Introduction to Algebraic Topology (Part - II)
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Lecture - 07
Compactly Generated Topology on Products

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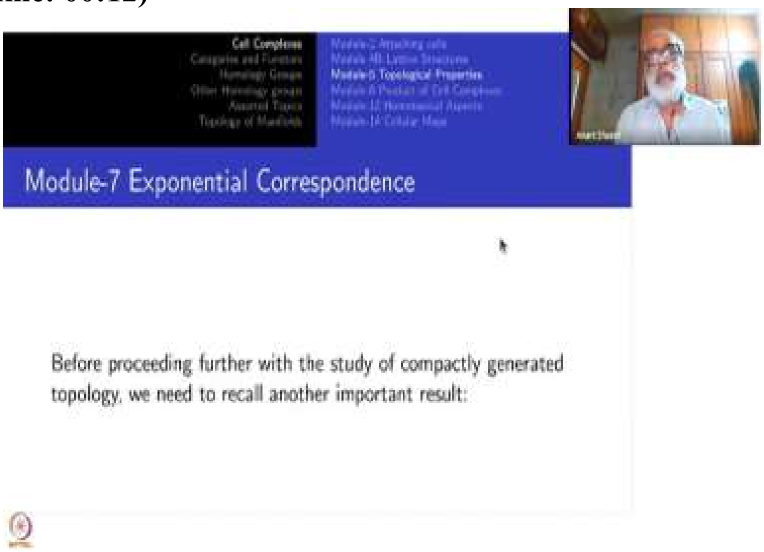

So today module 7. So we have been studying various topologies like induced topology especially compactly generated topology and so on. And one of the important things in studying function spaces is the exponential correspondence on the function spaces. Basically these things were done in part 1 very elaborately. Nevertheless because of the importance of this part, I will recall them as much as needed for our immediate purpose here.

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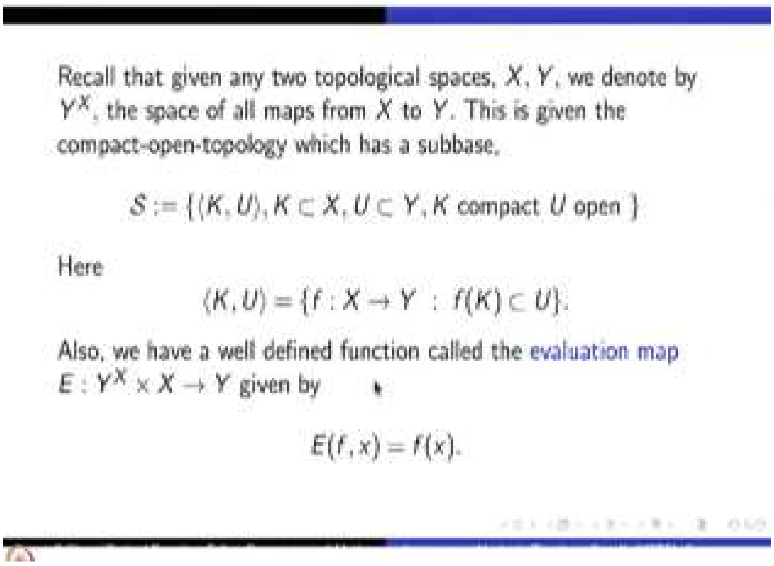

So, first let us recall what is the meaning of this compact open topology on the function spaces? Given any two topological spaces X and Y, we will denote by Y^X the space of all maps from X to Y, maps means continuous functions from X to Y. This is given the compact open topology which has a subbase, corresponding to each compact set K of X and an open subset U of Y, you define certain subsets $\langle K, U\rangle$ of Y^X and take all of them as a subbase.

So, what is this $\langle K, U\rangle$? It is all continuous functions f from X to Y which take K inside U, that is, $f(K)$ must be inside U; then you put f inside this $\langle K, U\rangle$. You take all such $\langle K, U\rangle$ okay? They are subsets of Y^X and they form a subbase for a topology on Y^X, okay? $\langle K, U\rangle$ is the set of all function f from X to Y such that $f(K)$ is contained inside U. A subset Y^X is open in this topology if and only if it is an arbitrary union of finite intersections of subsets of the form $\langle K, U\rangle$. i.e., $\langle K_1, U_1\rangle, \langle K_2, U_2\rangle, \ldots, \langle K_n, U_n\rangle$ and you take the intersection that will be an open set okay. So, arbitrary union of such things will be also open set. If we take only finite intersections like that, they will be form a base for this topology, okay?

There is an obvious map E from $Y^X \times X$ to Y given by $E(f, x)$ going to $f(x)$. If you fix one x, then f going to $f(x)$ is nothing but the x^{th}-coordinate projection. I would not like to call it that way though. Because now you are not thinking of Y^X as a subspace of product space Y taken X number of times. There is also the product topology topology no doubt. But now we are concentrating on the so called compact-open-topology. So, if you fix x, then f going to $f(x)$ is obviously continuous even in the product topology. But what is important is that: this E from $Y^X \times X$ to Y is a continuous function. That is what we want to have, okay? That is the first part of this theorem.

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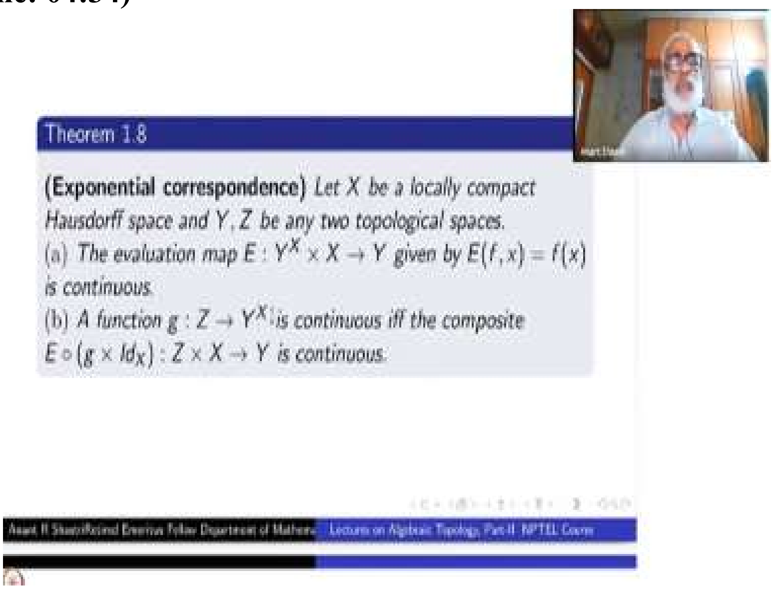

This map E is aptly called the evaluation map, Okay? What is the hypothesis on X? X is locally compact Hausdorff space. Y and Z could be any two topological spaces. Then the first part here is that E is continuous. The second part is corresponds to the name `exponential correspondence'. A function from Z to Y^X, where Z is any topological space is continuous if and only if $E \circ (g \times Id_X)$ from $Z \times X$ to to Y is continuous.

Notice if that g is continuous then $g \times Id_X$ is continuous. Therefore if E is continuous then the composite is continuous. The point here is in the converse, namely, if the composite is continuous why g should be continuous? In general it may not be so. The theorem says that it is continuous under this hypothesis, namely, X is locally compact and Hausdorff.

Proofs are not very difficult. Nevertheless, let us go through them so that you will become familiar with the concept of this compact open topology and the space Y^X etc.

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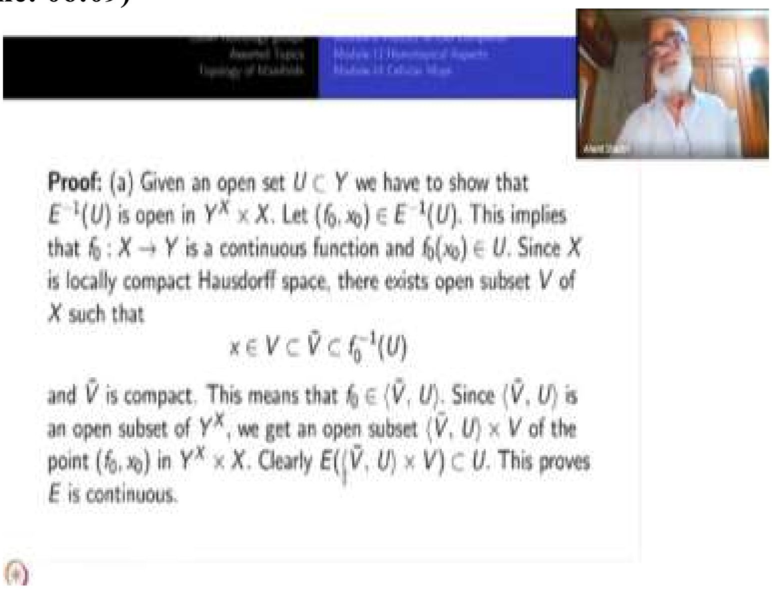

So, the first part is to show that E is continuous. Start with an open subset U inside Y and you want to show that $E^{-1}(U)$ is open in the product space of Y^X and X. Take a point (f_0, x_0) in $E^{-1}(U)$. What is the meaning of that? That means f_0 is a continuous function from X to Y and $f_0(x_0) = E(f_0, x_0)$ is inside U. But now U is open and f_0 is continuous means that there is neighborhood of x_0 which is taken inside U by f_0.

But now X is locally compact Hausdorff space. So, we can actually take an open subset V such that x_0 is inside V contained inside $\bar{V}$ contained in $f_0^{-1}(U)$, and such that $\bar{V}$ is compact. Inside every open neighbourhood, you can find a compact neighborhood by the local compactness hypothesis, okay? So, what is the meaning of this? This means that now $f_0(\bar{V})$ is contained inside U which is the same thing as saying that f_0 is in $\langle \bar{V}, U\rangle$.

$\bar{V}$ is compact, U is open and hence this is actually one an element of the subbase. Therefore, $\langle \bar{V}, U\rangle \times V$ is an open subset $Y^X \times X$, which will be neighborhood of (f_0, x_0). Now you look at $E(\langle \bar{V}, U\rangle) \times V$, okay? Take f here and take a point x, then $f(x)$ will be inside U why? Because x is inside V and hence inside $\bar{V}$ and $f(\bar{V})$ goes inside U. Therefore $E(f, x)$ will be inside U. This proves E is continuous. That is part (a), alright?

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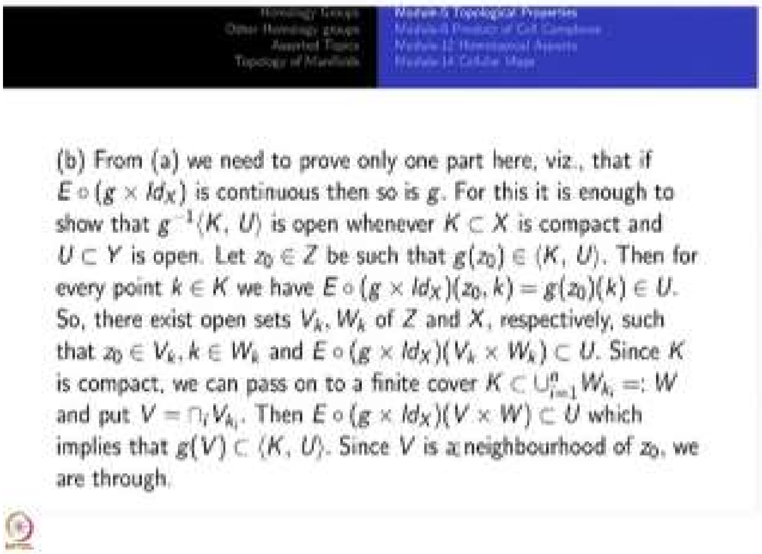

Let us go to part (b) alright? There are two ways. One way I already told you if g is continuous $g \times Id_X$ is continuous the composite with E will be continuous because we have just proved that E is continuous. So, now let us prove the converse. Okay? Now I assume that this composite is continuous. Then I want to show that g is continuous. For this, it is enough to show that g^{-1} of any of these sub basic open sets $\langle K, U\rangle$ is open, where K is compact and U is open. Okay?

Take only subbasic open sets, inverse image of those things are open will mean g is continuous. Okay? So, let z_0 belonging to Z, Z is an arbitrary space you remember. So let $g(z_0)$ belongs to $\langle K, U\rangle$. That is the meaning of saying that z_0 is in g^{-1} of that. We have is $E \circ (g \times Id_X)$ of (z_0, k) which is equal to $g(z_0)$ operation on k will be inside U for every $k \in K$.

By continuity of $E \circ (g \times Id_X)$, there exists open sets V_k, W_k of Z and X respectively, such that this z_0 is inside V_k and k is inside W_k, and $E \circ (g \times Id_X)$ of this $V_k \times W_k$ is contained inside U. So, this is the continuity of this composite function $E \circ (g \times Id_X)$, okay. Since K is compact so we can pass on to a finite cover. K is contained in the $\cup_{i=1}^n W_{k_i}$, because $\{W_k\}$ covers K as k ranges over all of K, since K is compact.

So, I extract a finite subcover for K. We call that as W. Correspondingly, I take V equal to intersection of these $V_{k_1}, \ldots, V_{k_n}$. It follows that $(E \circ (g \times Id_X))(V \times W)$ is contained inside U, okay? That just means that $g(V)$ is contained inside $\langle K, U \rangle$, okay? Since V is a neighbourhood of z_0, we are through.

So I have found neighbourhood V here contained in $g^{-1}(\langle K, U \rangle)$, which is therefore open. That proves this exponential correspondence theorem, Okay? Now I have come to the study of the compactly generated topology.

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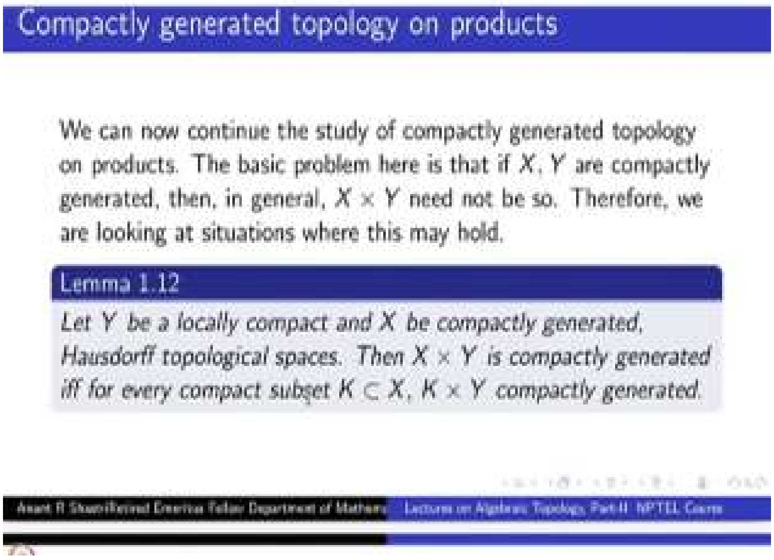

The basic problem here is that if X and Y are compactly generated, in general, the product topology need not be compactly generated. Therefore we are looking at situations where this maybe true. That means to put extra conditions on X and/or Y under which this may be true. However there is no if and only if statements, okay? So we will have to study whatever you know best we know, about various conditions okay.

So, the first lemma is: if Y is locally compact Hasudorff and X is compactly generated okay? (By the way, whenever I say compactly generated, all the time I assume the Hausdorffness also. So, both of them are Hausdorff spaces.) Then $X \times Y$ is compactly generated, if and only if for every compact subset K of X, we have $K \times Y$ is compactly generated. This lemma is only stepping stone for the results to come.

So, there is this extra hypothesis on the factor Y. Remember locally compact Hausdorff spaces are compactly generated. So, this is stronger condition than being just compactly generated okay?

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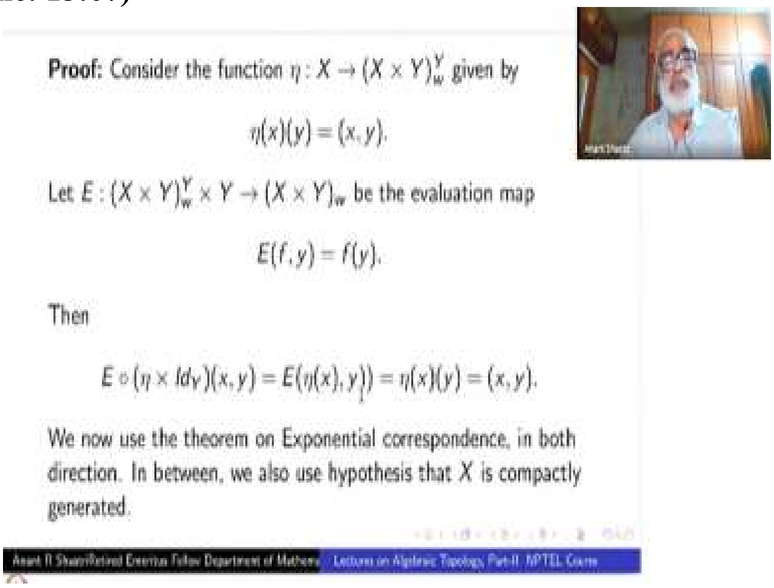

So a proof of this may be given by point-wise argument, going here going there going there and so on. But our exponential correspondence theorem helps you in proving this one somewhat elegantly, okay? Consider this function η from X going to $(X \times Y)^Y_w$, [here we take the product space $X \times Y$ and retopologise it with the weak topology, i.e., compatly generated topology and then take the function space $(X \times Y)^Y_w$ with the compact open topology. Remember that, we denote the weak topology by writing a suffix w.] Define $\eta(x)$ operating on y is equal to (x, y). Remember η is what? For each $x \in X, \eta(x)$ is the function from Y to $X \times Y$ which sends y to (x, y), okay? So this is the definition of η, okay?

Now look at the evaluation function from $(X \times Y)^Y_w$ to $X \times Y$. The base space in this case, instead of just X, is $(X \times Y)_w$, the exponent space is Y. So the evaluation map has its domain $(X \times Y)^Y_w \times Y$. Take a function f from Y to $(X \times Y)_w$ and a point y in Y, okay and evaluated f at y, you get a point in $X \times Y$. Because Y is locally compact Hausdorff space, by the previous theorem, this evaluation map is continuous, okay?

Now we will look at $E \circ (\eta \times Id_Y)$ operating on (x, y), Okay? This is equal to $E(\eta(x), y)$ which is equal, by definition if the evaluation map, to $\eta(x)$ operating on y which is nothing but (x, y). Therefore this composite function is continuous.

We now use the exponential correspondence theorem, in both directions, in an innovative way. Of course, we also use the hypothesis that X is compactly generated and Y is locally compact. This much hypothesis we have use okay.

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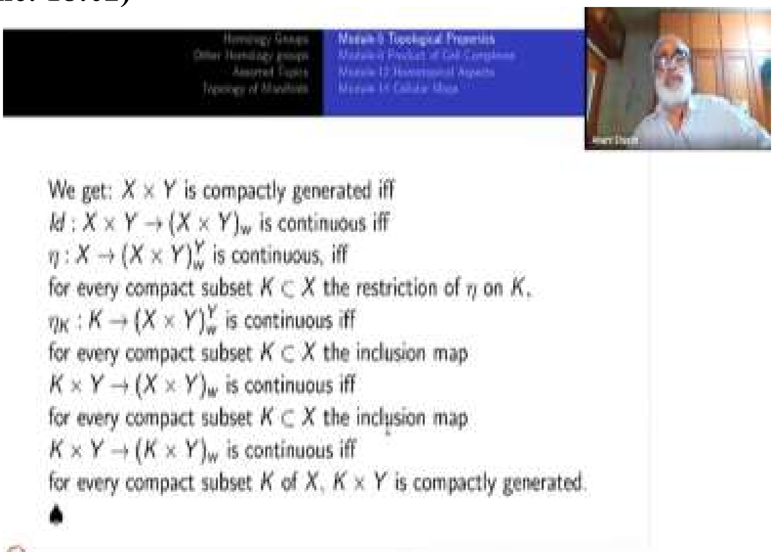

So, the statement we want to understand is that $X \times Y$ is compactly generated. This is true if and only if the identity map from $X \times Y$ to $(X \times Y)_w$ is continuous. We have seen that the identity map in the other direction is always continuous. So, identity map this way is also continuous iff $X \times Y$ is compactly generated, okay? So, the first step is just by definition itself.

The second step now is that this statement is equivalent to say that η from X to $(X \times Y)_w^Y$ is continuous. This is where the exponential correspondence is used, because this map corresponds to the identity map of $X \times Y$, right? $E \circ (\eta \times Id_Y)$ is the identity of $X \times Y$, which is continuous. Therefore η is continuous by the exponential correspondence theorem.

But now this is the same thing as saying that for every compact subset K of X, the restriction of η on K, let us call η_K from K to $(X \times Y)_w^Y$ is continuous, by the previous lemma, okay? So from an arbitrary space X, I have come to compact subspaces K of X, okay?

Now we go back, via exponential correspondence, this is true if and only if for every compact subset K of X, the inclusion map from $K \times Y$ to $(X \times Y)_w$ is continuous. Remember Y is locally compact, and inclusion map corresponds to η_K.

But then this inclusion map is taking value inside $(K \times Y)_w$. The codomain is the larger space $(X \times Y)_w$, but the subset $K \times Y$ with the subspace topology coincides with the weak topology $(K \times Y)_w$. Therefore, this inclusion is continuous iff the identity function from K cross Y to $(K \times Y)_w$ is continous, which in turn is equivalent to saying that $K \times Y$ is compactly generated for all compact subsets K of X. So this lemma is proved now.

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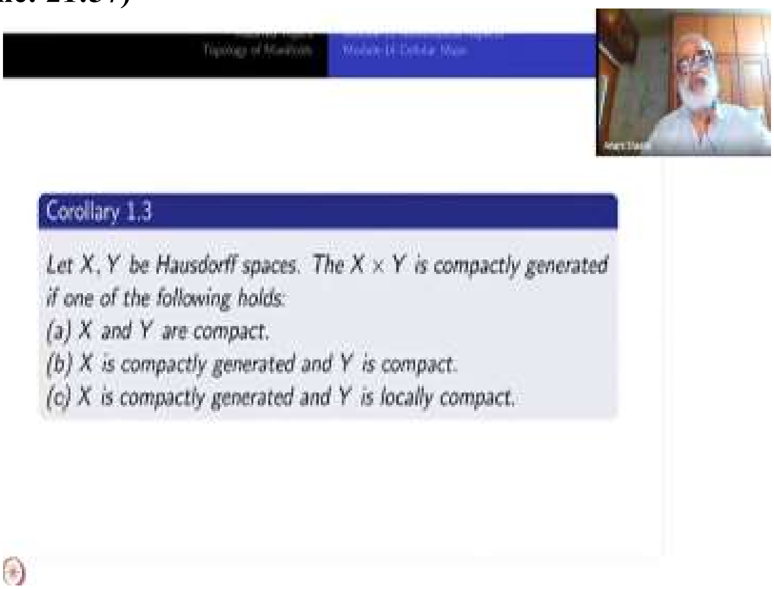

So now we keep using it beneficially to prove all these statements (a), (b), (c). They are all different statements, but somewhat similar. Common hypothesis is that X and Y are Hausdorff spaces. Conclusion is $X \times Y$ will be compactly generated under any one these easy to remember conditions.

(a) first condition is both X and Y are compact. No problem. The product itself is compact right? In a compact space, the compactly generated topology is the same topology. The first part does not need anybody any lemma, Okay?

Our final aim is part (c) in which instead of Y compact, Y is only locally compact. The first part was obvious. The second is also obvious but will needs some proof. Remember, in the lemma we had the statement if and only if something happens. So, we will prove that statement to prove part (c) here that is the whole idea. Okay?

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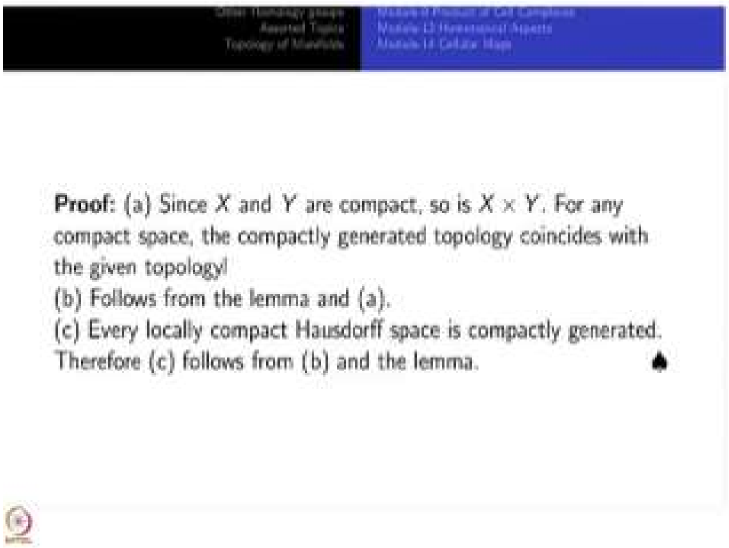

So, let us go through it again. Since X and Y are compact so is $X \times Y$. For any compact space compactly generated topology coincides with the given topology. That is part (a). (b) follows from the lemma and (a). See you start with the hypothesis that X is compactly generated and Y is compact, okay. Then what you have to do to use the lemma is that you take compact subsets K of X; cross with Y is compact and so is compactly generated by part (a). Since this is true for every compact subset of X, the product space $X \times Y$ is compactly generated by the lemma. So, the lemma gives you (b) immediately, okay?

Now we use this (b) and again the lemma. Now the role of X and Y will be interchanged to show that $X \times Y$ is compactly generated, okay? This time by taking compact subsets K of Y.

Every locally compact or Hausdorff space is compactly generated. Therefore (c) follows from (b) and the lemma. Interchange the role of X and Y. That is all a small trick but the true tool was exponential correspondence.

So, we have come to a stage where we can now study the topology of CW complexes namely when you take the product of CW complexes, i.e., when X and Y are CW-complexes, there is a canonical way of getting a product CW structure on $X \times Y$. The only problem is the topology on $X \times Y$, namely the product topology will be, in general, different from the CW topology, okay? So that is the point we want to study. We will do it next time. Thank you.