Introduction to Algebraic Topology (Part-II)

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Lecture – 06
Coinduced Topology

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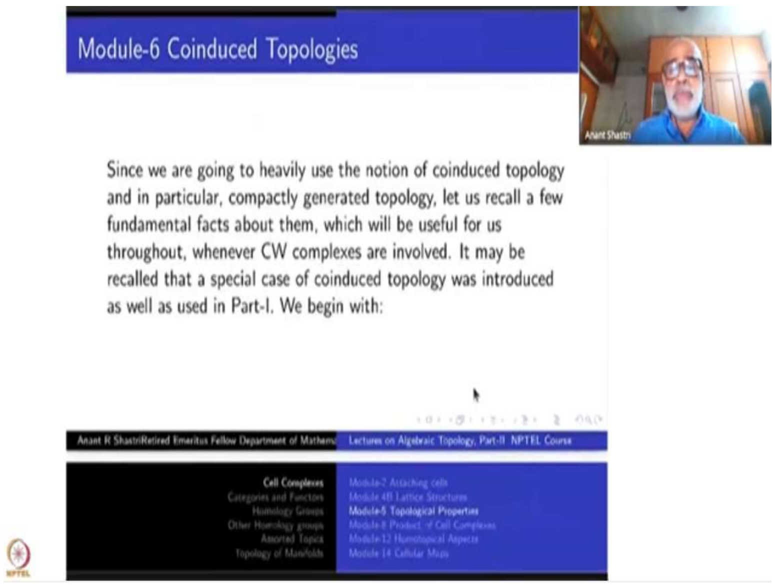

Since we are going to heavily use the notion of coinduced topology and in particular the so called compactly generated topology, let us recall a few fundamental facts about them which will be useful for us throughout the course and especially while dealing with CW-complexes. It may be recalled that a special case of coinduced topology was introduced as well used in part 1. So, we assume that you know these things, but let us recall these things in somewhat general fashion.

That is one of the reasons why the module 6 is released to you before module 5---we are going to use coinduced topology in model 5 okay. Yeah.

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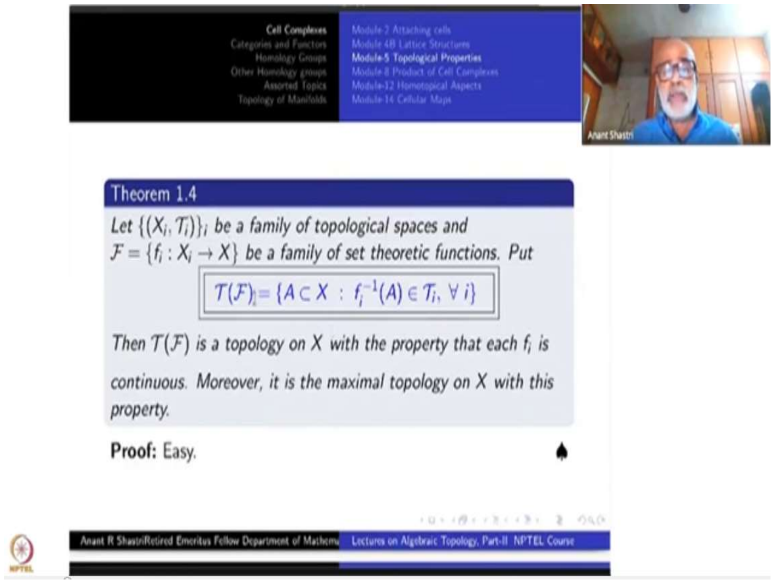

So, let us begin with the general theorem here, namely, start with a family of topological spaces. I shall denote them by $\{(X_i, \mathcal{T}_i)\}_i$, because, I want to specifically mention these topologies $\mathcal{T}_i$'s here. And look at a family of functions from X_i to X, denoted with $\mathcal{F}$. These are just set-theoretic functions first. I haven't taken any topology on X right now, okay?

Now, you put $\mathcal{T}(\mathcal{F})$ equal to all subsets of X such that for every i, the inverse image under f_i belongs to $\mathcal{T}_i$, i.e., inverse image must be an open subset in $\mathcal{T}_i$, for every i. $\mathcal{F}$ is such a collection. Then the statement is that $\mathcal{T}(\mathcal{F})$ is a topology on X with the property that each f_i is continuous. Moreover, it is the maximal topology on X with this property.

The proof is very straightforward and elementary. Straightforward means whatever you're supposed to do you have then straight, there is no tricky arguments here, okay?

So, for example, continuity of f_i follows because f_i inverse of any member of $\mathcal{T}(\mathcal{F})$ is open in $\mathcal{T}_i$ by the definition of members of $\mathcal{T}(\mathcal{F})$. So, it is very clear that if $\mathcal{T}(\mathcal{F})$ is a topology on X, then with respect to it, each f_i is continuous. Verifying that this is a topology and the maximality of this topology etc.

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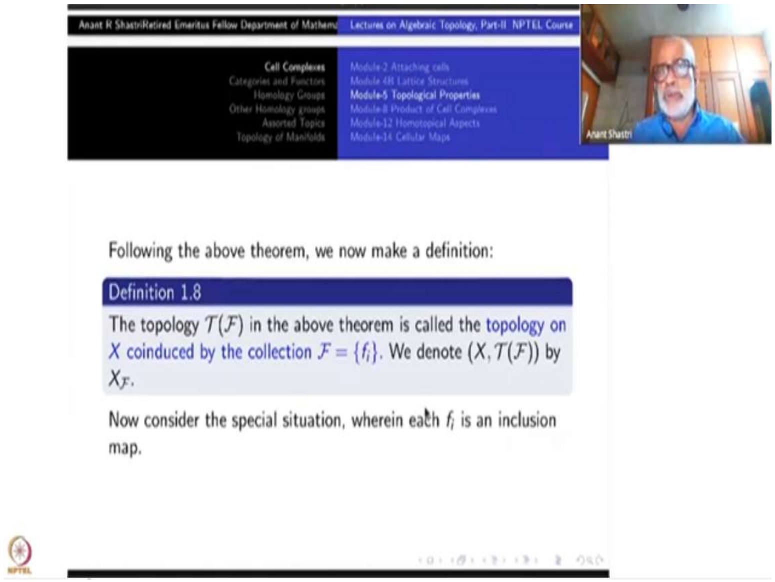

Following the above theorem we now make a new definition. The topology $\mathcal{T}(\mathcal{F})$ defined in the above theorem, is called topology on X coinduced by the collection $\mathcal{F}$. To indicate that it has something to do with this $\mathcal{F}$, we'll denote it by $(X, \mathcal{T}(\mathcal{F}))$ or simply by the shorter notation $X_{\mathcal{F}}$. Okay. Most of the time we will consider a special case wherein each f_i is an inclusion map of the sets, X_i to X, where X is a big set and all these X_i are subsets of X, with their own topologies. That is the situation that we want to apply this theorem.

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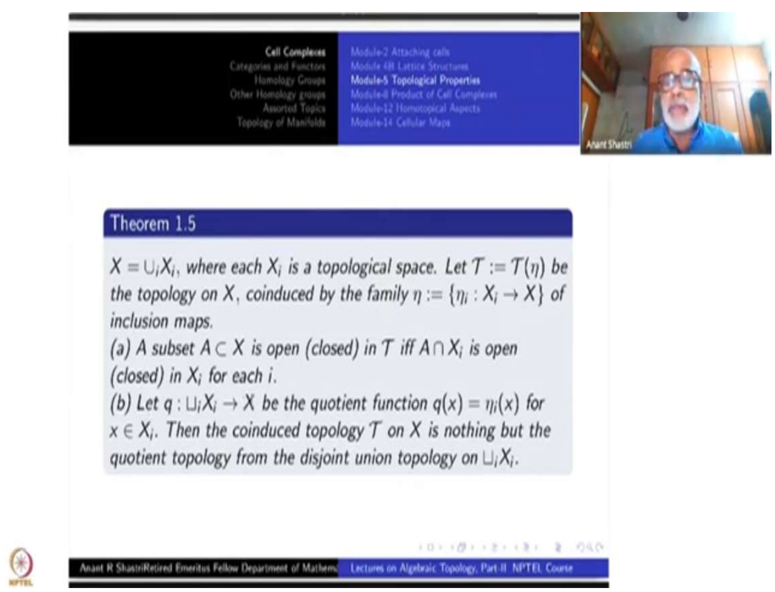

So, let X be the union of X_i's, where each X_i is a topological space. Let $\mathcal{T}$ equal to $\mathcal{T}(\eta)$ be the topology on X, where the family η consists of all inclusion maps η_i from X_i to X. I have assumed that the union is precisely equal to X here that is a special case here. A subset A of X is open (or respectively closed) in $\mathcal{T}$ if and only if $A \cap X_i$ is open (or closed, respectively) in X_i for each i. So, this is the first statement of the theorem.

The second statement says that if you take the disjoint union of X_i's then the ordinary union X will be automatically a quotient set, and the quotient function is a quotient map okay? So, that let us take the quotient map q which is nothing but inclusion map restricted to each X_i. Given any $x \in X$, it belongs to some X_i and hence q is surjective function. A surjective function is a quotient function set-theoretically.

But now, you can give the coinduced topology $\mathcal{T}$ on X which is nothing but the quotient topology coming from the disjoint union topology on disjoint union of X_i's. Recall that the disjoint union topology is nothing but the following: a subset is open if and only if its intersection with each X_i is open inside X_i. This disjoint union topology itself is a very special and simplest case of the coinduced topology. From there, we take an arbitrary union and get this one okay. Once again verifications of (a) and (b) are very straightforward.

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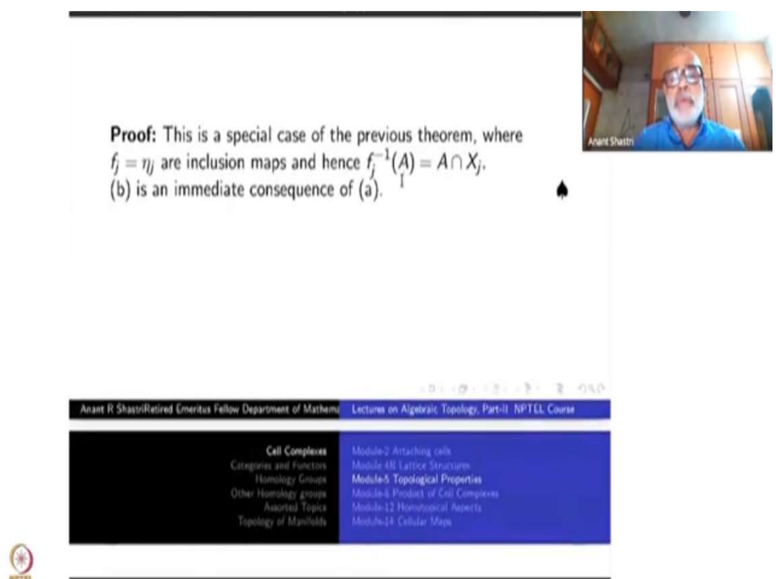

You just look at the meaning of $f_i^{-1}(A)$, when f_i is η_i, the inclusion map, it is nothing but exactly equal to $A \cap X_i$, okay? So, that is (a). And then (b) is an immediate consequence of (a) because what the definition of quotient topology? Something is open here if and only if its inverse image in the total disjoint union is open which means intersection with each X_i is open. So, you get (b) okay?

So now, we would like to take one more special case in which you can extend the neighborhoods, Okay?

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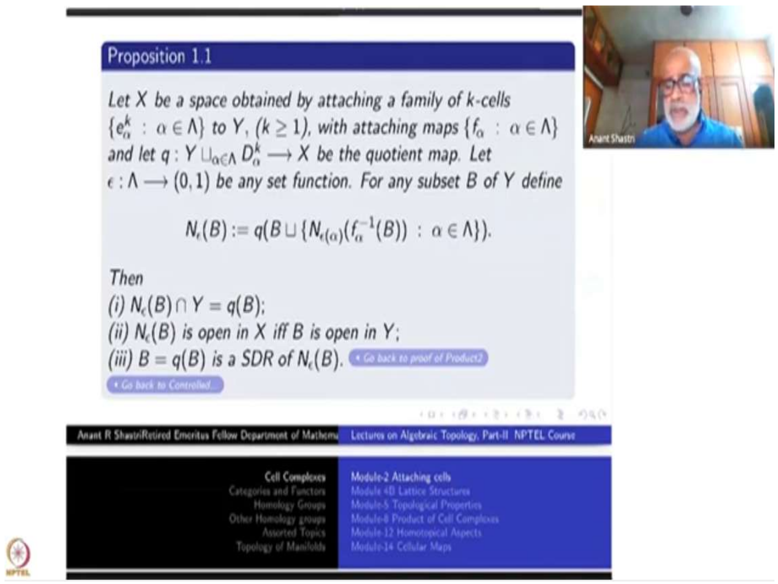

For that, you go back to the proposition 1.1. So, here we had this family of attaching cells attached to a space Y. So, X is obtained by attaching cells to Y right? Then we had these extensions of neighborhoods, quite elaborately stated. But there is a very simple extension of this one that purely topological spaces without any additional structures such as CW complex etc.

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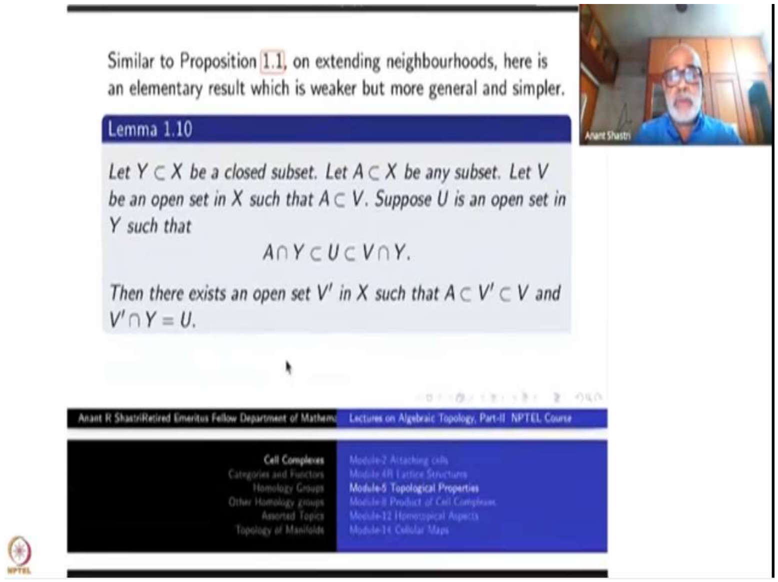

So, let us see what is this result. Let Y contained in X be a closed subset. Suppose A is a subset of X and V is an open subset of X such that A is inside V okay? Suppose U is an open set in Y such that $A \cap Y$ is contained inside U. Okay?

Then, there exists an open subset V' in X such that A is contained inside V' contained inside V and V' intersection Y is precisely U. What I have done is that for the part of A inside Y, U was a nbd inside Y, that U gets extended to a neighborhood of A inside the whole of X, but contained inside the given neighborhood V. That's the meaning of this V' is contained inside V.

So, this is what I mean by extending neighborhoods you can extend the neighborhood, but at the same time can control it also. That it is not too large. So, it is contained inside an already chosen open set. Okay? So, this is an ordinary statement for subsets from a closed set from Y to X, which we can now generalize to a large extent. So, that is what we are doing date. So, the proof of this one is very straightforward, though it looks somewhat complicated and all that.

(Added by the reviewer) The exact statement and the proof has been given in Part-I lemma 2.8 So, we shall skip it. It is also proved in Part-I by giving a counter example, that the next result is completely wrong. So, we shall skip that also and go directly to theorem 1.7.

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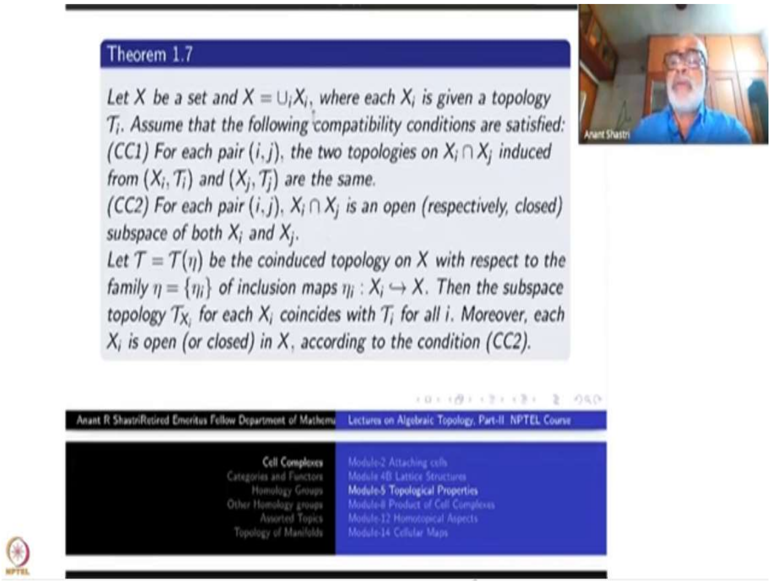

Now we shall study what more relation if at all, is there between the topology on X and the coinduced topology on X by a family of subspaces, though we may not get a complete answer here. Suppose that we start with a topology on X and these X_i's are subspaces covering X, and you give the coinduced topology $\mathcal{T}$ on X. Is there some relation between $\mathcal{T}$ and the original topology on X? This is what we want to study but we are not going to get a complete answer here okay? But we get a set of sufficient conditions which ensure that the two topologies are equal. This is the next theorem.

So, start with X equal to union X_i's, where each X_i is given a topology $\mathcal{T}_i$, okay? Assume that the following compatibility conditions are satisfied, namely:
(CC1): For each pair of indices i, j, the two topologies on $X_i \cap X_j$ induced from $\mathcal{T}_i$ and $\mathcal{T}_j$ are the same. So, the subspace topologies on the intersections coming from the two different subsets must be the same. That is the first condition. The second condition is more generous condition.
(CC2) For each pair i, j, $X_i \cap X_j$ is an open subset of both X_i and X_j.

You can replace `open subset' by  `closed subset'. So, that is a separate statement: for all of i, j, you have to put the same condition okay. So, either you take all of them open or all of them closed, so, that is (CC2).

Now, let $\mathcal{T}$ be the coinduced topology on X with respect to the family $\eta = \{\eta_i\}$, the inclusion maps of X_i into X. Then the subspace topology $\mathcal{T}$ restricted to X_i is equal the already given topology $\mathcal{T}_i$ on X_i. Moreover, each X_i is open (or closed) in X according to what you have chosen in condition (CC2).

So, this is the beginning of our attempt to understand the relation between the two topologies on X. Okay? So, what we did? We started with some topologies one each X_i, gave the coinduced topology on X, and then looked at the subspace topologies on X_i. The question is whether they are equal to the orginal topologies $\mathcal{T}_i$?

If these are subspace topologies on each X_i, then when you go to the intersections what happens? There will be two different ways of doing this, namely, first come to X_i and then go to X_i intersection X_j, or secondly, first come to X_j here and then go to the intersection. Both of them will be subspace topologies from the topology on X and therefore must coincide. Therefore condition (CC1) is a must. However, (CC2) is not a necessary condition.

This is a generous condition. This will actually imply whatever we want, but whatever we want may be there without this strong condition okay? So, we have to be somewhat apologetic about (CC2), which is a bit stronger than necessary.

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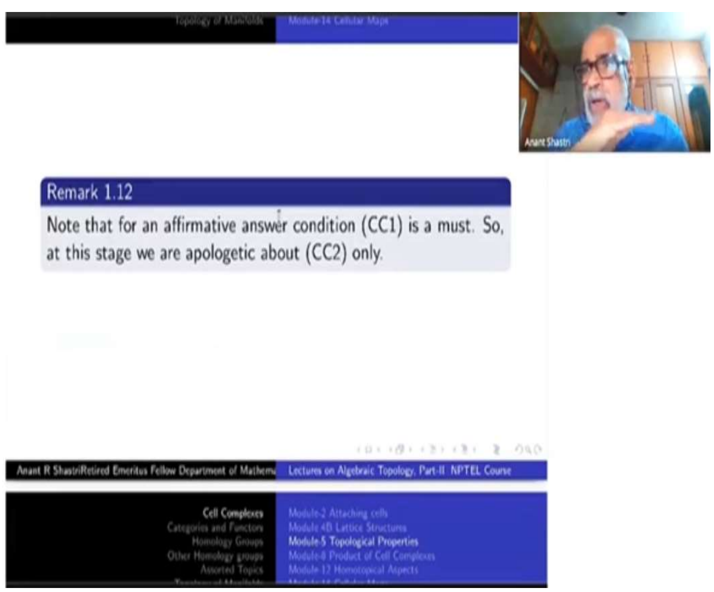

That is such an affirmative answer condition (CC1) is a must it is necessary condition. The second condition is not necessary, but it is sufficient it is it is stronger condition.

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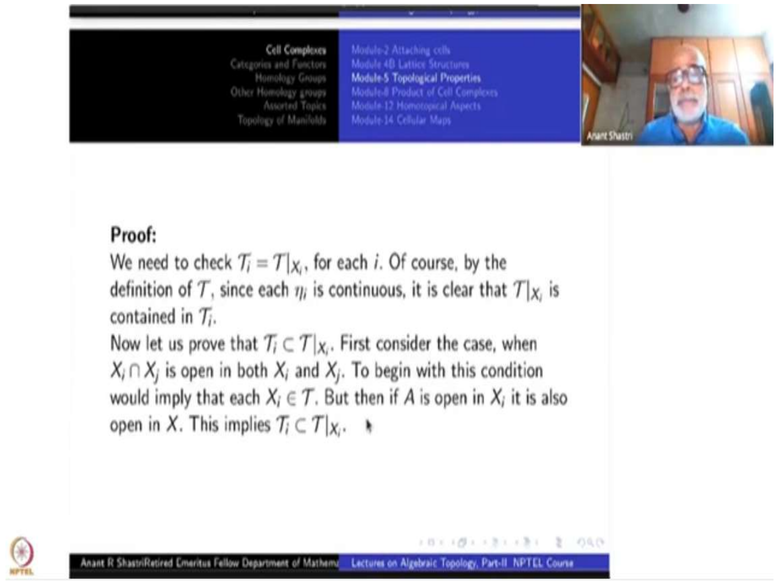

The proof is very straightforward. We need to check that the original topology $\mathcal{T}_i$ is the same thing as $\mathcal{T}(\eta)$ restricted to X_i for each i. By the definition of $\mathcal{T}(\eta)$, each $\mathcal{T}_i$ is a continuous function from where to where from original $(X_i, \mathcal{T}_i)$ to $(X, \mathcal{T}(\eta))$. Therefore, $\mathcal{T}$ restricted X_i is contained inside $\mathcal{T}_i$. Okay?

Now, let us prove that $\mathcal{T}_i$ is contained inside $\mathcal{T}$ restriction X_i, okay? First consider the case when $X_i \cap X_j$ is open in both X_i and X_j. So, we are using the second condition also. Second condition has two different cases, (1) all of them open (2) all of them closed. So, let us take the case when $X_i \cap X_j$ are open in both X_i and X_j.

To begin with this condition itself will imply that each X_i is open in $\mathcal{T}$. Right? Because fixing X_i, it is open in the coinduced topology. By its very definition, if intersection with each X_j must be open in X_j and that is what it is. So, each X_i is open in $\mathcal{T}$. But then if A is open in X_i, it will be open in X also right? Open subset of an open subset is open. If A is an open subset of X_i, it is to open in X also. So then this $\mathcal{T}_i$ is contained inside in $\mathcal{T}$ restricted to X_i. Okay?

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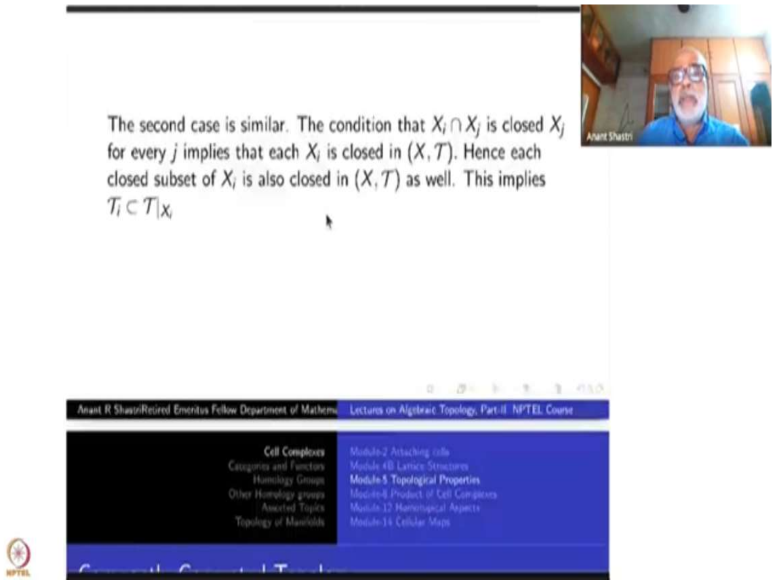

The same way now you can use closedness also. Okay. The second case is now that $X_i \cap X_j$ is closed in X_j for every j implies that X_i is closed in $\mathcal{T}$. Hence each closed subset of X_i is a closed subset of X also. Okay? similar proof. Alright?

So, there are nice theories here about the first question that I raised here, but we will not need them. So I'm skipping them.

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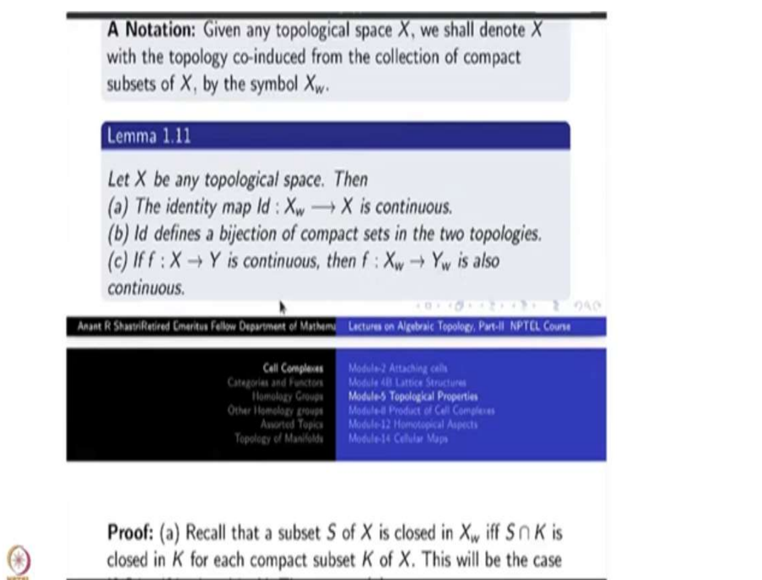

Let us now go to another special case of this coinduced topology, namely compactly generated topology, okay? So, we make a notation here. Take any topological space to start with, okay?

Then look at all the compact subsets with their induced topologies, subspace topologies. So, there's a family of topological spaces and you look at the inclusion of maps, those maps will coinduced a topology on X. That topology is denoted by X_w.

This is the classical notation. It referred to the fact that we are taking weak-topology. So, that's why the notation X_w. It is considered to be a weak topology, weaker than what? Topology coinduced by the set of all compact subsets of X okay? So, what is the relation between the original topology on X and X_w? That is what we would like to understand. This is going to be very important. So, please pay your attention to this concept so that you can use it later.

(a) The very first thing to notice that is that identity map from X_w to X is continuous.

What is the meaning of this? Every set which is open in X is already open in X_w also. So, in other words, there are more open sets in X_w. So, X_w is finer than X. Though it is called weak topology, don't be under the impression that it is weaker than the original topology on X, it is actually finer. Okay?

(b) The Identity map defines a bijection of compact sets in the two topologies. Though there are more open sets here in the domain, the set of all compact subsets is the same on either side. That is the beauty of this weak topology. So, that is part (b) here.

(c) The third property is, the third statement is that if f from X to Y is continuous, then the same map f (but change the topologies on both sides to weak topology) from X_w to Y_w is also continuous, okay?

So, let us see how (a) and (b) work out. Then I will leave (c) to you to work it out. Alright?

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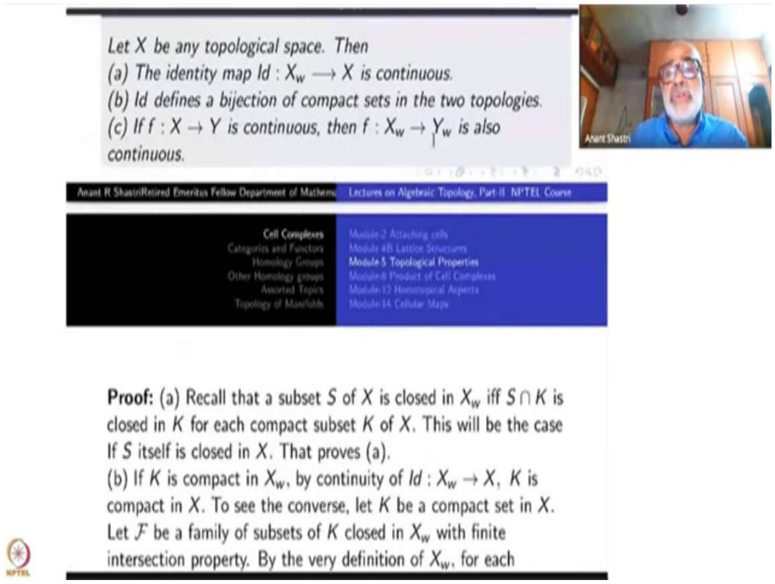

Recall that a subset S of X is closed in X_w, (I'm just recalling the definition of weak topology here) if and only if $S \cap K$, where K is a compact subset of X is closed in K. This should happen for every compact K. That is the meaning of something is closed in X_w, the compactly generated topology or the topology coinduced by the family of compact sets.

If S itself is closed in the whole space X intersection with each subspace will be closed in that subspace.

In particular intersection with each compact set K is closed in K. Hence S is closed in X_w. Okay? Therefore, this part (a) follows by taking the complements, it is open in X then it is open in X_w. Okay? Now the second one (b). If K is compact in X_w, we want to show that it's compact in X. But the identity map is continuous, image of a compact sets is compact. Therefore, K is compact in X also. Excellent!

The converse is important. Suppose K is a compact subset of X, why it is compact in X_w that is what you have to show, Yeah? Let $\mathcal{F}$ be a family of subsets of K, closed in X_w and with finite intersection property.

(Now, remember that to show that K is compact, I have to show that intersection of all these set in $\mathcal{F}$ is non-empty, okay? To prove this itself, you have to just appeal to DeMorgan's Law. So, I am going to use that property.)

Let $\mathcal{F}$ be a family of subsets of K closed in X_w and with finite intersection property. By the very definition of X_w, for each F in $\mathcal{F}$, $F \cap K$ is closed in K under the topology induced by the original topology of X. But each F is a subset of K and hence $F \cap K$ is equal to F itself. Okay?

Since K is compact in the topology induced from the orignal topology on X, it follows that follows that the intersection of all F in $\mathcal{F}$ is non-empty. That is all we wanted to prove, alright?

So, we leave the proof of part (c) to you, okay?

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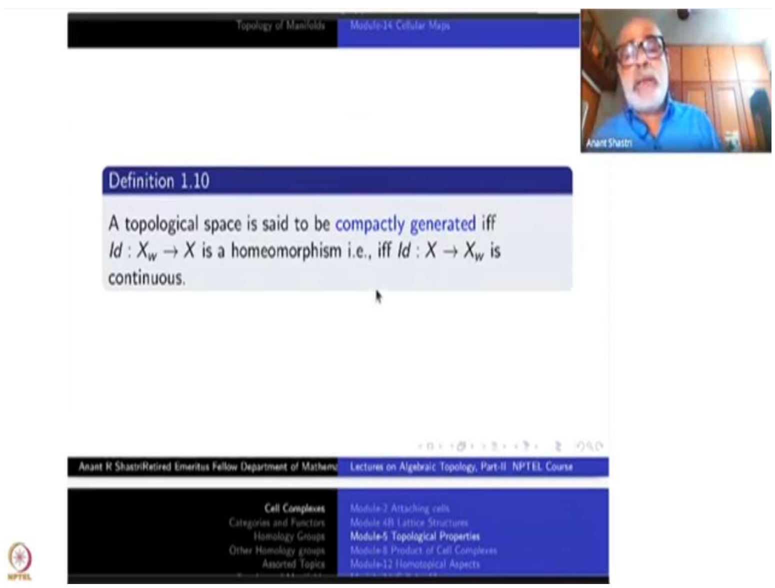

However, let us make this definition now. A space is said to be compactly generated if and only if this identity map in (a), (which is we have seen is continuous) must be a homeomorphism. That means that identity map the other way round is also continuous, which just means, in turn, that the two topologies on X are the same, okay?

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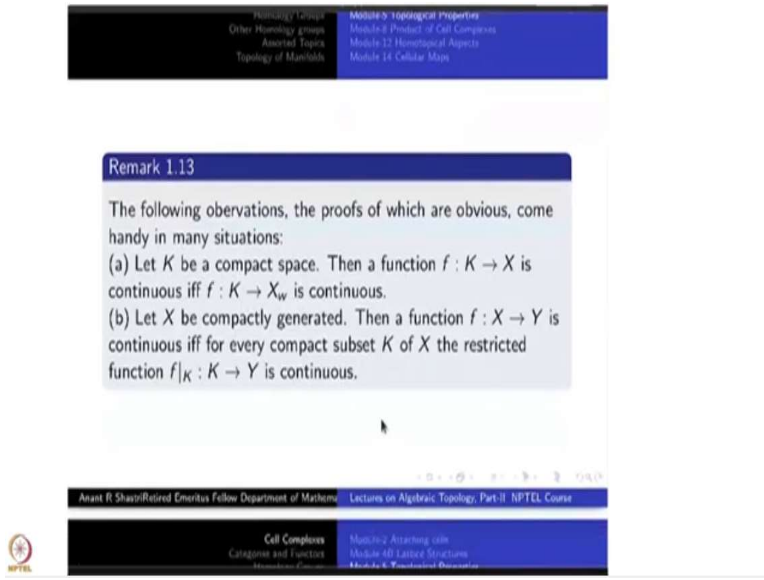

So, here are two remarks. If you learn them properly, everything with weak topology will be easy for you. So, what are these remarks?

(a) Take a compact space K. Then a function f from K to X is continuous if and only if the same function f from K to X_w is continuous, okay? K can be any compact space and f could be any function.

The next one (b) is: Suppose our X is compactly generated. Then a function f from X to Y is continuous if and only if for every compact subset K of X, the restricted function f from K to Y is continuous.

This fact (b) is going to be the key to understand compactly generated topology and its use in constructing continuous functions, okay? So, this will be used again and again in the study of CW-complexes. I have emphasized this one several times.

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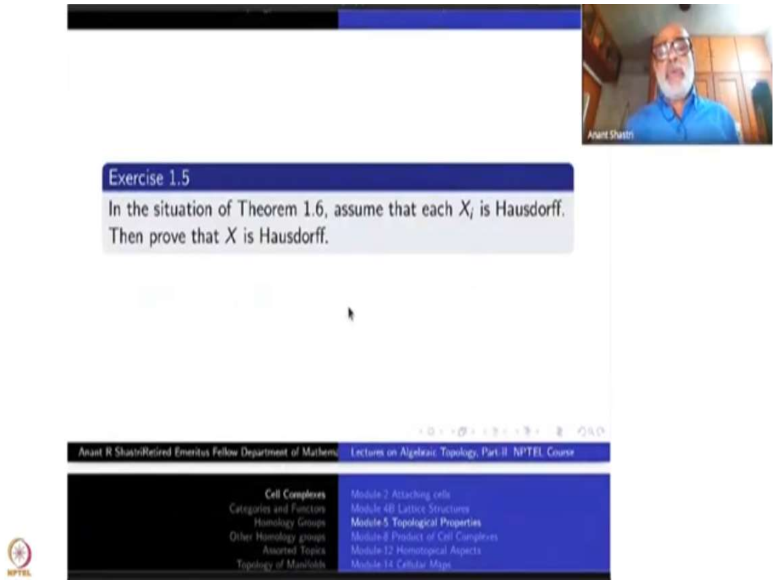

Consider the situation of theorem 1.6. I want to leave this as an exercise to you. Assume that each X_i is a Hausdorff space. Prove that X is Hausdorff. It's not very difficult but we have to workout. So, that is roughly what we wanted to tell you about coinduced and compactly generated topologies, Yes. Thank you.