Introduction to Algebraic Topology (Part - II)
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Lecture - 59
Proof of Part B

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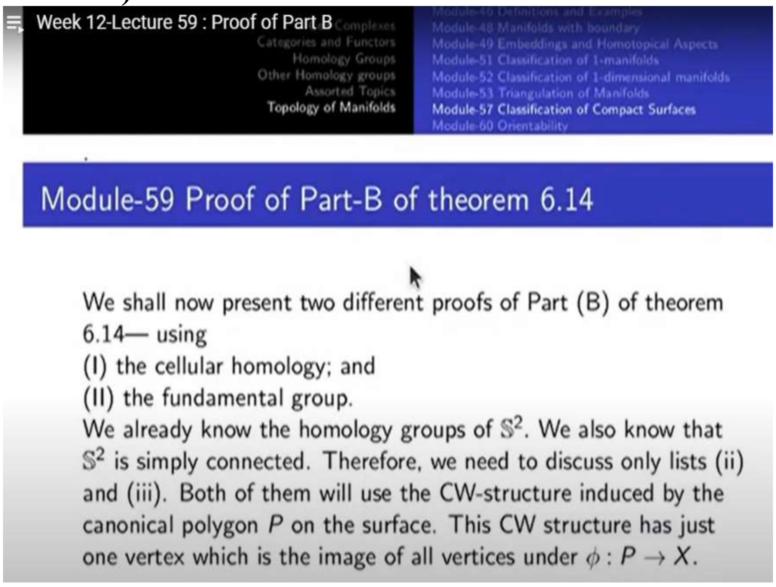

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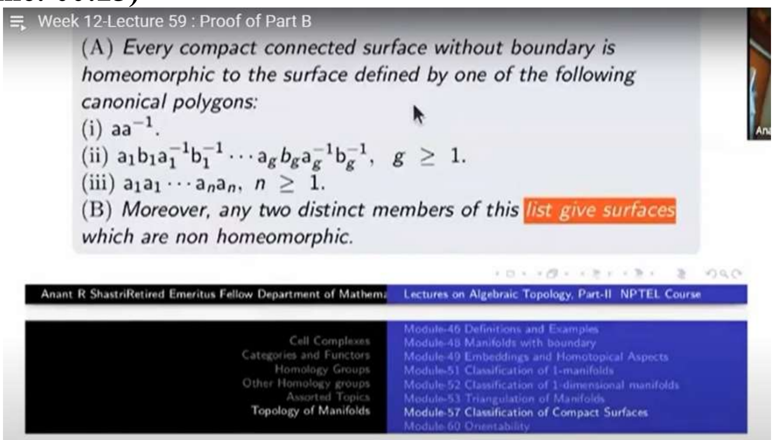

So, today, we shall complete the second part, part B of theorem 6.4. So, recall that theorem 6.4 gives you three sets of canonical polygons. The first one consists of just one member. This represents the 2-sphere. The second and third ones give a sequence of members, one for each natural number g or m. So, part B says that any two distinct members belonging to this list represent surfaces of different homeomorphism classes. So, that is what we have to prove, that will complete the proof of this theorem. So, let us go back to the today's slides.

We have two methods, two different proofs, one using cellular homology and other using fundamental group. Since, we already know the homology of $\mathbb{S}^2$ as well as its fundamental group, ($\pi_1(\mathbb{S}^2) = (1)$), we have to consider only the list (ii) and list (iii) for this purpose. Both the proofs will give a very special CW structure on the surface. Namely the one which is induced by the canonical polygon itself. This CW-structure has just one vertex which is the image of all vertices of $\mathcal{P}$, under the quotient map q from $\mathcal{P}$ to X. Right? So, that is what we have seen.

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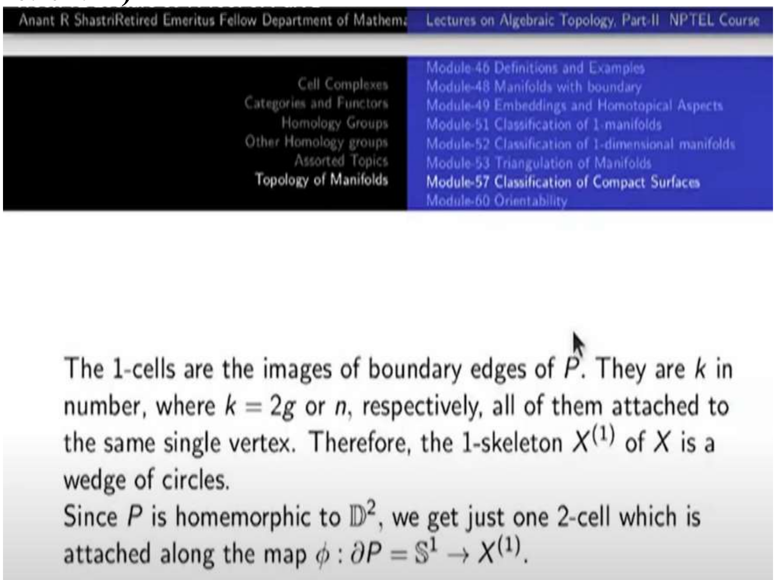

So, the 0-skeleton of X is just a single vertex. What the 1-cells? They are the images of the boundary edges of $\mathcal{P}$, each edge becomes a circle below in X, because only the two end points get identified. So, how many of them are there? They are k in number where $k = 2g$ in the case of (ii) and equal to m in case of (iii) respectively, because edges in the boundary of $\mathcal{P}$ are identified pair-wise. Therefore $X^{(1)}$, the 1-skeleton of X is nothing but a wedge of k-circles, a bouquet of k circles you can call. Now, since $\mathcal{P}$ is homeomorphic $\mathbb{D}^2$, being a convex polygon in $\mathbb{R}^2$, its image defines a 2-cell in X which covers the entire of X. What is that attaching map here.? The quotient map ϕ restricted the boundary of $\mathcal{P}$, and the whole thing goes into $X^{(1)}$. So this describes the CW structure on X. We shall use this CW-structure for both methods.

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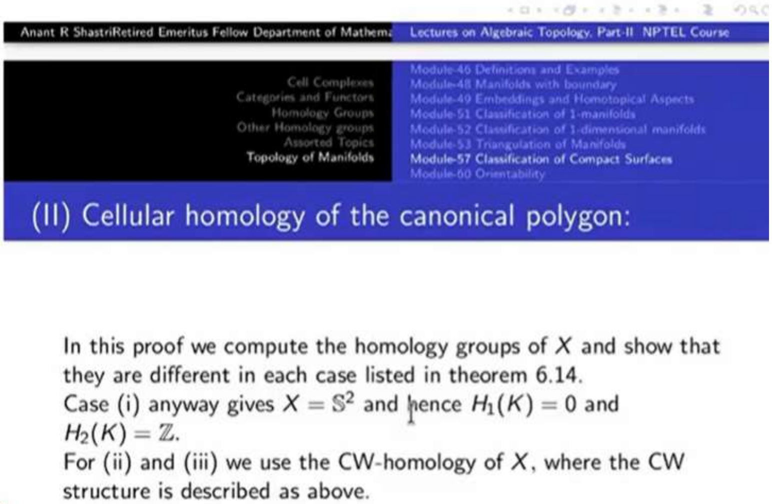

So in the first method, let us compute the singular homology Okay? So I repeat in this proof, we compute the homology groups of X and show that the listed things in the theorem 6.4 have all different homology groups. In case (i), anyway, X is $\mathbb{S}^2$ and we do not have to worry about that. So, $H_1(X)$ is 0 and $H_2(X)$ is $\mathbb{Z}$.

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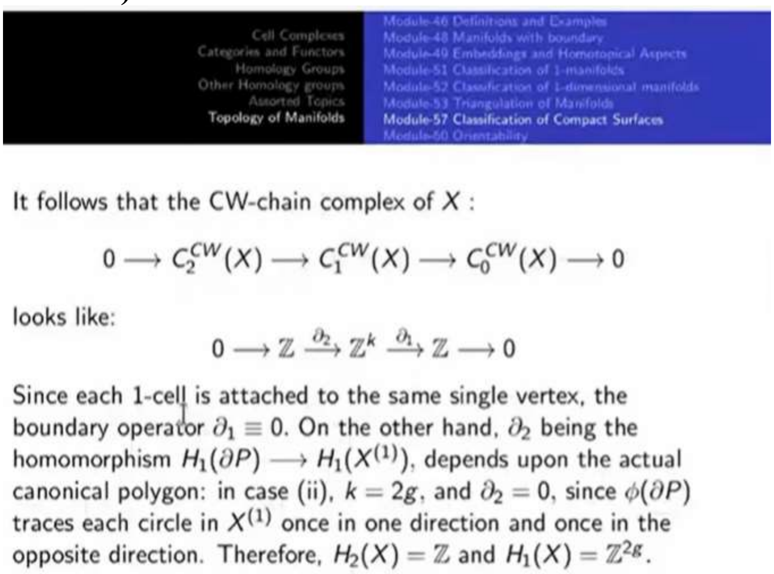

For the case (ii) and (iii), we use the elaborate CW-structure given by the canonical polygons, as described above. The singular homology can be computed using the cellular chain complex, $C_\cdot = C_\cdot^{CW}(X)$. So, C_0 will be the infinite cyclic group generated by the single vertex. C_1 is the free abelian group generated by k circles, and so, this is $\mathbb{Z}^k$. C_2 is again infinite cyclic generated by the single 2-cell which is the image of $\mathcal{P}$. So, the entire chain complex looks like $0 \to \mathbb{Z} \to \mathbb{Z}^k \to \mathbb{Z}$. So, it remains to determine what are these boundary operators ∂_2 and ∂_1, okay?

So, let us go through this one. Since each 1-cell is attached to the same single vertex, the boundary operator ∂_1 maps each generator to 0 and hence is identically zero, okay?

What happens to ∂_2? The generator of C_2 are represented by the entire boundary in the relative homology of $H_2(X^{(2)}, X^(1))$. And the attaching map is the quotient map ϕ restricted to the boundary, which traces each of the k circles twice.

In case (ii), the tracing is done once in each direction, so the degree of the projection composed ϕ onto each circle is 0. It follows that ∂_2 is identically 0. It follows that $H_1(X)$ is $\mathbb{Z}^{2g}$ and $H_2(X)$ is $\mathbb{Z}$.

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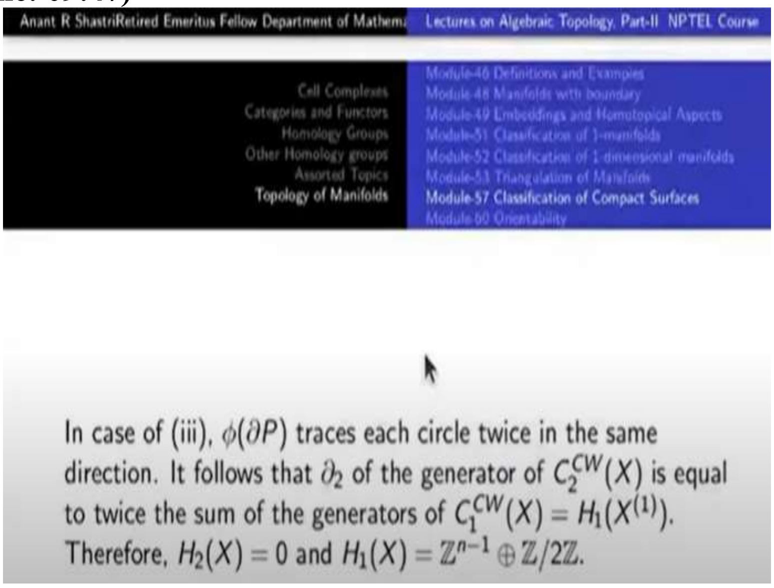

Let us see what happens in case (iii). Here, the q traces each circle twice but in the same direction. Therefore the degree of the projection composed q is 2, for each circle. Therefore ∂_2 is injective and its image is twice the sum of all the generators. It follows that $H_1(X)$ is $\mathbb{Z}^{m-1} \oplus \mathbb{Z}/2\mathbb{Z}$ and $H_2(X) = 0$. Now just by looking at H_1, you will see that they are all distinct for each member in the list. That completes a proof of part (B).

Nonetheless, you can also keep in mind that all members of (ii) and (iii) are distinguished by H_2 itself. In (ii), all of them have H_2 infinite cyclic whereas in (iii) all of them have H_2 equal to 0. However, in (i) also we have $H_2(\mathbb{S}^2)$ infinite cyclic. This property corresponds to the geometric notion of orientability, which we take as definition of orientability, here. Accordingly, all memebrs in (iii) are non orientable. This fact is also recognized by H_1, viz., a surface is orientable iff its H_1 has no 2 torsion.

Since homology is a homotopy invariant, we actually get a stronger conclusion here, viz., the list in the theorem gives distinct spaces even upto homotopy type. For topological spaces, singular homology is a topological invariant, it is also a homotopy invariant. Let us see now, just for fun, another proof of part (B) which is even simpler than this proof. Okay? Though it gives you less information apparently, but the statement here is much more stronger for some other reason. We also keep in mind that third series H_2 is 0 the second series H_2 is infinite cyclic.

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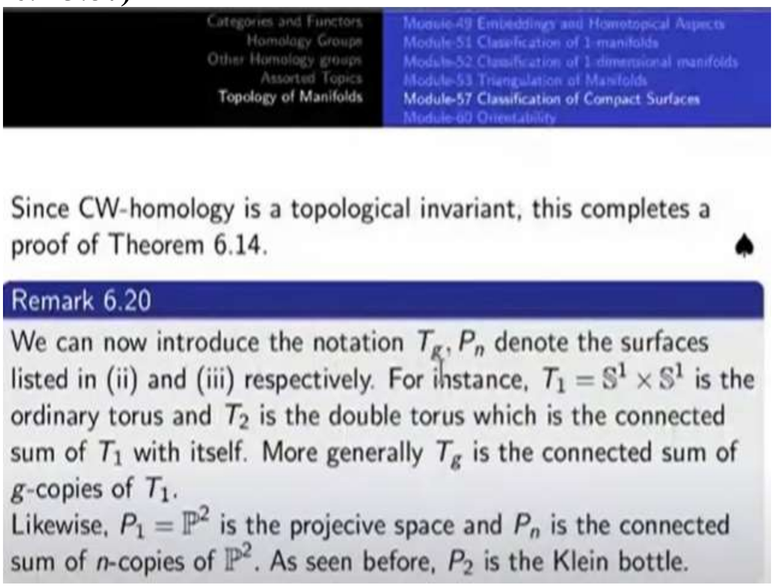

So, we can now introduce a notation before going to the second proof. T_g denotes the members of (ii) and P_m denotes the members of (iii). Of course we can use T_0 to denote (i) i.e, $T_0 = \mathbb{S}^2$ as well. It follows that T_g is the connected sum of g copies of T_1 which is the torus. Likewise P_1 is nothing but the projective space and P_n is the connected sum of n copies of $\mathbb{P}^2$. We have also proved that P_2 is nothing but the Klein bottle, okay? These are sum familiar surfaces.

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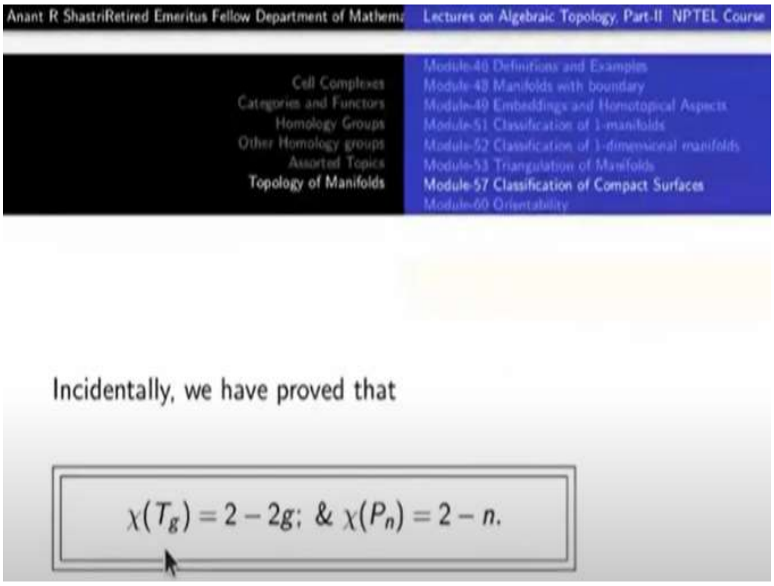

Incidentally what we have proved is that $\chi(T_g) = 2 - 2g$, and $\chi(P_n) = 2 - n$. Go back here and see in case (iii) the ranks of H_0, H_1 and H_2 are respecitively, $1, n-1$ and 0. Therefore $\chi(P_n) = 2 - n$. Likewise, in (ii) the ranks of H_0, H_1 and H_2 are resectively, are $1, 2g$, and 1. Therefore, within each series, just the Euler charactristic is enough to distinguish the members. Unfortunately, it will not work across the two series. Members of both first series and second series may have the same Euler characteristic where if $m = 2g$. So, there are elements here which have same Euler characteristics. Alright. So Euler characteristic together with orientability will distinguish surfaces.

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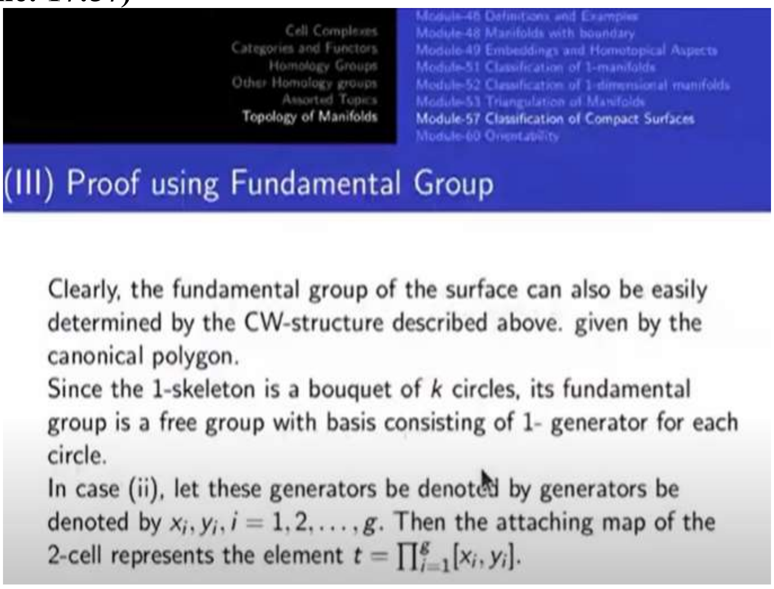

Now, let us go to the second proof using the fundamental group. Clearly the fundamental group of a surface can also be easily determined by the CW-structure described above given by the canonical polygon. Since the 1-skeleton is a bouquet of k circles, its fundamental group is a free group with bases consisting of one generator for each circle. This we have computed several times. In the case (ii), let these generators be denoted by the letters x_i, y_i corresponding to the image of the edges a_i, b_i respectively, $i = 1, \ldots, g$. It follows that the attaching map of the 2-cell represents the element $z = [x_1, y_1], \ldots, [x_g, y_g]$, the product of the commutators $[x_i, y_i] = x_i y_i x_i^{-1} y_i^{-1}$, taken in that order. Therefore, the fundamental group of X is the quotient of the free group by the normal subgroup generated by this one element z.

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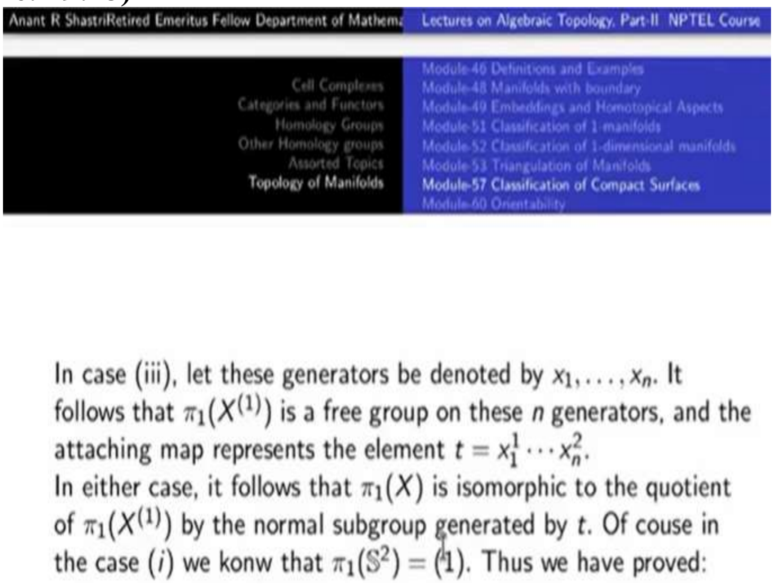

Let us see what happens in the case (iii). Here we may use the notation for the generators for $\pi_1(X^{(1)})$ to be $x_1, x_2, \ldots, x_n$, corresponding to the image of the edges a_i. Then the attaching map of the 2-cell represents the element $t = x_1^2 \ldots x_n^2$, the product taken in that order and the fundamental group of X is the quotient of the free group by the normal subgroup generated by t. Any way we also know that $\pi_1(\mathbb{S}^2)$ the trivial group.

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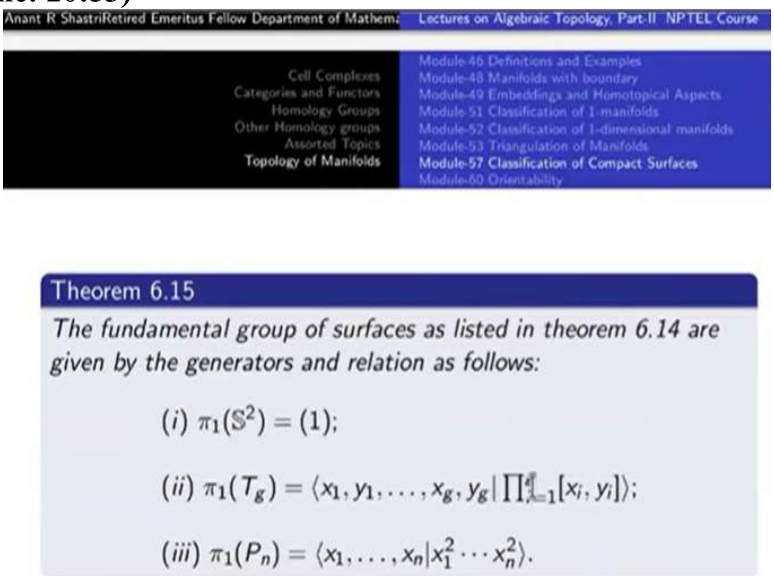

This we have summed up in theorem 6.15 below. This is the standard way of writing a group given by generators and relations. The first one is $\pi_1(\mathbb{S}^2) = (1)$. That means trivial group. (ii) $\pi_1(T_g)$ is generated by $x_1, y_1, \ldots, x_g, y_g$ and then there is a vertical separator here you know a divider line and next to it we write the elements which generate the normal subgroup. In this case, it is the product of the commutators $[x_i, y_i]$. And then entire thing put inside this angular brackets.

(iii) Here $\pi_1(P_m)$ is generated by m elements $x_1, x_2, \ldots, x_n$ and there is just one relation $x_1^2, \ldots, x_n^2$, okay? So, this is the theorem that we have deduced from our knowledge of the CW-complex structure given by a canonical polygon on X.

I want to say that this is enough now, to distinguish these members. The first one, in (i) the fundamental group here is trivial. In any of these other cases it will not be trivial. Why? What you are going to do here? Why the quotient of this free group by the normal subgroup is non trivial? Just take the abelianization of this group. As soon as you go to the abelianization, the image of the generators commute.

Thereore in (ii), the product of the commutators becomes the trivial element. The abelianization of a free group is a free abelian group of the same rank. Therefore the abelianzation of the quotient is now equal to the quotient of the free abelian group by the trivial group and is is equal to the free abelian group itself, which is actually of rank $2g$. Simialrly, in (iii) the abelianization of π_1 is nothing but the quotient of free abelian group over $x_1, x_2, \ldots, x_n$ modulo the element $2(x_1 + \cdots + x_n)$. This much information is enough to distinguish the members of the entire list.

Indeed, this is another way of computing H_1, because we have seen that the abelianization of π_1 is nothing but H_1.

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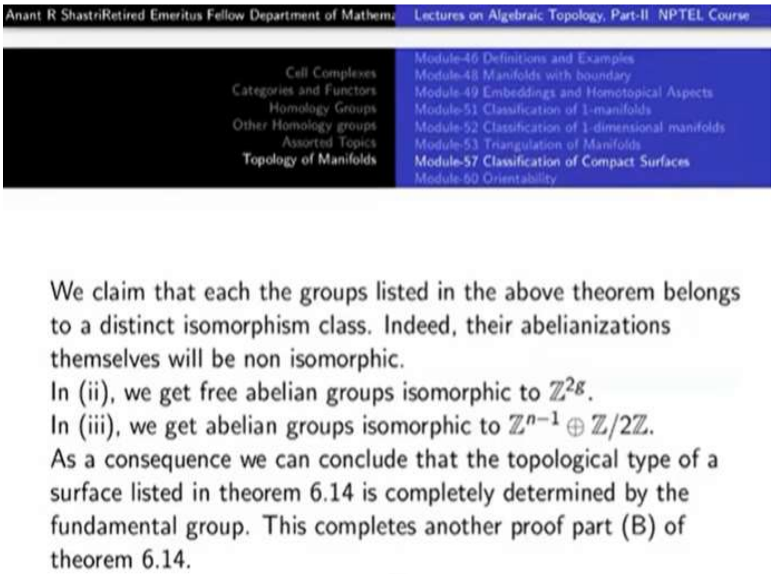

That completes two of the proofs of part (B). We can give yet another proof by putting directly a simplicial structure on X and using simplicial homology, which is not much different but more complicated. So, we shall skip that proof. Now, I have a few comments to make.

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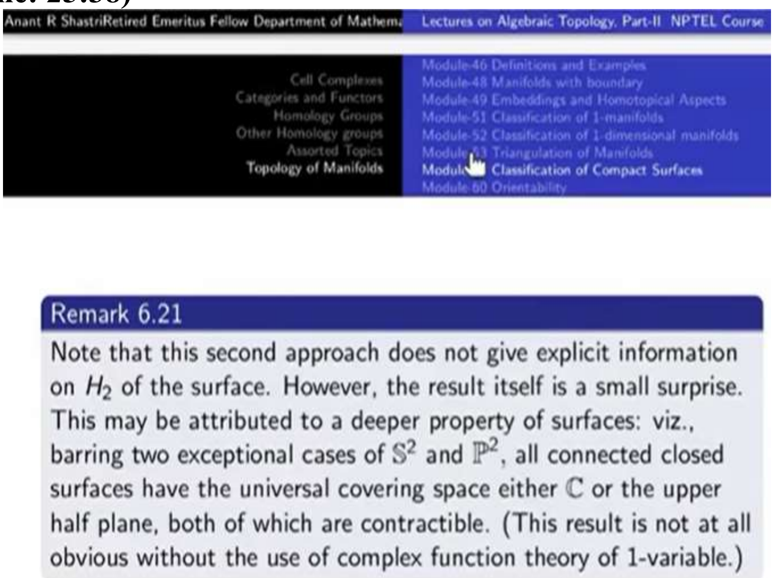

Note that this second proof does not use explicitly the information on H_2. Of course all spaces are connected and hence H_0 gives you nothing special. However, H_2 detects orientability of a surface. The same can be detected by π_1, namely existence of 2-torsion elements is equivalent to non orientability of the surface.

So, the fact that just the fundamental group of the surface is enough to determine the homotopy type is a small surprise. Actually even the diffeomorphism type can be determined though we have not proved it here. This is attributed to a deeper property of surfaces namely, barring just two cases $X = \mathbb{S}^2$ or $= \mathbb{P}^2$, in which case, the fundamental groups are finite, all other fundamental groups are infinite and all connected closed surfaces have the universal covering space either the complex plane or the upper half plane, both of which are contractible. So, all surfaces except two are covered by contractible spaces. That is something very special.

However, whatever I have stated just now is not at all obvious. Only for torus and the Klein bottle, we can easily see that the universal cover is the complex plane. For all those double torus, triple torus etc, there are no easy methods to detect their universal covers. Of course, it is all classical result but the only way so far is to use deep function theory of one complex variable. You have to use the group $SL(2;\mathbb{Z})$ and its action on the upper half plane via Mobius transformations, a very very entertaining and rich source of mathematical literature.

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You can ask whether such a result is true in higher dimensional manifolds, namely, suppose we have a n-manifold X covered by a contractible space. Even more restricted way, let us say it is covered by an Euclidean space. Is the homeomorphism type of X determined by the fundamental group? This problem goes under the name Borel's conjecture. It is at least 50 year old conjecture, which has been verified in every known case. A complete solution is yet to come. Okay? For more details, you can see Farrell[2002] which is there in ICTP notes.

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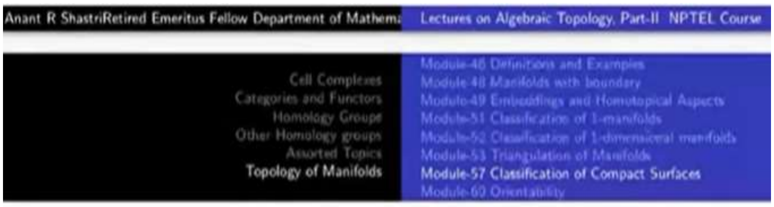

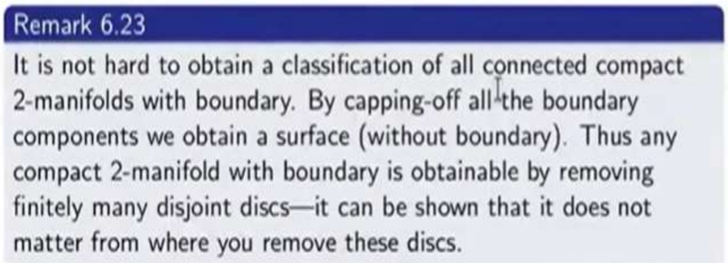

It is not hard to obtain classification all compact connected 2-manifolds with the boundary as well. Recall so far we have taken only boundaryless case. How do you go about it? Because it is compact there will be only finitely many boundary components. What are these boundary components? Each boundary component is a connected 1-dimensional compact manifold which does not have any boundary, and therefore it must be the circle. In each of the circle, you can just put a 2-disc and obtain a connected 2-manifold which has no boundary. This operation is called `capping of boundary spheres'. Thus, in principle, any compact manifold with boundary is got by starting with a corresponding manifold without boundary and then making a number of holes in it by removing small disks disjoint from each other, okay? So, this is the picture for all compact 2-manifolds.

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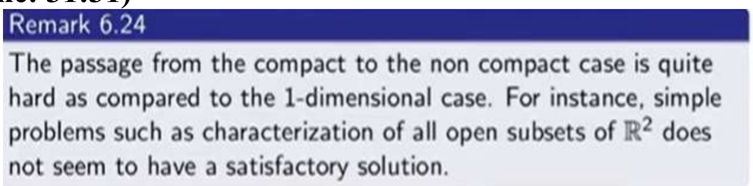

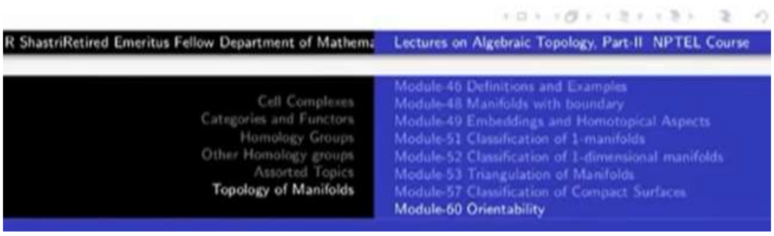

Now, the passage from compact to non compactness. That is a different game altogether. In dimension 1, we did not have so much of trouble in the proof of classification of 1-manifolds.

However, non compact manifolds of dimension 2 are still not well understood at all. People are trying to claim partial results here, partial results there, claims and counter-claims are going on. Even the classification of homeomorphism types of all open subsets of $\mathbb{R}^2$ is not known. Like this by putting additional conditions, by restricting the classes, some people have obtained positive results. So, you can also try your hand. Of course, first you look into the literature for whatever is known and try to go ahead okay?

So, that brings us to the end of this theorem, the classification of surfaces. Next time we shall consider yet another concept, which will be the last module in this course. Thank you.