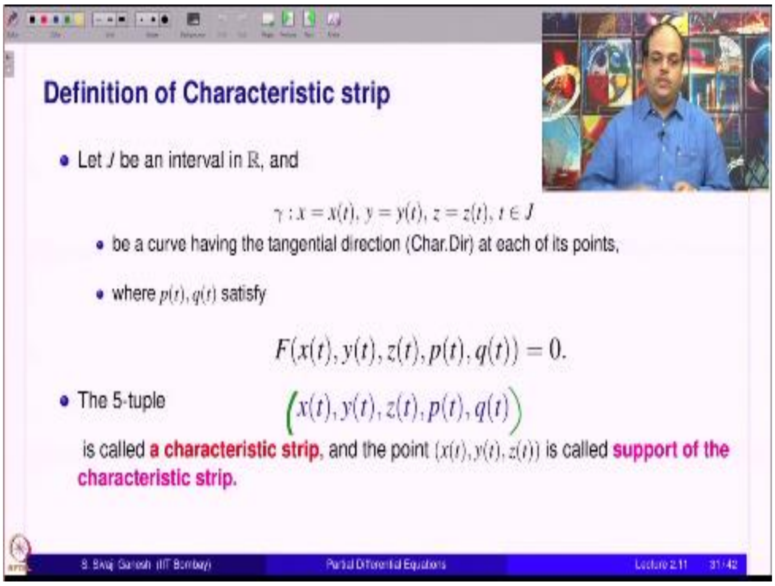

So, let J be an interval in R and gamma be a curve given by x = x t, y = y t, z = z t be a curve having tangential direction as Char.Dir that is characteristic direction at each of its points, where p t, q t satisfy this equation F of x t, y t, z t, p t, q t at t = 0. The 5 tuple x t, y t, z t, p t, q t is called a characteristic strip. Maybe you may put like that, because we are saying tuple, but it is okay. Even without that it is fine.

All these 5 together is called a 5 tuple. That is called a characteristic strip and the point x t, y t, z t is called support of the characteristic strip. Because that is a point at which you are putting a plane with the normal p t, q t - 1.

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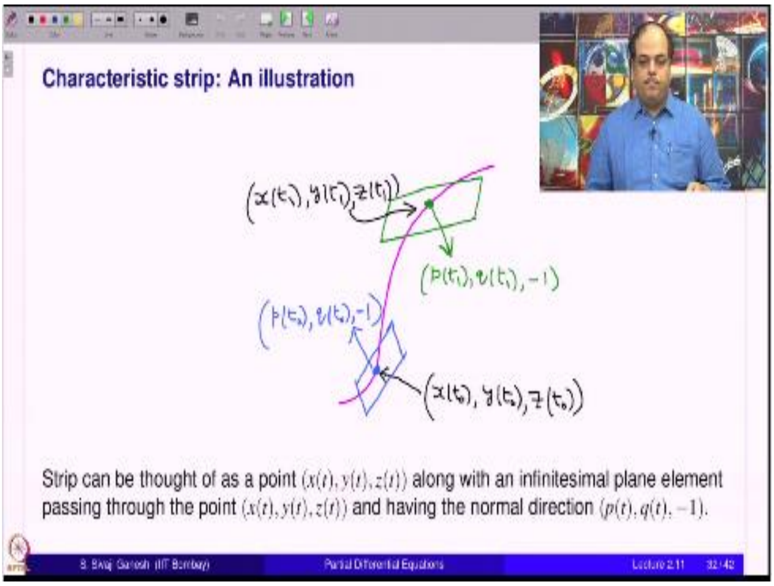

So, let us illustrate that pictorially. So, here there are 2 points I have considered indexed by t = t 0 here and t = t one. So, x t 0, y t 0, z t 0 is a point. These are tangent. This is a plane with the normal p, q – 1. This is another point where the normal is p, q - 1. So, it can be, a strip can be thought of as a point along with an infinitely small plane element passing through that point x t, y t, z t with the normal direction p t, q t - 1.

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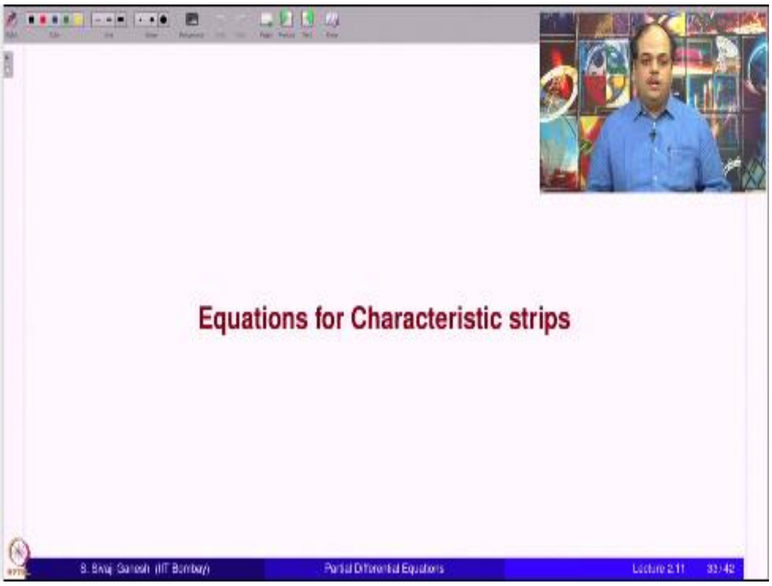

Now, how do you get equations for p t, q t.

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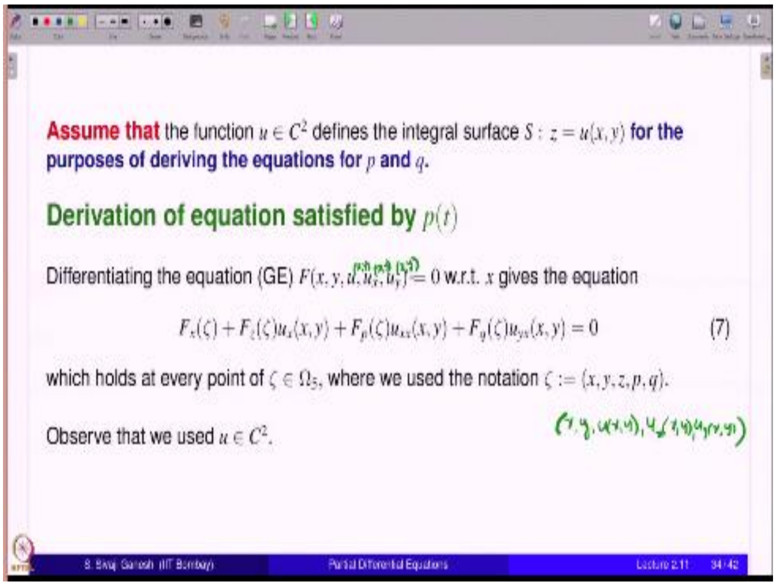

So, here we assume some more nice properties about the solution u. See u is supposed to be a solution to first order PDE. We have not yet found. But we are assuming that it is C 2. Is it okay? That is a question. It is okay because we are going to use this only to derive certain equations and for deriving the equation we assume what you want does not matter, but after getting the equations then you should show that solution exists.

There you should not suppose that C 2 etc. We will comment on this in the next lecture also. So, this is only to derive the equations for p t, q t that we are assuming u in C 2. So, how do we derive the equation for p t? What are that at our hands? This equation F of x, y, u, u x, u y = 0. Differentiate that with respect to x. So, first is F x. Here I am using the zeta to stand for x, y, z, p, q.. So, F x and then with respect to in z you have u x, y.

Therefore, you need to differentiate F z and u with respect to x, then with here also p F p and then whichever is here with respect to x which is u xx and here it is F q and u xy or u yx = 0 at every point in omega 5, where we use this notation zeta = x, y, z, p, q. Yep, now actually this is zeta is not x, y, z, p, q. It should be x y, we are to substitute x, y, z, p, q = x, y, u xy, u x xy, u y xy. That is what it is because we are going to differentiate here.

This u is a function of x y, u x is a function of x y, u y is also a function of x y. So, that is why we get this by chain rule.

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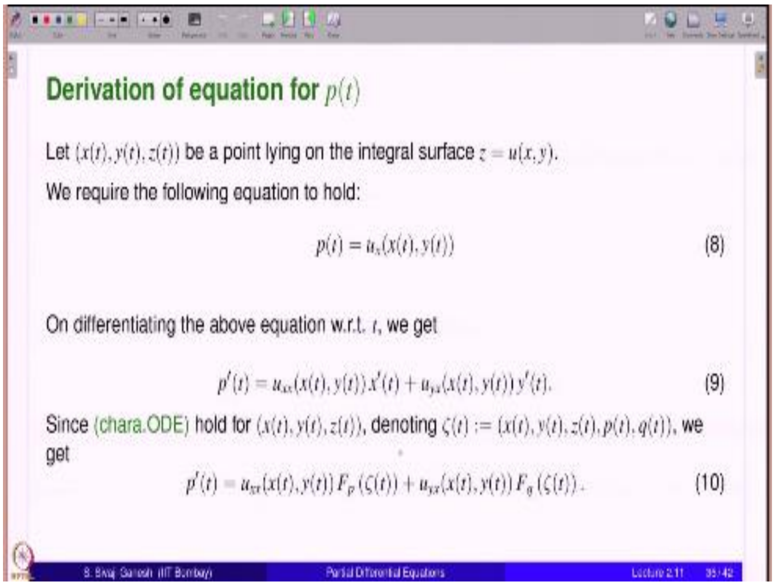

And x t, y t, z t is a point lying on an integral surface z = xy. We require the following thing to hold namely p of t I want it to be like u x. So p of t = u x of x t, y t. We are demanding this.. Because p t, q t – 1 should be such that F of x t, y t, z t at p t, q t should be equal to 0. So p is supposed to play the role of u x along the curve x t, y t. So, that is my p t. We want this.

So on differentiating the above equation with respect to t and using chain rule we get p prime t is equal to, t appears in both variables. So, differentiate this with respect to x, u xx at the point x t, y t into derivative of x with respect to t that is x prime t. And differentiate this with respect to y that is u y x at the point x t, y t into derivative of y t which is y prime t. Since characteristic ODE hold for x t, y t, z t denoting zeta t equal to this 5 tuple.

x t, y t, z t, p t, q t we get p prime t is = u xx, x prime is F p, y prime is F q. So, we have got this equation for p prime.

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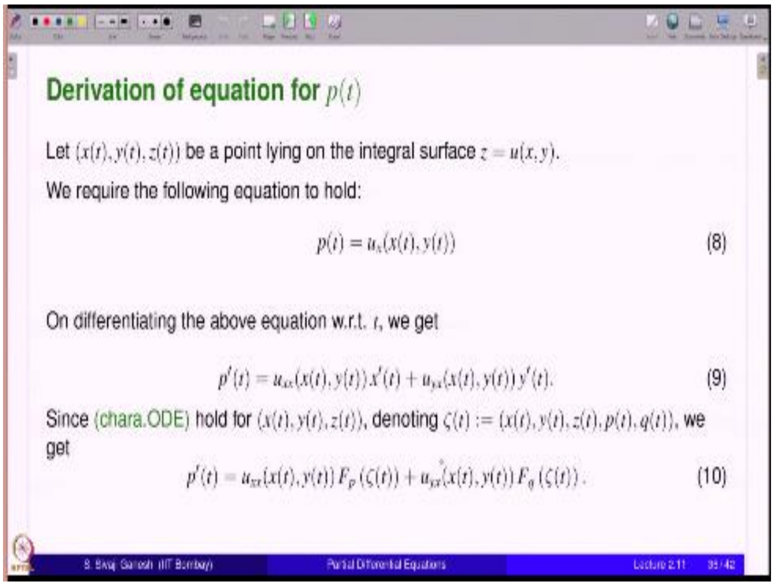

And from the equation 7 that is the one we got after differentiating F of x y. This one, this equation. And here we almost got an equation for p prime. The only problem is there is u xx and u yx. We do not want that. Equation of p should involve only F, it can involve x t, y t, z t that no problem but not u xx es and u yx.

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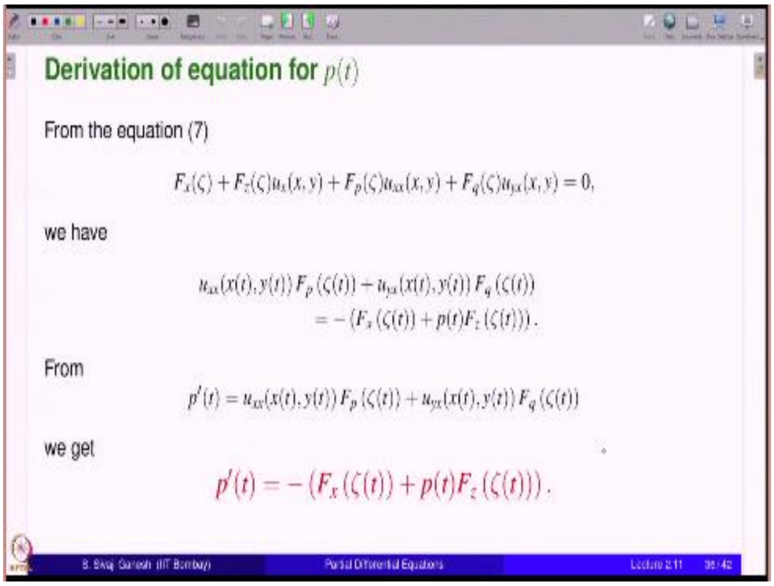

So, that needs to be removed now. So, we solve for that, which we do not want in terms of what we know F x, p is what we want to find, F z, which we know. Therefore, p prime t equal to this and now becomes p prime t = - F x + p F z. This is the equation for p prime.

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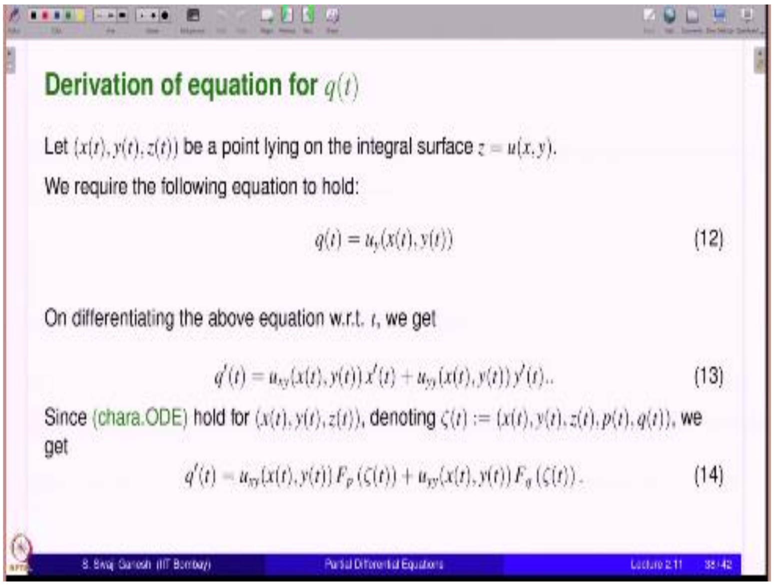

Similarly, we do for q. What do we do? We look at the same equation GE. Differentiate that with respect to y we get something. Then we propose the value for q t. q t is supposed to be u y of x t y t. Differentiate this with respect to t it will involve this u x y and u y y. Eliminate this using the previous equation, I think is equation 11 and then you get an equation for q prime.

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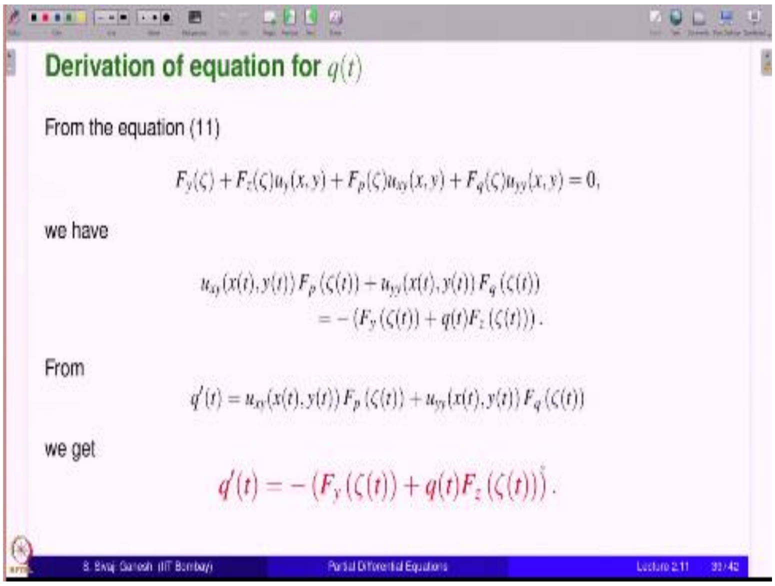

x is replaced by y. That is the only change. p is replaced by q. So, this is the equation for q prime. So, we got the equation for Q prime also.

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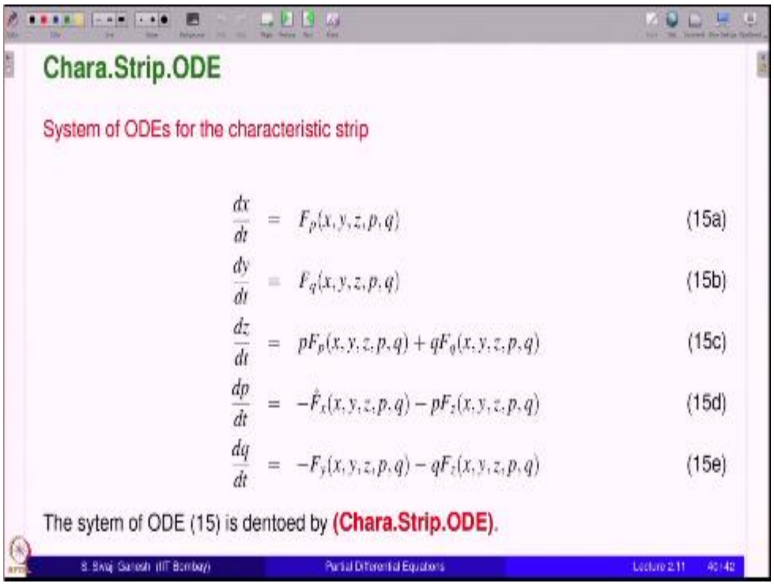

So, this is the system where the first 3 we have a time or we have proposed using characteristic direction. We realised that p q also depend on x t, y t, z t and we said we have to find equation for p t, q t and we have got. We have appended Chara.ODE with these 2 equations. Now it is called Chara.Strip.ODE equations for the characteristic strip. Now, there should be no problem because x, y, z, p, q here also x y, z, p, q.

Anything else F, we know F. Therefore, we know we have F p, F q. So, no problem. Of course, it is a nonlinear system of ordinary differential equations, how to solve this? We have to solve this to get the characteristic strip and that we will do in the next class.

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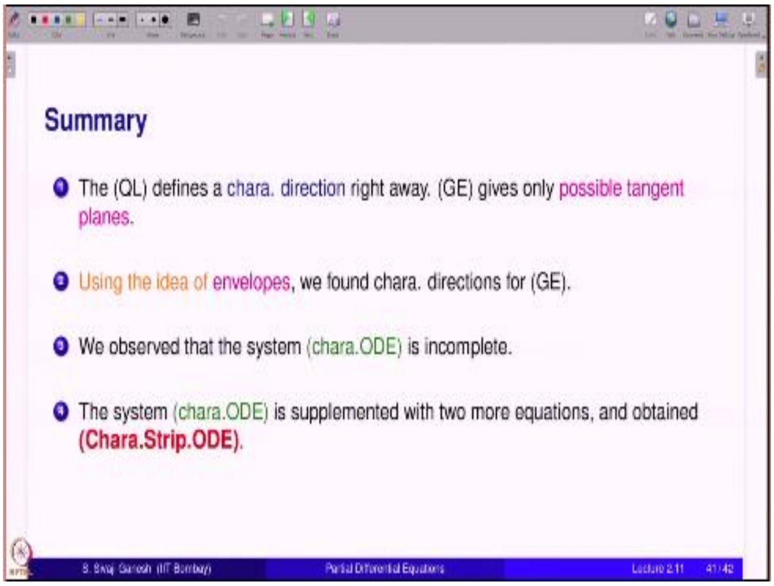

So, let us summarize what happened so far. We saw that the Quasilinear equations has a characteristic direction right away given by the equation QL. GE gives only possible tangent planes. And using the idea of envelopes, we found characteristic directions for GE and we observed that the system Chara.ODE is incomplete. So, the system is supplemented with 2 more equations and we got characteristic strip equations, ordinary differential equations for characteristic strip.

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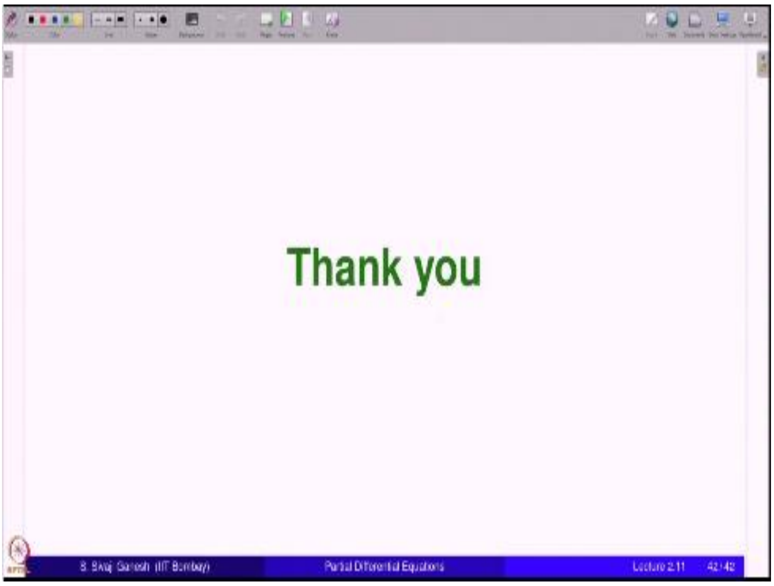

So, in the next lecture, we are going to take up the Chara.Strip.ODE system, which is system of 5 equations, nonlinear equations and we need to solve them and what is the strategy in causing these equations? We take the datum curve, take a point on the datum curve and pass a characteristic curve through that. So, now, if you take datum curve what is known is only x, y, z values on the datum curves namely F s, G s, H s which are given to us.

We will use them as initial conditions in the characteristic system of ODE in the positive case. But in the general nonlinear equations case, we have 2 more equations which is p t and q t equations for them. Therefore, we should know on the datum curve what should be the values of this p t, q t on the datum curve. So, that is what is called initial strip. So, initial data or datum curve is given that must be extended to a strip and that is called initial strip.
Using initial strip and using the initial conditions coming from the initial strip, we will solve

characteristic strip ODEs and get characteristic strip. We will determine characteristic strip and from there will come characteristic curves and from there we try to take the union and get the integral surface. So, these steps will be implemented in the forthcoming lectures. Thank you