**Introduction to Algebraic Topology (Part – II)**
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**Lecture - 60**
**Orientability**

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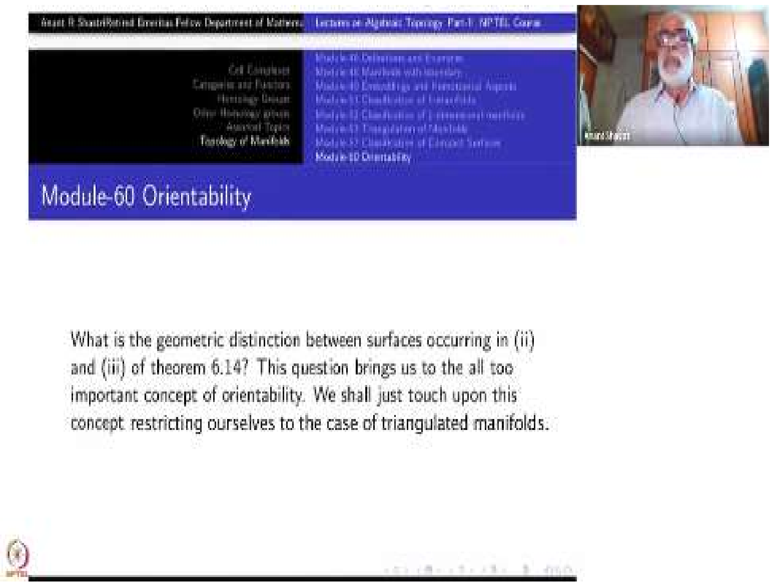


So, we come to the last module of this course. What is the geometry distinction between surfaces occurring in the list (ii) and list (iii) namely all those surfaces which are obtained as connected sums of a number of torus or the are connected sums of a number of projective spaces. We know that the algebra either homology or the fundamental group distinguishes them. We have studied them thoroughly. But now, we are asking, what is the geometric concept behind this, that is happening here, okay?

This question brings us to the all important concept of orientability. However, we shall just touch upon this concept restricting ourselves to the case of triangulated manifolds, with the classical approach, which in a sense, somewhat turns out to be somewhat weaker than other approaches okay?

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So, by an orientation on an $n$-simplex $n \geq 1$, we mean an equivalence class of labeling the vertices. That means we are choosing a total order on the vertex set and define two such ordering to be equivalent if one is obtainable from the other via an even the permutation okay? Once there is a labeling, another labeling corresponds to a permutation of the set, if that permutation is of even, namely the signature of that permutation is $1$, then we identify the two permutation okay.

So, it turns out that there are exactly two such equivalence classes, on a set with $(n+1)$ elements $n$ positive. Therefore, there are exactly two orientations on any simplex. For instance, given any on ordering, select two vertices and interchange their positions, keeping rest of the vertices undisturbed, that will give you an ordering in the other equivalence class. So, you would not get any other class. So, an orientation on a simplex means choosing a total order, upto an even permutation.

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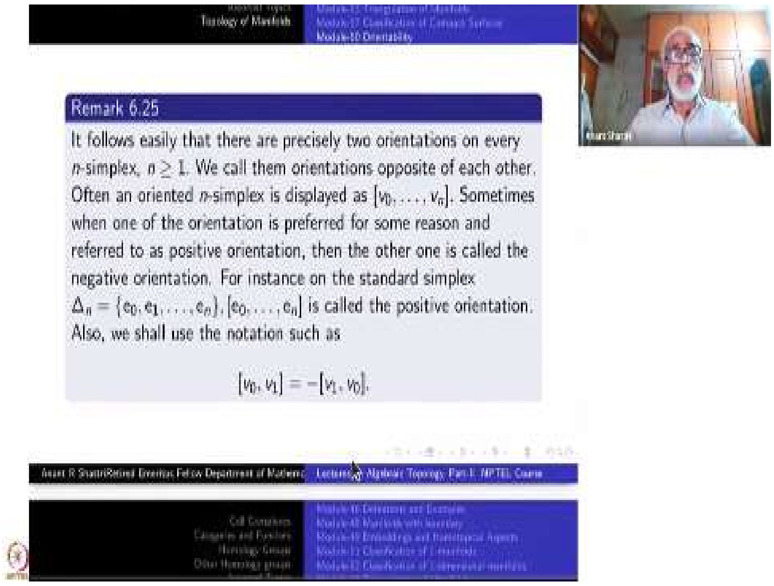


The two possible choices are called orientations opposite to each other. Okay? Suppose you fix one of them and call it positive orientation, then the other will be negative orientation. It is just like two square roots of $-1$, viz., $\pm i$. Okay, there is nothing like a positive square root and a negative square root of $-1$. That is just a joke okay? So, there is no positivity negativity, but people do use this kind of loose terminologies you must understand that. That is all okay. Better terminology would be to say that one is the opposite of the other.

Often an oriented n-simplex is displayed by putting the sequence inside a square bracket such as $[v_0, v_1, \ldots, v_n]$ an an oriented $n$-simplex $\sigma = \{v_0, v_1, ..., v_n\}$ which has $(n+1)$ vertices. We have followed this convention earlier in the construction of the chain complex corresponding to a simplicial complex.

Sometimes when one of the orientation is preferred for some reason and referred to as a positive orientation, then the other one is called negative orientation, or the opposite orientation. Like the anti-clockwise and clockwise orientations. For instance, on the standard $n$-simplex $\Delta_n = \{e_0, e_1, \ldots, e_n\}$, where $e_i$ are the standard basic elements of $\mathbb{R}^{n+1}$, $[e_0, e_1, ..., e_n]$ is called the positive orientation. Then the other orientation whatever you take will be the negative orientation, that is all.

So, we also use this notation $[e_0, e_1] = -[e_1, e_0]$, and this notation is justified because of our integration theory and because of our homology theory okay? We have seen that the 1-chain $[e_0, e_1] + [e_1, e_0]$ is null homologous. So, we can use this notation. Integration along this edge and integration in the opposite direction, they are related by this relation that one is the negative of the other. That is the strong reason why this terminology is in vogue. Okay?

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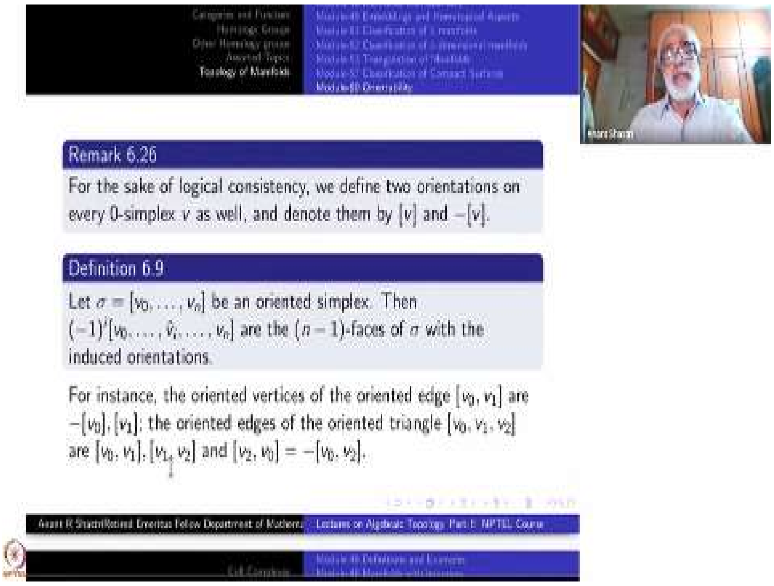


For the sake of logical consistency, we shall introduce two orientations classes on 0-simplexes also. Here the permutation group is trivial and hence there is no way we can use it. But just for logical consistency, what we will do is we will declare that each vertex also has two orientations, namely $[v_0]$ and $-[v_0]$. This is consistant with the algebra that we do, viz., the set of all 0-chains on a single vertex is an infinite cyclic group and has two generators, one is the negative of the other.

Let us now make a definition. Let $\sigma = [v_0, v_1, \ldots, v_n]$ be an oriented $n$-simplex. If you take $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$, wherein the vertex $v_i$ is omitted, remember what is this? it is $i$-th $(n-1)$-face of $\sigma$. We now put the correct sign $(-1)^i$ along with it, viz., $(-1)^i[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$ and call it the $i$-th face of $\sigma$ with the induced orientation. It then follows that the boundary of an oriented simplex is the sum of all its $(n-1)$-faces with the induced orientations.

For instance, $\partial[v_0, v_1]$ is $[v_0] - [v_1]$. And for the oriented triangle, $\partial[v_0, v_1, v_2] = [v_1, v_2] + [v_2, v_0] + [v_0, v_1]$, which tells you how the edges are got by tracing the boundary of the triangle in the anti-clockwise sense.

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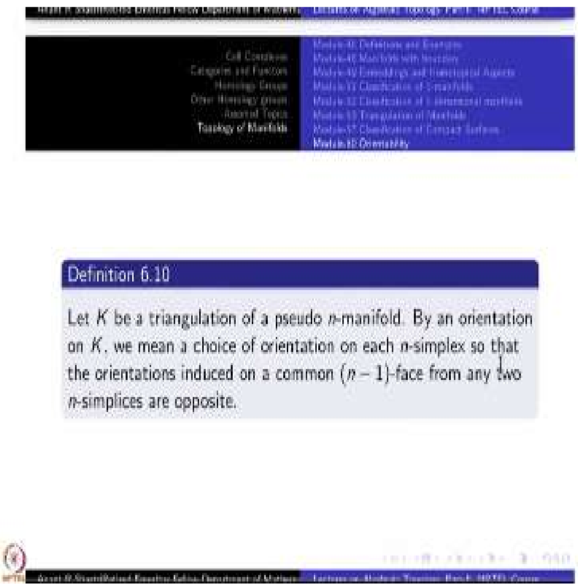


Let $K$ be a triangulated pseudo $n$-manifold. You know what is pseudo $n$-manifold? First of all, it is a simplicial complex, and it satisfies a particular fundamental property which a triangulated $n$-manifold satisfies. For example, it is pure of dimension $n$ and every $(n-1)$-simplex is the face of exactly two $n$-simplexes.

By an orientation on $K$, we mean a choice of orientation on each $n$-simplex such that the orientation induced on a common $(n-1)$-face of two of the n simplex (an $(n-1)$ face also called a facet) must be opposite of each other. okay? If you have an orientation on $K$, then $K$ is called orientable. If you fix an orientation on it then it will be called an oriented pseudo $n$-manifold.

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So, not all pseudo manifolds may be orientable. This condition is not a trivial one. It turns out that given a pseudo manifold $X$ its orientability depends just on the homotopy type of underlying topological space okay? This is a deep remark which you will not bother to prove here, in general, but for surfaces, we have already a proof here. So, I want to indicate that.

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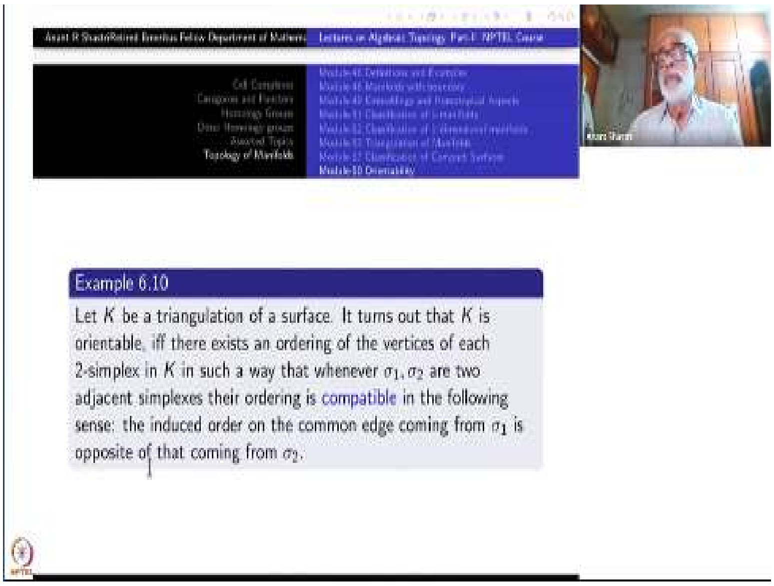


Let $K$ be a triangulation of a surface. It turns out and easy to that $K$ is orientable if and only if there exists a total ordering of the vertices of each 2-simplex in $K$(which is the same as choosing orientation on each 2-simplex) such that whenever $\sigma_1, \sigma_2$ are two adjacent 2-simplices, the two ordering on them are compatible (this is the word I want to use in the following sense) i.e., the induced orderings coming from $\sigma_i$ on the common edge must be opposite of each other. That is just reformulating the above general definition, in this special case. I am just recalling this definition here in this special case. The word `compatible' is used in this sense.

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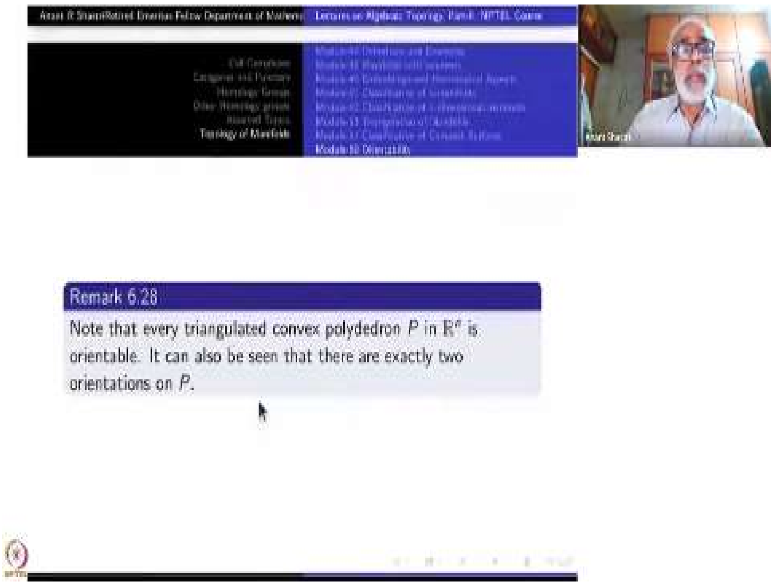


Note that every triangulated, convex polydedron $P$ in $\mathbb{R}^n$ is orientable okay? This is not very difficult to see. For $n = 1$ or 2, it is totally obvious, okay, it can also be seen that there are exactly 2 orientations on $P$, Okay? In $\mathbb{R}^2$, for instance you can follow the convention that all 2 -simplexes are oriented anticlockwise. That will automatically satisfy the compatibility condition. The same thing you can do in $\mathbb{R}^n$ also.

For a general pseudo $n$-manifold, what you can try do is that you start with one simplex whichever order you want. Then look at any one of the facets with the induced orientation. There is exactly one other $n$-simplex of which it is a facet and that $n$-simplex has exactly one extra vertex. So, you can change the orientation of the facet and extend that orientation over to the $n$-simplex in a unique way. Keep going on like this till you hit upon a $n$-simplex of which more than one facets are already carry orientations induced by the orientations of $n$-simplexes that you have fixed so far. It is then not clear whether there will be a compatible way of extending all these to an orientation of that $n$-simplex. That is the problem.

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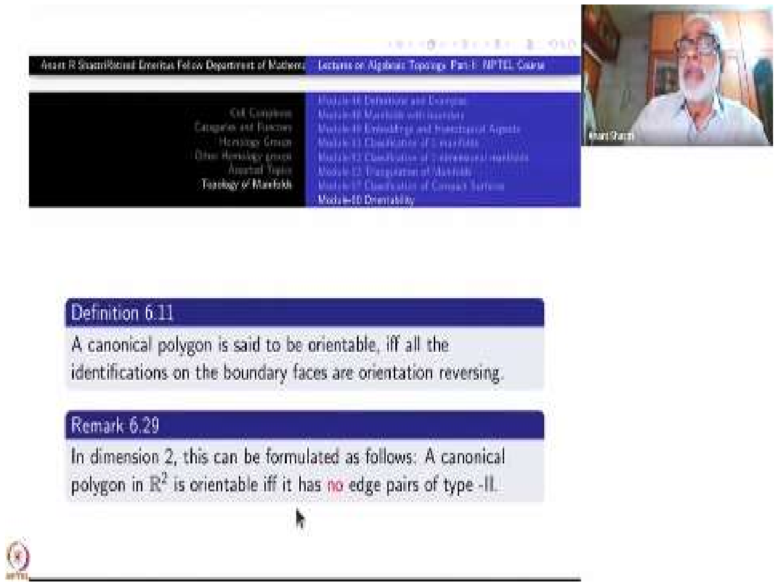


So now, I am making a different definition here. A convex polygon in $\mathbb{R}^2$ is always orientable right? I am taking a canonical polygon what is a canonical polygon? Canonical polygon means that there is a sequence of edges on the boundary such that each edge is identified with exactly one other edge. So we have edge pairs remember that, that is the definition of a canonical polygon. So, a canonical polygon is said to be orientable if and only if all the identifications namely pair wise identifications of the edges are orientation reversing okay? That is the definition of a canonical polygon to be orientable okay? (So, this definition can be taken in any in $\mathbb{R}^n$ also instead of polygon.) In dimension 2, this can be reformulated as follows. A canonical polygon $\mathbb{R}^2$ is orientable if and only it has no edge pairs of type II. That means no edge is identified with another edge by an orientation preserving isomorphism. Why this artificial looking definition? You will see that this is the correct one for us, to go about. Okay?

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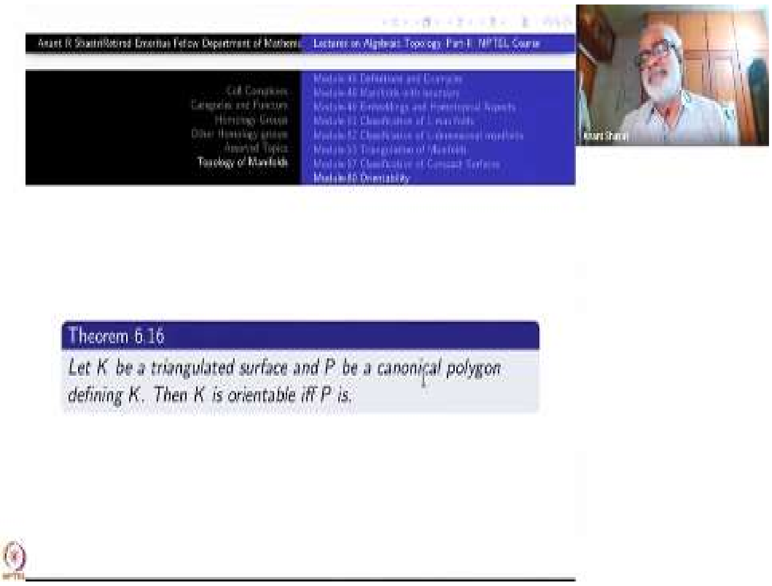


So, this is a theorem. Let $K$ be a triangulated surface, okay? And let $P$ be the triangulated canonical polygon associated to $K$. Then $K$ as a triangulated surface is orientable if and only if $P$ as a canonical polygon is orientable. So this is the theorem. This is what made us make this definition okay.

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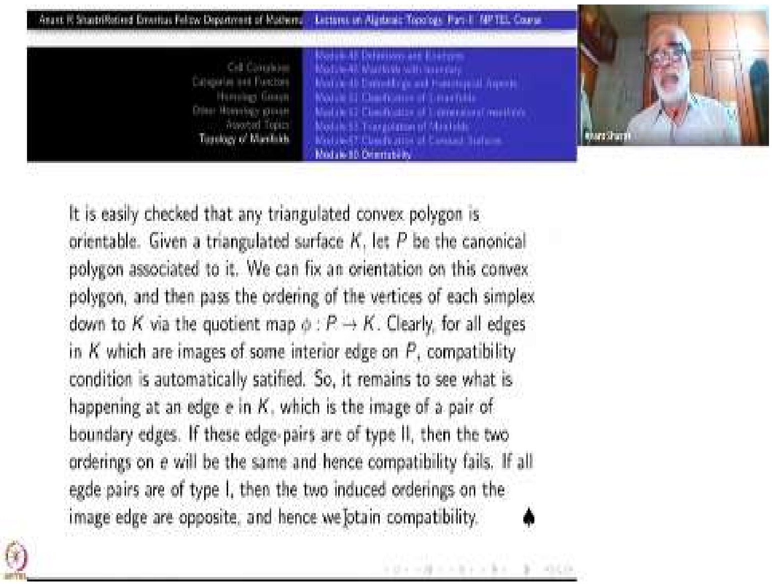


So, how does it work? Indeed, we take the standard orientation on each simplex in $P$. Since the quotient map $\phi$ from $P$ to $K$ is bijective on each simplex, and defines a bijection of 2-simplexes in $P$ and $K$, all that we do is to put the order defined by $\phi$ itself one $\phi(\sigma)$ for each $\sigma$ in $P$.

Given a triangulated surface $K$, let $P$ be the triangulated canonical polygon associated to it you got my point. So, to start with we can fix an orientation on this convex polygon then pass the ordering of the vertices of each simplex down to $K$ via the quotient map $\phi$ from $P$ to $K$. Clearly for all edges in $K$ which are images of some interior edges $P$, there is no problem of compatibity, because the same holds inside $P$ already.

So, the problem is only at the edges $K$, which are images of a pair of boundary edges in $P$. Okay, if these edge pairs are all of type II, then the 2 orderings on the edge will be the same coming from 2 different edges, and that will create problem, compatibility fails. If on the other hand all edge pair are of type I then there is no problem okay? Compatibility is over. So this is the explanation for this theorem Okay. In fact, only after observing this one has formulated this theorem okay?

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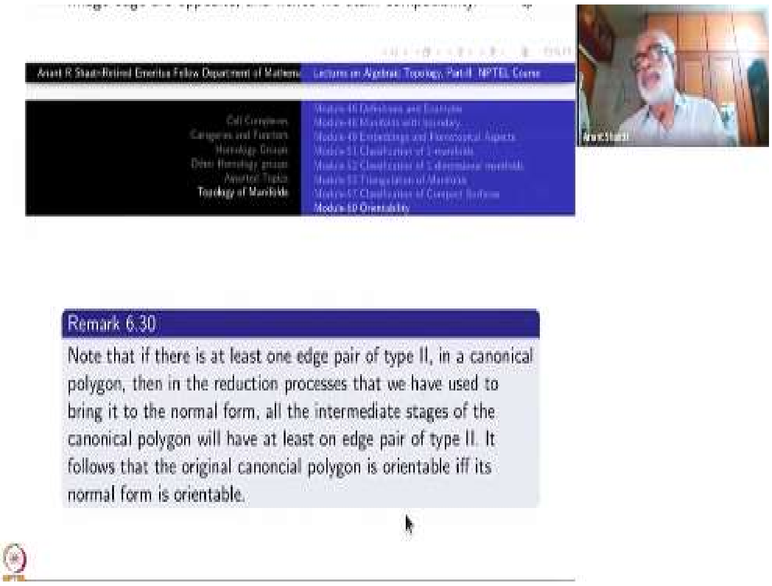


Note that if there is at least one edge pair of type 2 in a canonical polygon then in the reduction process from a canonical polygon to the normal form, the various steps involved will never get get rid of a pair of type II. Type 1 pairs are sometimes cancelled out. A type two II pair may disappear only to to introduce another one of the same type.

So, if a canonical polygon has an edge pair of type II, then its associated normal form will be in the sublist (iii). Therefore, it follows that original canonical polygon is orientable if and only if its normal form is orientable. You can also say that the former is not orientable if the lattes is not orientable. It is the same thing.

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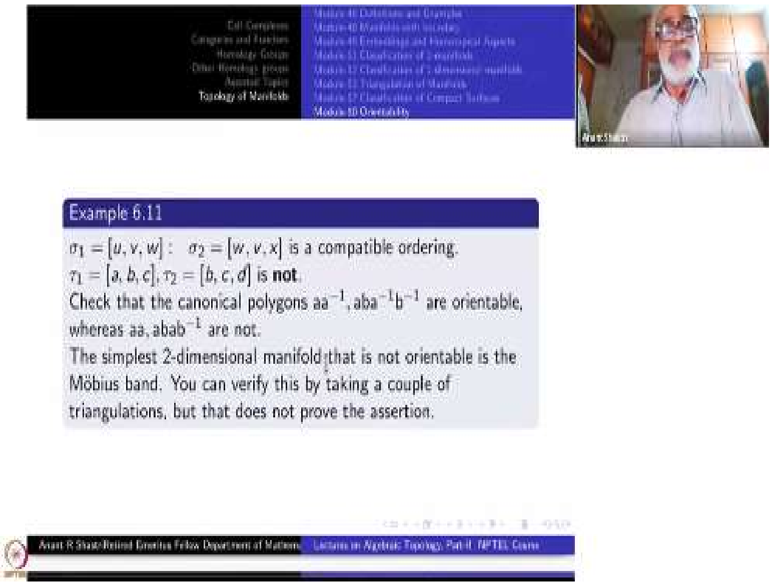


So, I will give you one more example here. Let us $\sigma_1 = [u, v, w]$, and $\sigma_2 = [w, v, x]$. What is the common edge $\{v, w\}$ right? So, what are the two induced orders. From $\sigma_1$, when you drop out $u$, get $[v, w]$. From $\sigma_2$, when you drop out $x$ you get $(-1)^2[w, v] = [w, v]$. So these two are opposite to each other. So, the simplicial complex which is the union of $\sigma_1$ and $\sigma_2$ is orientable.

Now, suppose these two triangles are taken and you are messing it up namely, $\tau_1 = [a, b, c]$ and $\tau_2 = [b, c, d]$ okay? Check that this is not a compatible ordering. Next check that the canonical polygon canonical polygons $aa^{-1}$ and $aba^{-1}b^{-1}$ are orientable, whereas $aa$, and $abab^{-1}$ are not orientable. These canonical polygons respectively define the sphere, the torus, the projective space and the Klein bottle.

The simplest surface that is not orientable is the projective space; if allow boundary then it is the Mobius band. We had some great experience with them in one of the live sessions. The Mobius band is not orientable you may try to verify this by taking a couple of triangulations of it. But such verifications will never prove the assertion, because, one may argue that there is some other triangulation for which you have not yet verified the same. So, to prove that it is not orientable you have to have a different device. So, with this definition, it is not that easy to demonstrate that something is not orientable. So, in the live session we examined what happen when you the Mobius band. Okay?

On the other hand, you can demonstrate something is orientable by just producing one triangulation which is orientable. But to say that underlying space is not orientable that is rather difficult you give me a triangulation, I can verify that it is not orientable that is possible. But no triangulation is orientable is not easy to alright. So, but here in the case of surfaces, because of our homology and so on, we have got a complete understanding of the orientability okay, this is just a lucky part so there it comes so, easily for us.

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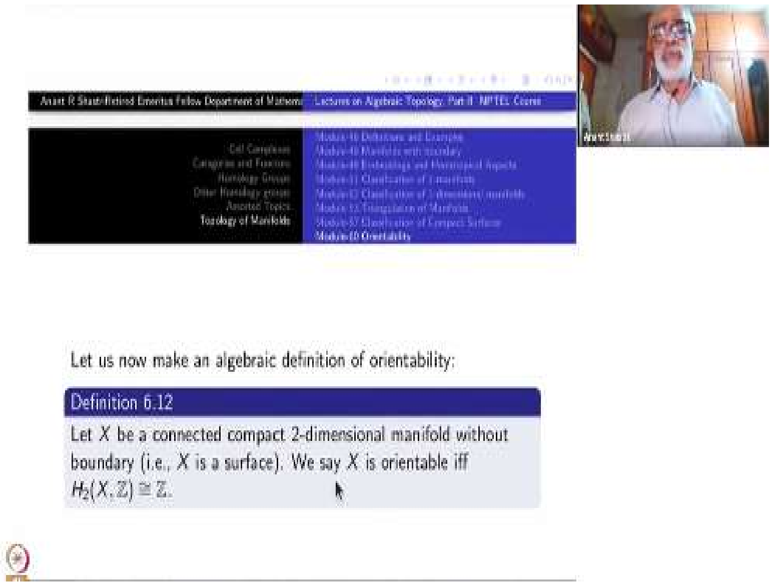


So, let us make an algebraic definition of orientability and then see that, that is a topological invariant. From there we can deduce all these things. So, what is the definition? Let $X$ be a connected compact 2-dimensional manifold without boundary, Okay? There is no triangulation here now. Earlier we defined orientability using a triangulation. Let us call that combinatorial orientation. Now we want to define algebraic orientation okay? So, we say $X$ is orientable if and only if the second homology of $X$, with respect to integer coefficients, $H_2(X; Z)$, is isomorphic to $Z$. This looks like very artificial definition, but this is motivated by the geometry of surfaces. And this definition can be taken for all manifolds not only just surfaces. Instead of $H_2$ here, you put $H_n$ here. That definition is very strong. Of course, it can be further generalized.

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So, since, we are assuming that every surface $X$ is triangulable, okay? (not proved that one okay) it follows from theorem 6.14 and our computation of $H_2$, that $X$ is orientable if and only if it belongs to list (i) (namely, $\mathbb{S}^2$) or (ii) namely connected sums of tori. Thus the new algebraic definition and the old combinatorial definition of orientability coincide here okay? So, in both cases, the associated canoncial polygon should not have edge pairs of type II.

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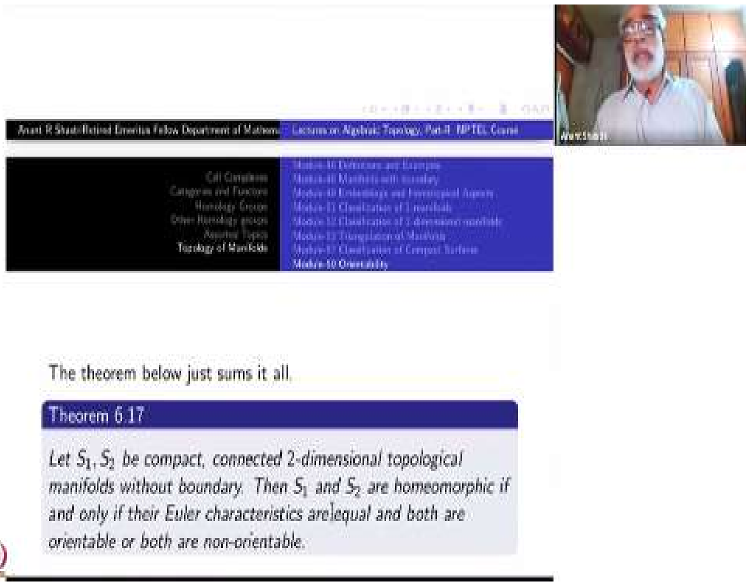


The theorem below just sums it all. Here, I want to tell you something without using the triangulation at all okay. Let $S_1$ and $S_2$ be any two compact connected 2-dimensional topological manifolds without boundary. Then $S_1$ and $S_2$ are homeomorphic to each other if and only if there Euler characteristics are equal and both are orientable or both are non orientable.

We have already seen this result. And we have seen that Euler characteristic itself cannot distinguish between the second series and the third series. Members of the third series can have same Euler characteristic as some members in the second series. Okay? But put one more condition: orientability, the second and third series get distinguished, okay? So, the proof uses is triangulation. Finally we have a statement without reference to any triangulation okay? Only Euler characteristic and orientability will give you the classification. For the definition of orientability, we can use the algebraic definition, $H_2$ must be infinite cyclic.

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So, let me make a general remark here. In general, consider a smooth manifold $M$ with or without boundary and compact or non compact. That is quite a general stuff, but for the smoothness condition, we can talk about orientability in at least 4 different ways. The above combinatorial one is valid there because a smooth manifold is always triangulable. There is something called a tangent bundle for a smooth manifold okay? And the concept of orienting vector bundles, which gives another definition. Then there are things called differential n forms on $M$, existence of a non trivial differential $n$-form gives yet another definition, okay? And then finally, of course there is this algebraic definition. Also there are many more. But I want to tell you is that (at least these four) all of them coincide. Okay? The algebraic definition gives you that orientability is indeed a homotopy invariant, whereas, the differential topological ones they will give you that orientability is only diffeomorphism invariant. The combinatorial one is the most difficult to handle for invariance, but easy to perceive, easy to understand and easy to compute in special cases, but in proving general theorems, it is difficult okay. So, we have to study that so, that computation and easy to understand is for that it is it is very helpful. Okay?

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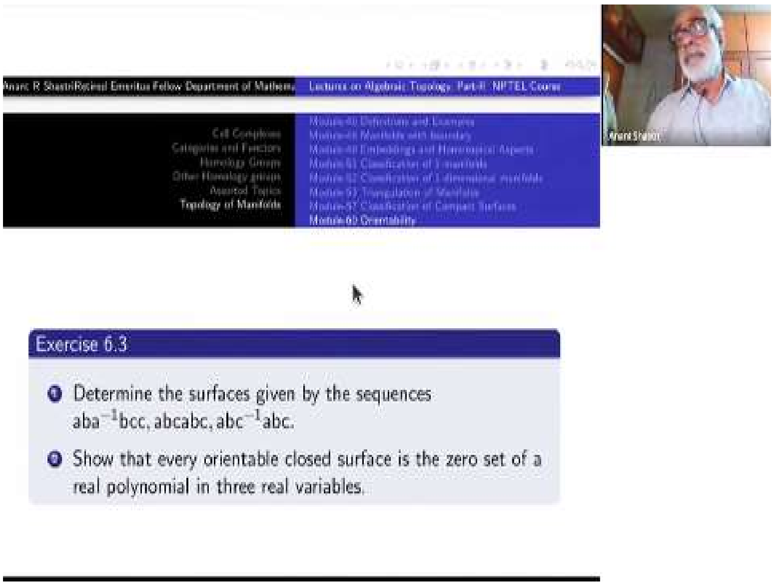


So, here are a few exercises these exercises will be again updated and given to you in the form of PDF files. Okay? You have to work them out. You have to work out exercises and some of them you have to submit also. Okay? I have enjoyed lecturing to you. See you some other time. Thank you.