Partial Differential Equations
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Lecture – 2.9
First Order Partial Differential Equations
Tutorial of Quasilinear Equations

Welcome to a tutorial on Cauchy linear equations. In this, we are going to solve some Cauchy problems for Quasilinear equations.

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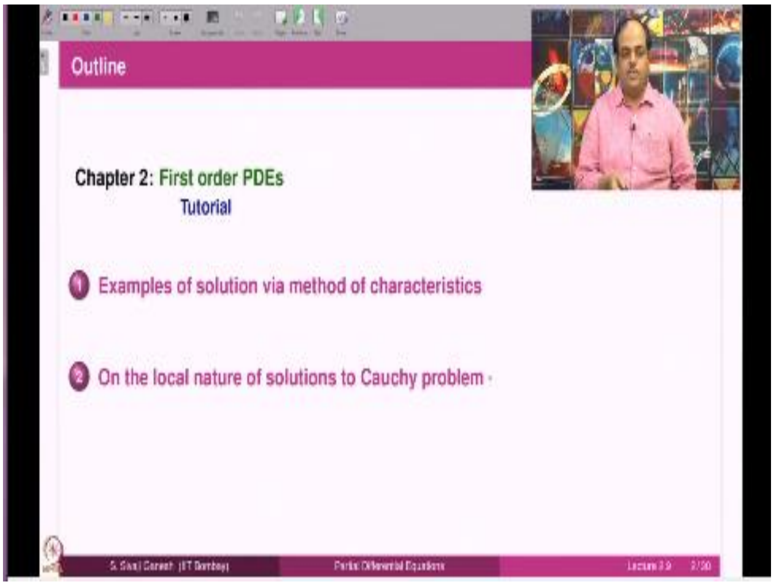

And we also explained through examples, the local nature of solutions to Cauchy problem. Recall that the existence and uniqueness theorem gave us existence of a solution only nearby any fixed point on the datum curve. So, is that all that can be expected or can we get a solution whose integral surface consists of the entire datum curve or defined on entire domain omega 2?

These are the questions; we are going to discuss under this heading on the local nature of solutions to Cauchy problem.

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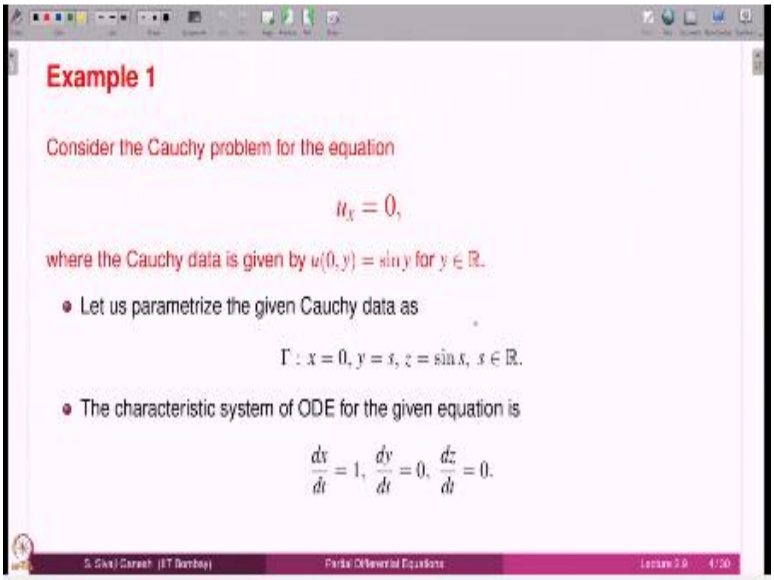

So, let us start some problems. Examples, this is the simplest example that we considered in the beginning u x = 0 and data is u 0 y = sin y. So, how do we solve this? We need to first parameterize the given Cauchy data x = 0, y = s, z = sin s and y in R, therefore, s in R. So, this is our datum curve. Then we need to look at the characteristics system of ODE for the given equation.

Recall dx by dt = a in this example, a u x that is a is 1, b and c are 0. So, dy by dt = b, which is 0; dz by dt = c, which is 0. So, this is a characteristic system of ODE associated to this equation. Now, we need to solve these ODEs, system of ODEs so, with the initial condition. So, that at t = 0, we are at a point on gamma.

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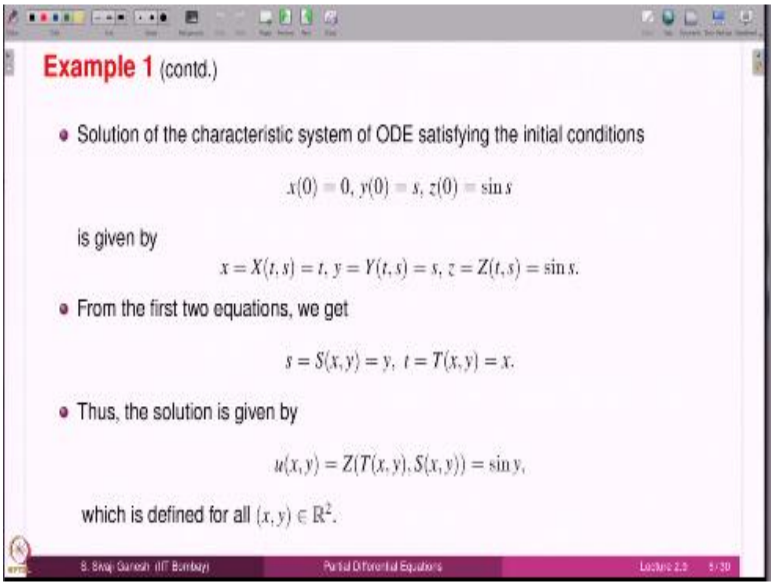

So, x of 0, when time is 0 = 0, y of 0 is s that have 0 equal to sin s, because 0, s, sin s is an arbitrary point on gamma, the datum curve. Solution is very simple to obtain. So, we get x = X of t s = t, y = Y of t s = s and z = Z of t s = sin s. That is very easy to see, because see here, dx by dt = 1, therefore, x has to be t + constant. At t = 0, x must be 0 therefore, this is x = t.

Here, dy by dt = 0 that means y is constant. At t = 0, it must be s. Therefore, y = s. dz by dt = 0 that means z is constant with respect to t, but at t = 0, it should be sin x. That will give us the solutions. Now, we need to eliminate or we need to solve for t and s in terms of x and y from the first 2 equations, which is obvious in this example. t is T of x y = x, s = X of x y = y. So, we have worked.

Now, we need to substitute in this and we get a solution. So, u x y = sin y, which is defined for all x y in R 2.

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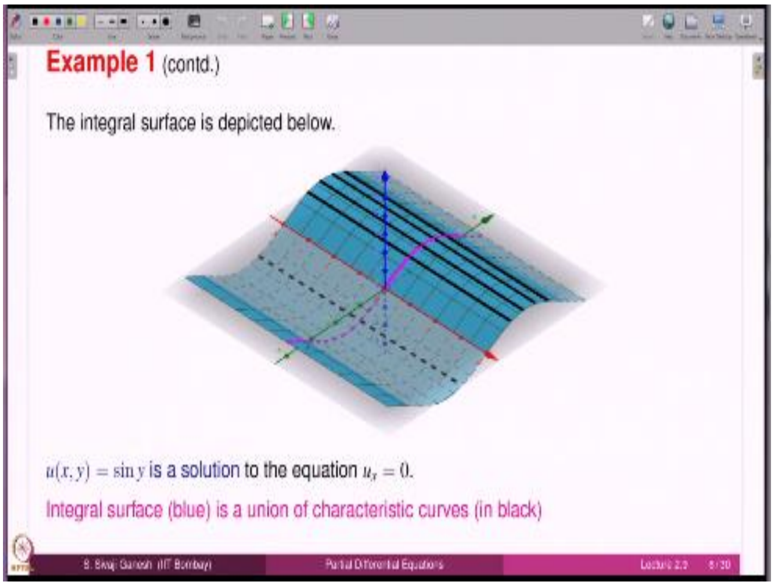

These are pictures that we have already seen. The red is x axis; green is y axis and this is z axis. And this one which is in magenta colour is the initial data u 0 y = sin y. So, 0, s, sin s as varies in R, you get this curve. And integral surface is a blue colour one, these are the characteristic curves; here, they are straight lines. We already saw that.

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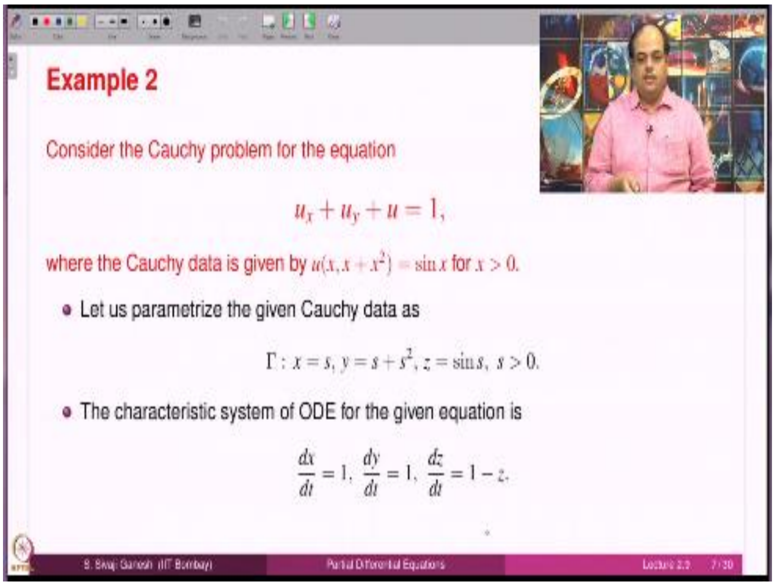

Let us look at another example. This is also linear equation, u x + u y + u = 1 and the Cauchy data is of this nature, it is prescribed on some curve u of x, x + x square = sin x. So, as before, you need to parameterize Cauchy data. So, s, x = s, y = s + s square and z = sin s and for s positive, done. And we need to write characteristic system of ODE. dx by dt = a, in this example, a is 1; dv by dt is b, b is 1; dz by dt is c, remember c is anybody who is on the other side.

So, it will be 1 – z. You have to be careful there. Do not think, it is 1. It is 1 – z, because the equation is of the form a u x + b u y = 2 c. So, dz by dt = c. Therefore, it is 1 – z.

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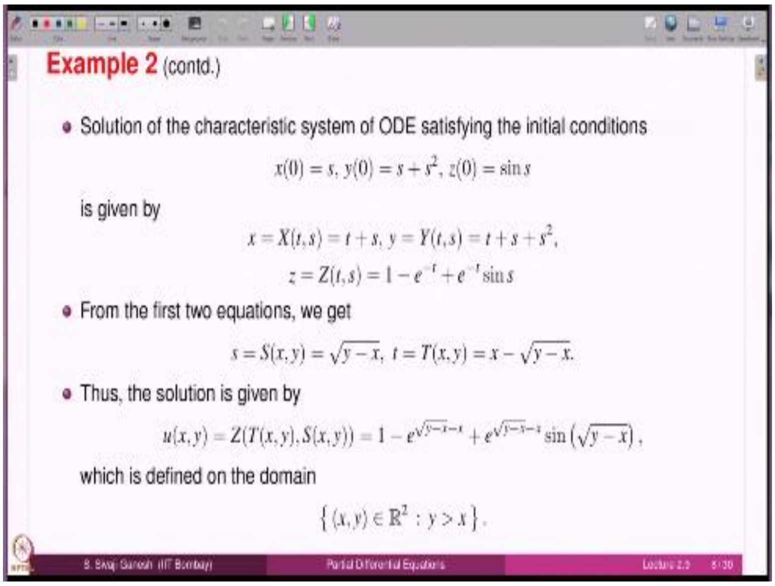

Now, we need to solve the system of characteristic ODE with the initial conditions which are given here. And we get solutions as this, x is t + s; y is t + s + s square; z = 1 – e power – t + e power – t sin s, fine. Now we need to find the t and s as functions of x and y using the first 2 equations, x = t + s and y is = t + s + s square. It is not clear to me how to get it, but let us ask whether it is possible to get at all.

So, therefore, here it looks like it is possible to get. So, t + s is actually x, therefore, x square is equal to y – x, therefore, s = root y – x, because s is positive that I am not taking minus so, root y – x. So, once you know s, t can be obtained from here x – s that is where x – root y – x. So, it is possible to get. And then substitute for t and s in this formula for z, you will get a solution. It is the solution.

And now question is, we have got the formula, ask what is the domain on which it is defined? First of all, you need that y – x should be positive because it is square root that is it. Everything else is fine. So, y – x is positive that is the restriction which is y is bigger than x. So, that means it defines a solution in this domain x y in R 2 such that y is bigger than x.

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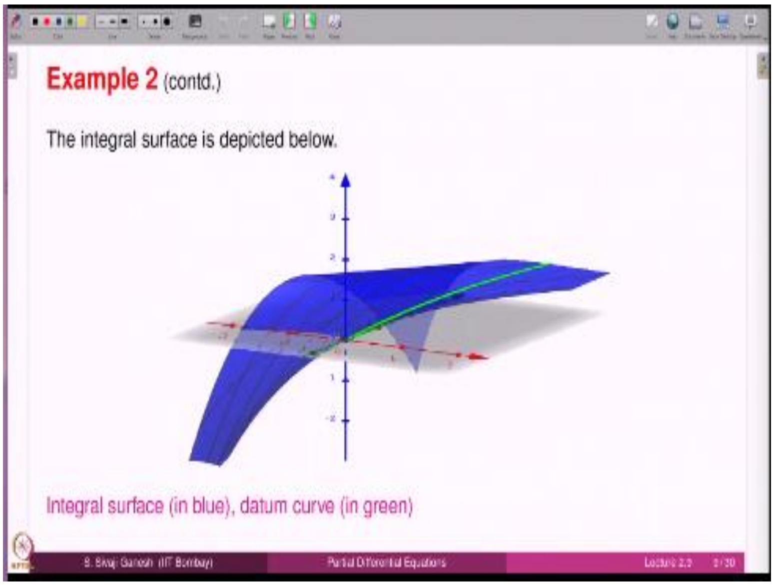

So, this is the integral surface. It is in blue. Datum curve is in green. So, along this line, y = x, the formula has a problem, right? So, you will see that corresponding trouble here when y = x on the line y = x.

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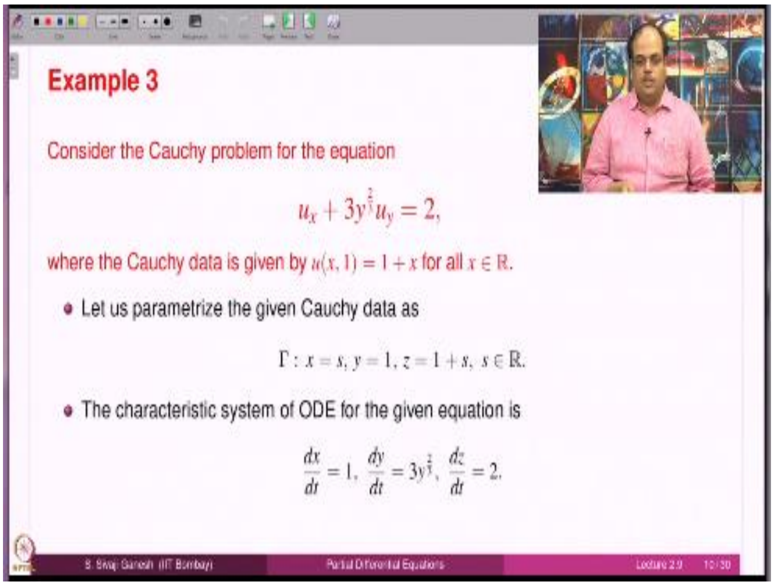

Let us look at another example. In this example, what is happening is u x + 3 y power 2 by 3 u y = 2. Now, a is 1, b is 3 y power 2 by 3, this is not a C 1 function. Our theorem requires C 1 function, right? It is not a C 1 function. Let us see what happens. c, of course, is 2, constant, no problem. Cauchy data is u of x 1 = 1 + x. So, first thing as always is to write gamma in the parametric form, is this and characteristic system of ODE is this. Now, we need to solve the system of ODE with initial conditions.

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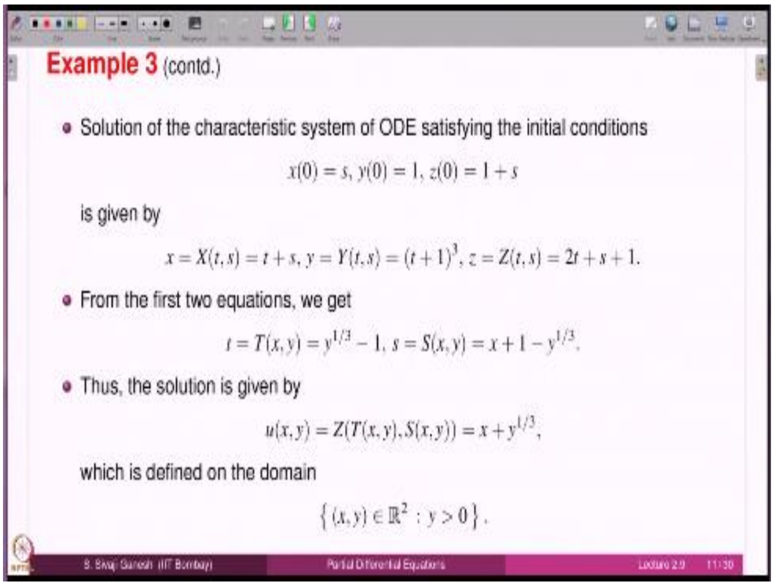

This solution is this x = t + s; y = t + 1 whole cube, z = 2 t + s + 1. Now, for the first 2 equations, we get t = y power 1 by 3 – 1 and s = x + 1 – y power 1 by 3. In fact, for t, you use this equation, because it does not have s. So, from here, we get t; once you get t, substitute here, we get for s that is why we got this. Now, go back and substitute in this formula, x + y power 1 by 3.

And where is it defined? It is defined everywhere. But then actually ever R 2 because this is just cube root of y. Cube root of any real number makes sense. But the problem here is that it is not differentiable at y = 0. So, we have to choose either y positive or y negative. We have chosen y positive.

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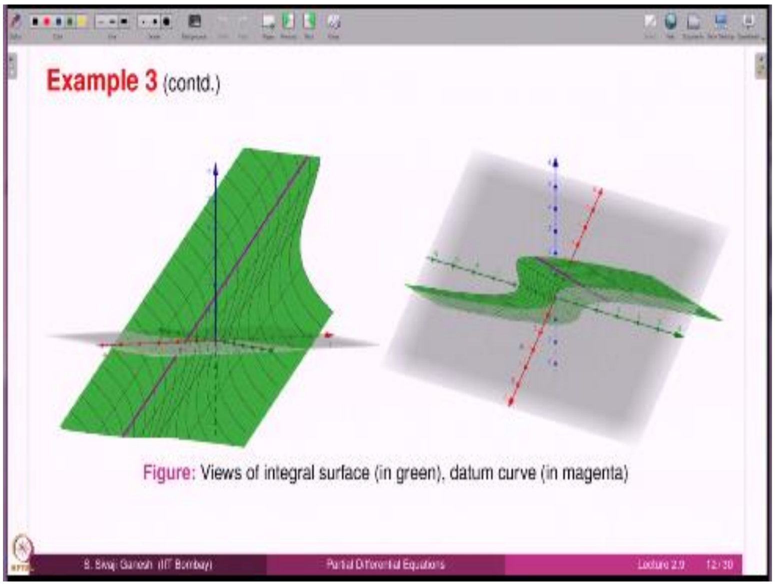

And these are the views of the integral surface with the different orientations. Remember, always the axis, red is x, green is y and the blue is z axis. So, you see that some steeping is happening here, around y = 0; should happen, because y power 1 by 3 is there. It is not differentiable. Something, it should be reflected in the picture.

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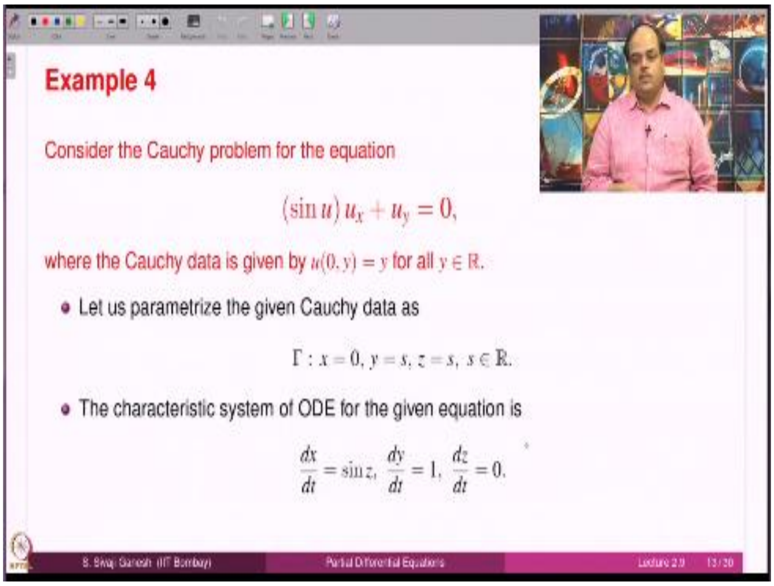

Let us look at another example. This is a very non standard example. This was constructed with something else in mind. And but it turned out to be, it is a very good example. It is in Quasilinear equation. So far, we have considered only a linear equation. This is Quasilinear equation. sin u u x + u y = 0. Here a and b, a is sin u. If a is 0, b should be nonzero, but b is always 1. So, it is, a square + b square is nonzero, fine.

Cauchy data given is u of 0 y = y for all y in R that is all. Parameterize a Cauchy data, 0, s, s; s in R. Characteristic system of ODE is this. Now, because of the Quasilinear nature, equation for x actually involves z now, whereas y and z does not involve any other variables. So from here, you can see that along in characteristic curve, z is constant because dz by dt is 0 and what is; dy by dt = 1 means y = t plus constant.

Therefore, because a t = 0 should be s, y is t + s, z is constant and that constant has to be s. Therefore, you put the s here and integrate this.

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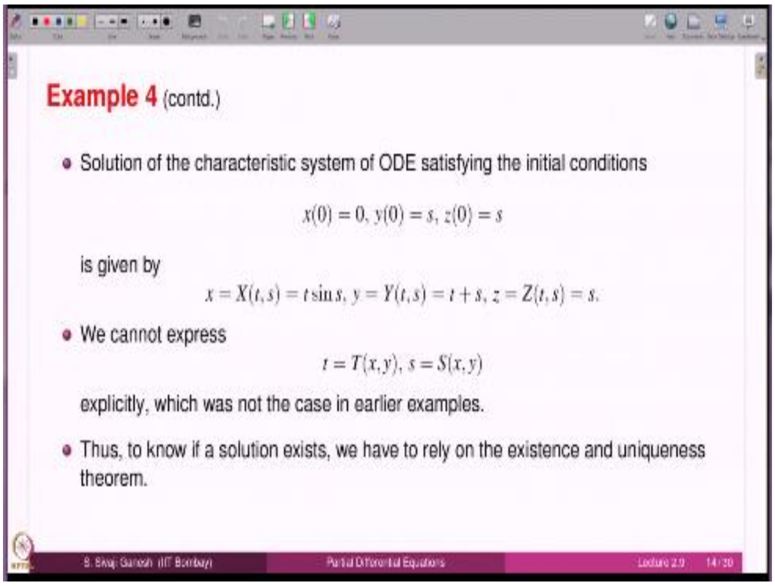

So, we get sin s into t and t = 0, it should be 0. So, this is a solution. y is t + s, z is s. Now, using x and y equations, we have to get an expression for t and s. But, you see, I do not think, it is possible because t sin s is there. This is t + s. So, this is okay nice t + s but there is a sin here. So, we cannot express, I cannot express explicitly, then I asked, is it possible for anybody to express at all? Which means, is the inverse function theorem applicable?

We will check that. This was not the case so far. In all earlier cases, we could solve, maybe it is a bad function of x and y; it does not matter, but we could explicitly solve. Here explicitly, we are not able to solve, fine. So, to know if a solution exists, we have to rely on the existence uniqueness theorem. Now, we have no choice.

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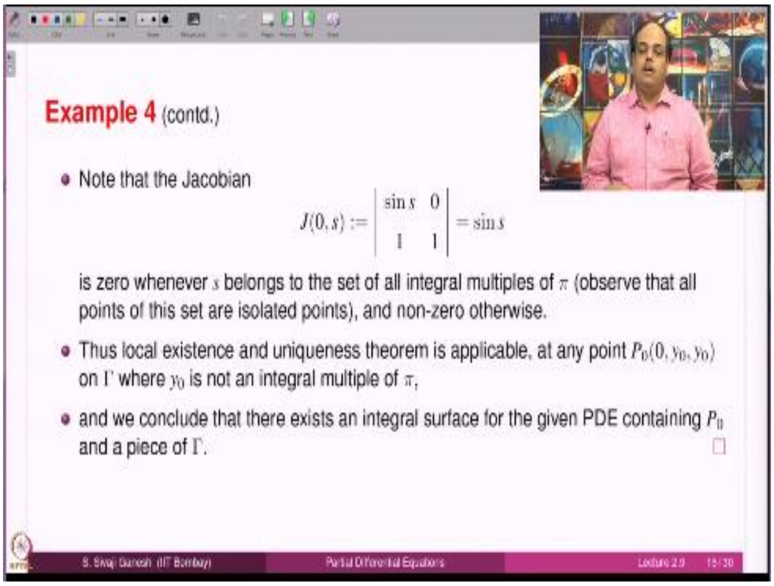

So, in this example, we are not going to get explicit form of the solution because we are unable to inverter. We are unable to write t and s as functions of x y. So, when you look at the whether it is possible at all, the J 0 s turns out to be sin s. Of course, sin s is 0, whenever s is a multiple of I, all integral multiples of I. Other than that, it is always nonzero. So, if s is equal to k I for some k integer, this Jacobian is 0.

If it is not like a pair for some k, then is always nonzero. These are all isolated points. So, Jacobean if you remember, we have pointed out the ways of failure of transversality condition and there, we said that it is a possibility that you have a sequence s n along which J is 0, but here and converging to some point. Here, it is not happening. These are all isolated points. There is no convergent subsequence of these multiples of I.

Otherwise, Jacobean is nonzero. Therefore, local existence and uniqueness theorem is applicable whenever s is not a multiple of I. So, in terms of y 0, y 0 is not an integral multiple of I. And we conclude that there exists an integral surface for a given PDE containing P 0 and a piece of gamma. Of course, question remains what happens when s is an integral multiple of I that is to be explored.

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So, now, let us look at some examples which illustrate the local nature of solutions to the Cauchy problem.

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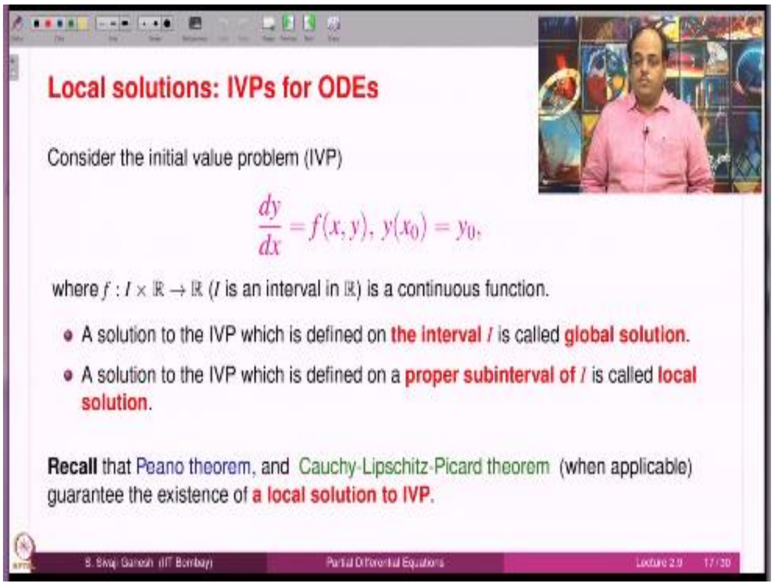

Before that, let us revise the notion of local solution. In initial value problems for ODEs, this is the equation, we consider dy by dx = f of x, y and y x 0 = y 0, this is initial condition. So, both equation and initial condition together is called initial value problem called IVP in short. Of course, we need to assume something on f. Let us assume that f is a continuous function.

Now, a solution to the IVP which is defined on the interval I is called global solution. What is I? I is here. This ODE makes sense for x in I. And if you are a solution, which is different for every x in I, we call it global solution. Imagine, it is not the case and solution is defined only on a subinterval of I, the proper subinterval of I, then that is called local solution. Now, recall that Peano’s theorem and Cauchy Lipschitz Picard’s theorem whenever it is applicable, always guarantee the existence of a local solution to IVP.

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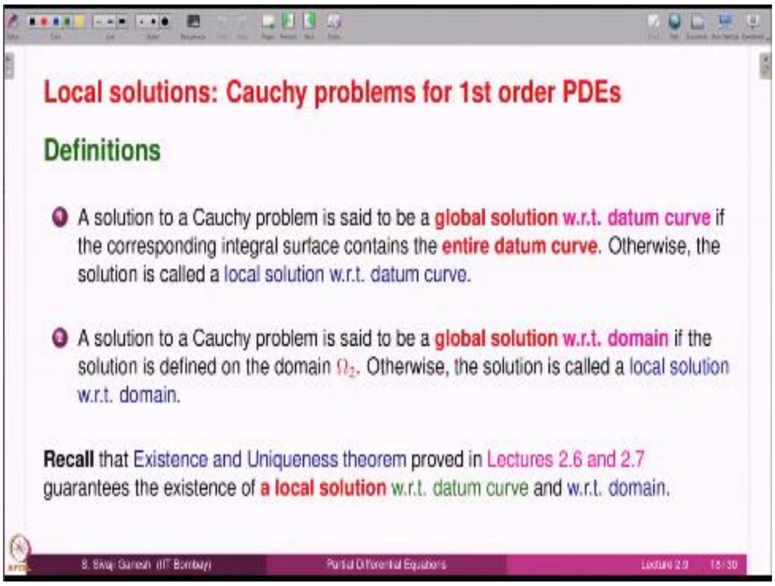

They do not talk about global existence. There are other theorems about global existence and there is a full understanding of what happens if a local solution can be extended to make it a global solution. If you fail somewhere, what are the precise reasons? Why you cannot extend it to a global solution? So, that is very much understood for initial value problems or ODEs, but that is not the case in my opinion for partial differential equations, I have not come across such results.

Let us now define for partial differential equations. A solution to a Cauchy problem for a Quasilinear equation, it can be for any equation, first order PDE because we are in this first order PDE setup and these are tutorial on Quasilinear equations, we can as well assume for Quasilinear equations, otherwise concepts are quite general. So, a solution to a Cauchy problem is said to be a global solution if, this is where something comes with respect to datum curve.

If the corresponding integral surface contains the entire datum curve, you have a solution. Then look at the integral surface z = dx y, entire gamma if it is on that, we say the global solution with respect to datum curve. Otherwise, the solution is called local solution with respect to datum curve. We have another related notion, a solution to Cauchy problem is said to be a global solution with respect to domain, with respect to domain if the solution is defined on the domain omega 2.

What is omega 2? Omega 3 is the set on which the Quasilinear equation was defined, the coefficients a b c. They are defined on omega 3. Projection of omega 3 to x y plane is omega 2. So, you would expect that solution should be defined throughout omega 2. If it is so, we are happy and we will call it global solution with respect to domain. Otherwise, solution is called local solution with respect to domain.

Recall the existence and uniqueness theorem proved in lectures 2.6 and 2.7, they guarantee, the theorem guarantees the existence of a local solution with respect to datum current and with respect to domain, both.

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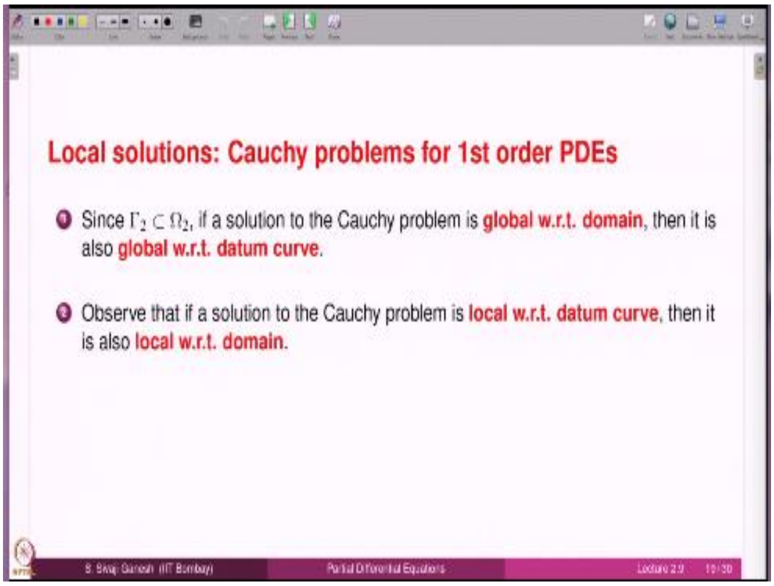

Now, since gamma 2 is a subset of omega 2, if a solution to the Cauchy problem is global with respect to domain that means, it is defined throughout omega 2, it is also different to a gamma 2; gamma 2 is a projection of gamma. So, it should be global with respect to datum curve. Observed that if a solution to the; Cauchy problem is local with respect to datum curve, then it is also local with respect to domain.

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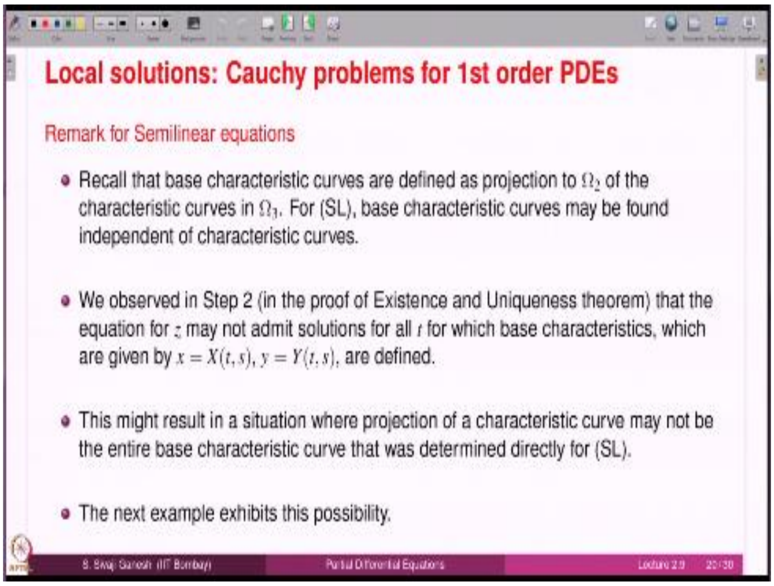

Now, this remark applicable for semilinear equations. Recall that the base characteristic curves are defined as projection to omega 2 of the characteristic curves in omega 3 in the context of general Quasilinear equation. But in a semilinear equation, what happens is that base characteristic curves can be found out independent of the characteristic curves because the equations governing the base characteristic curves namely dx by dt = a and dy by dt = b involve only x and y; a and b are functions of x y only. It does not depend on z.

Therefore, base characters can be found independent of characteristic curves. Now, we observed in step 2 in namely in the proof of the existence uniqueness theorem, there, we observed that the equation for z may not admit solutions for all t for which base characteristics are defined simply because the dz by dt is a nonlinear equation. For a general semilinear equation, dz by dt is a nonlinear equation and solutions to nonlinear equations, as we said, as a rule are only local solutions.

So, it can even cut out some portion of this base characteristic curves. So, this might result in a situation where projection of a characteristic curve may not be the entire base characteristic curve that you already found otherwise. Let us assume all these curves are the longest possible things that we have found. We will see an example. It would be obvious.

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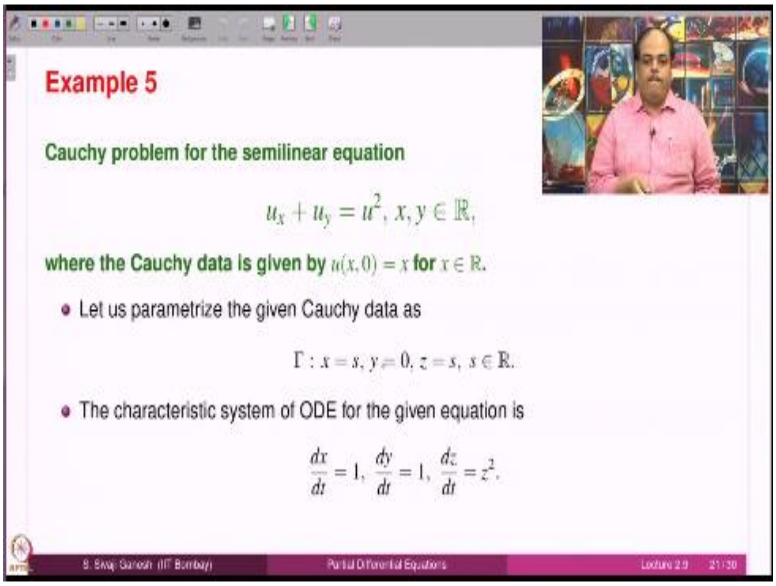

So, the next example exhibits this possibility. u x + u y = u square and Cauchy data is u x 0 = x. I think, this is the simplest complicated semilinear equation, because u square is the first equation that we learn even in ODE dy by dx = y square. That is the first nonlinear equation that we will come across in first order ODEs. So, let us parameterize a given Cauchy data. x = s, y = 0, z = s, s in R.

The characteristic system of ODE is dx by dt = 1, dy by dt = 1, dz by dt = x square. Now, when we compute the base characteristics, x = x of t s = t + s. Because it is t plus constant, it has to be t + s. y is just t.

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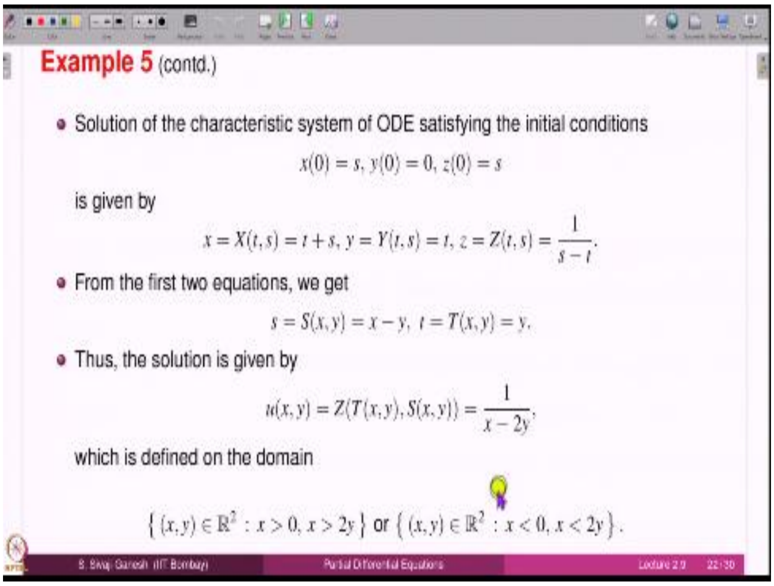

Of course, Z also can be integrated, we get 1 by s – t. From the first 2 equations, we can solve for t and s in terms of x and y. Because these are just linear equations, very easy to solve, u equal to this, 1 by x – 2 y. It is defined on the domain whenever x – 2y is nonzero. So, we have to stick to one of them. Because I do not want my domain x = 2 y happening. So, x is greater than 2y one option; x is less than 2y is another option, but when x is greater than 2y, I take x positive or x is less than 2y, I take x negative.

Why is that? Because only this domain is in contact with the datum curve on which, datum curve, a projection of the datum curve is intersecting only this part or this part. It is not intersecting uniformly x bigger than 2y.

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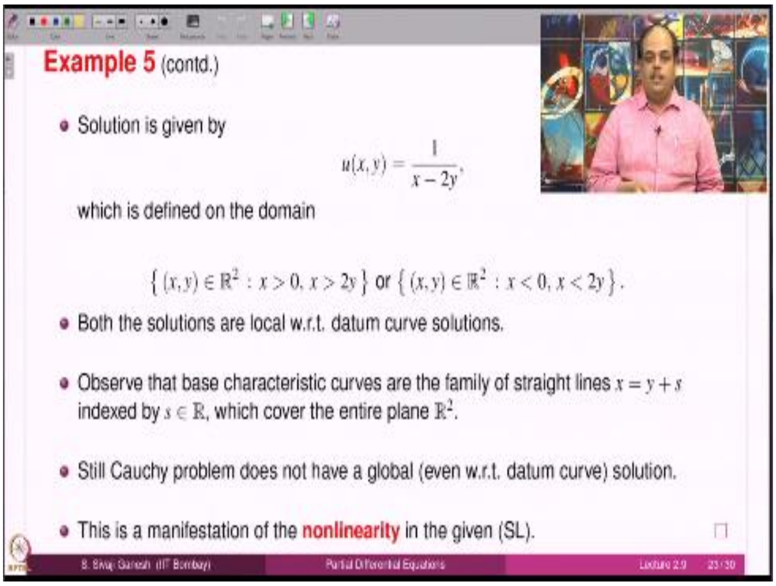

Now, both solutions are local with respect to datum curve solutions, so, local solution with respect to datum curve. Observed that base characteristics are the family of straight lines x = y + s. See here, y = t. So, y + x, x – y = s that is a family of base characteristics and they fill entire claim. Still Cauchy problem does not have a global with respect to domain. Forgot about it. Even with respect to datum curve does not have a global solution. So, this is a manifestation of the non linearity in the right hand side namely, u square in the PDE.

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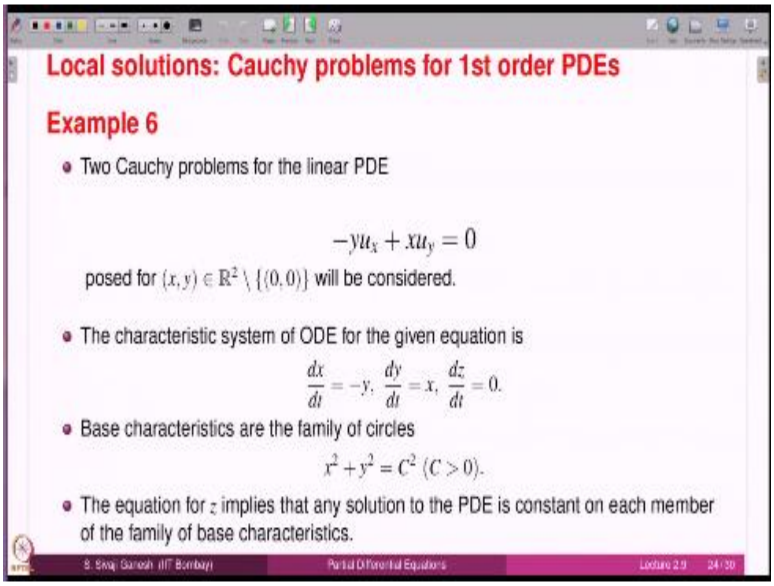

Let us look at example 6. Here, we consider 2 Cauchy problems for the linear PDE. Linear PDE, minus y u x + x u y = 0 post, of course, I do not want coefficients x and y to vanish simultaneously, which happens at the origin. So, I remove the origin. On that domain I consider this equation and the characteristics system of ODE can directly write down and base characteristics because of this nature, dx by dt is minus y and dy dt is x.

So, if you compute one more derivative, d 2 x by dt square is minus dy by dt that is equal to minus x. Therefore, dy dx by dt square + x = 0. Similarly, one can do with y. So, solutions of x and y are going to be solutions of y double dash + y = 0, which are combination of cosine and sine. And the trajectories will be circles. So, base characteristics are the family of circles x square + y square = c square.

Since, it is positive number, we write c square because we do not want write true c in some other place, so, we write c square and always make sure make mention this that c is positive. So, that we do not get confused later. Now, the equation for z implies that any solution to the PDE is constant along each of the base characteristics because z is constant. dx by dt is 0 on solutions of this that means on each circle, the solution is constant.

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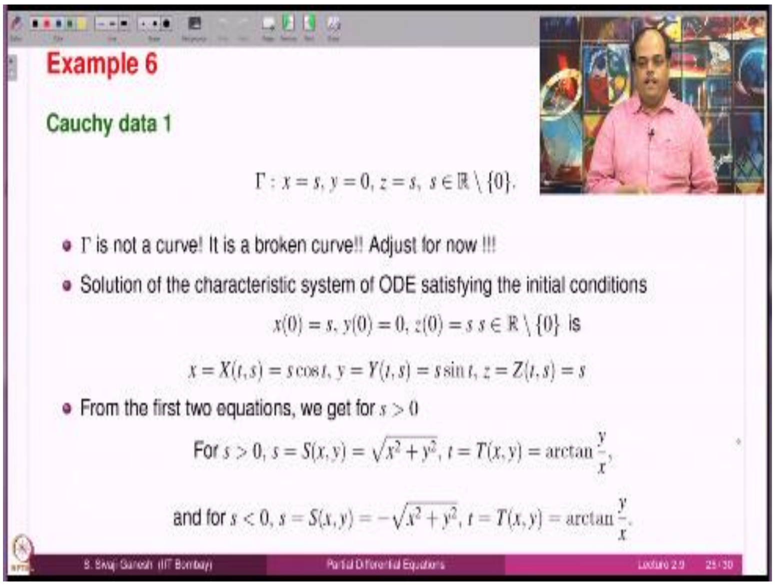

So, if you know at one point on the circle, what is the value of the solution? Then it is the same constant throughout on the circle. Cauchy data 1, we consider x = s, y = 0, z = s and s, R – 0. This is not that what we like but it is okay. We continue with this the computation. Gamma is not curve obviously, it is 2 pieces. But never mind, I just for now, I guess one can create similar conditions, but then they look more complicated than this, this is very easy for computation.

So, let me allow me this. So, I am going to compute with this. So, this is the initial conditions and then solutions as I said, cos t, sin t the feature and z is s. Now, from the first 2 equations for positive s, I can eliminate or I can express, I think, I should not use a word eliminate, I can express s and t in terms of x and y, I get this and for s negative, I get this expression for s that is why negative, for s negative, positive; for s positive, t remains same.

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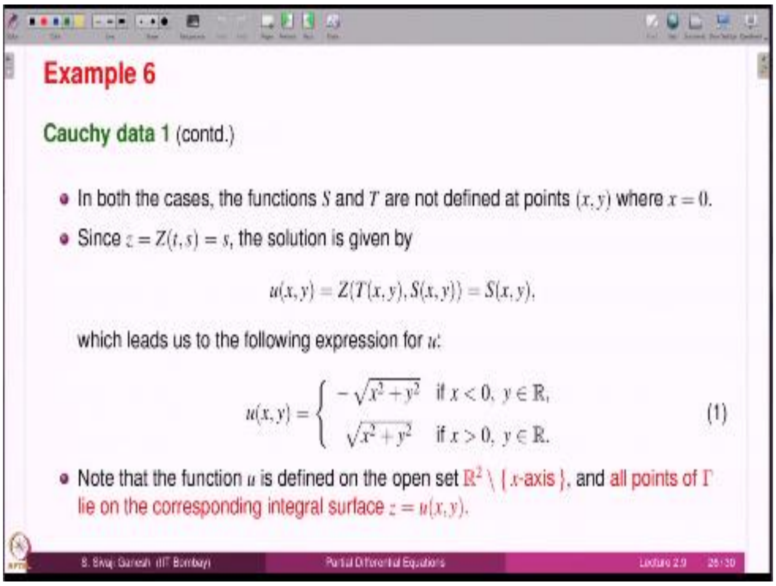

So, in both the cases, the function s and t are not defined at points, where x is 0, because when x is 0, there is a trouble which not defined. So, since z = s, the solution is given by u x y = s x y. Therefore, u x y equal to this if x is negative, x positive. Now, it is defined on R 2 – x axis and all points of gamma lie on the corresponding integral surface. So, this is global with respect to datum curve, not global with respect to domain because it is not defined everywhere and R 2 – 0,0. No, it is defined R 2 minus x axis.

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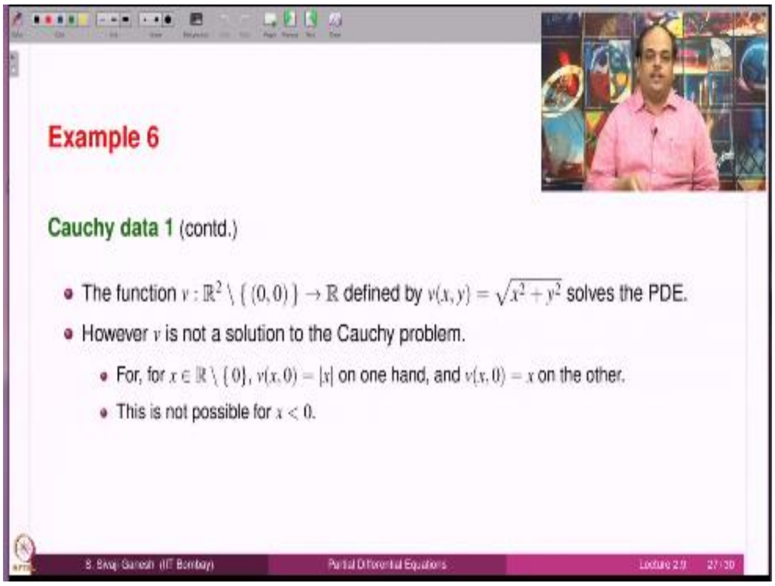

Now, another function v defined on entire domain R 2 – 0, 0 given by this formula, it is also PDE, a problem is; it is not a solution to the Cauchy problem, why? For x in R – 0, v x 0 is root x square that is mod x on one hand, but v x 0 has to be x on the other hand because that is the initial condition. So, both cannot happen. Particularly for x negative, it is not possible.

So, it is not a solution to the Cauchy problem throughout. Yes, if you restrict for x positive side, then yes.

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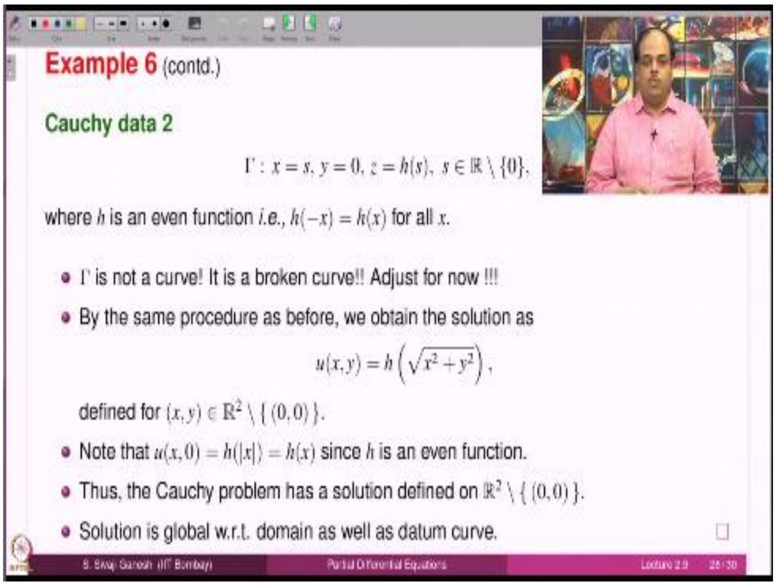

Cauchy data 2: Here, we look at x = s, y = 0, z = h s. So, it is like a correction for the previous thing. Here, mod x is an even function. And here given data is not even function that is why there is a problem. So, now I am going to change it to even function, h s, where s is an even function. Still the same problem, but again, I just same procedure as above, we get u x y equal to a function h of root x square + y square. This is defined whenever x y is different from origin and a smooth function.

If h is differentiable C 1 function, then this being a composition of C 1 functions, it is C 1 function and it will give a solution. See, now, u x 0 is h of mod x and that is equal to h s because it is an even function. Thus, Cauchy problem has a solution defined on R 2 minus origin. So, it has a global with respect to domain solution therefore, global with respect to datum well. Solution is global with respect to both.

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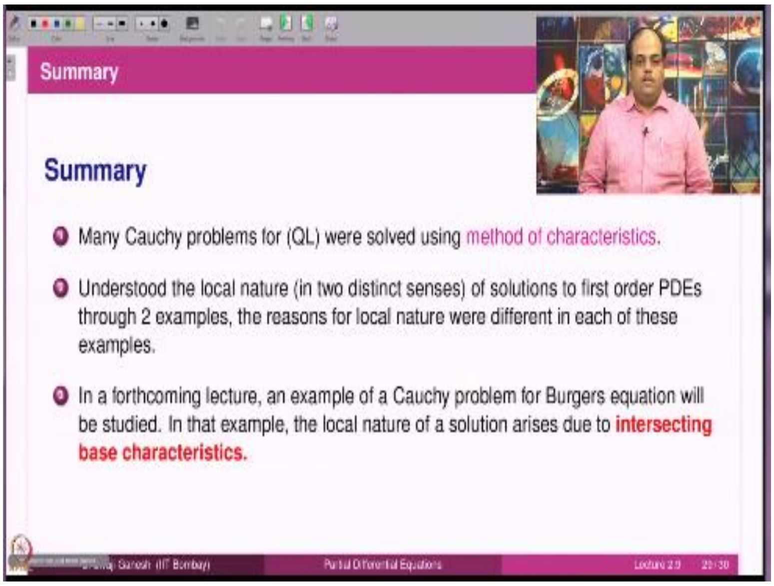

So, let us summarise. Many Cauchy problems are solved using methods of characteristics. Understood the local nature of solutions to first order PDEs; this, we understood local nature can be in the sense, two different senses. One is with respect to the datum curve. Another is with respect to the domain. Of course, through 2 examples, we have understood. And reasons are different in each of these examples.

And in a forthcoming lecture, an example of a Cauchy problem for Burgers equation will be studied. In that example, the local nature of a solution arises due to intersecting base characteristics that we will see in a forthcoming lecture.

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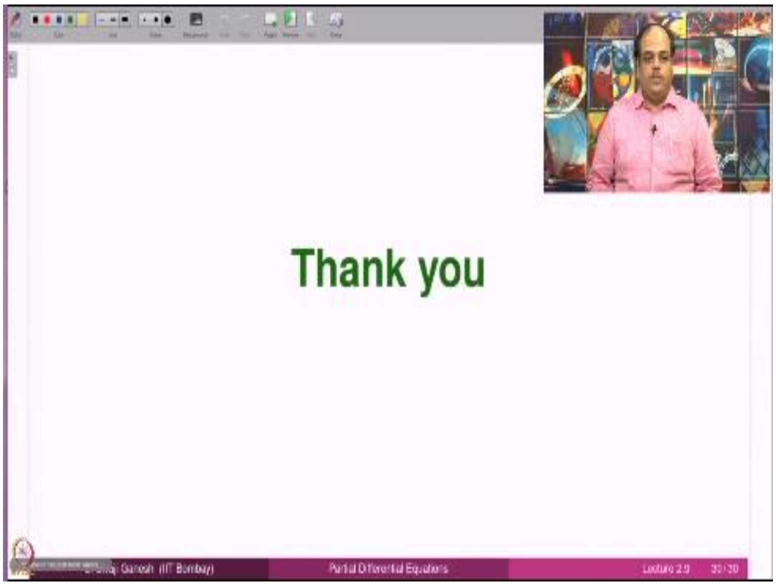

So, with this, we come to the end of Quasilinear equations. And we will then start with the general nonlinear equation once again Cauchy problem. We will be making regular comparison to what we did in the Cauchy problem for Quasilinear equations. You might say that Cauchy linear equation is a special case of a general equation. Why do 2 times? Why repeation? Why do not you do the general thing first?

No, because, Quasilinear equation always, when you do not understand something, you would like to understand with a special case, Quasilinear is one such special case where things are easily understood. Now, we try to extend these ideas to the general case. That is what is the natural progression; in solving problems in mathematics. Thank you.