Introduction to Algebraic Topology (Part –II)
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Lecture - 58
Final Reduction – Completion of the Proof

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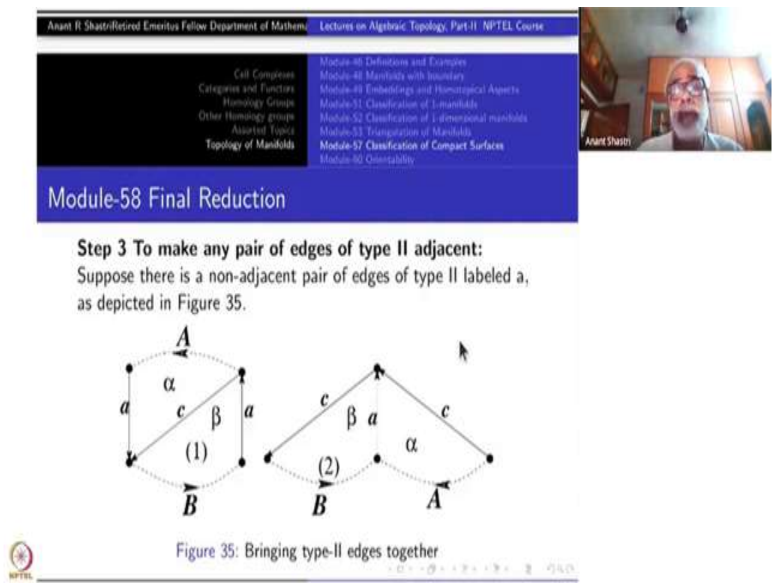

So, we are in the process of establishing classification of surfaces and last time we began with the canonical polygons and carried out two of the steps in the reduction process. One was to get pairs of type I which are adjacent eliminated. The second one was to see that there is only one vertex class. All the vertices on the canonical polygon should be identified to a single point. This we have achieved. Now, the next step 3 here is to make any pair of edges of type II adjacent. okay?

So, that means that we are assuming there is a pair of type II which is not adjacent. If it is adjacent there is nothing to do.) Assuming it is not adjacent what is the meaning of that? There will be an edge a here and then there will be some sequence of other edges okay? Then the edge a occurs again, followed by another sequence of edges. Cyclically, we can express the sequence in the form $AaBa$ as in the following figure, with A and B non empty.

So, what I do now? I start with the tip of first edge a to the tip of second edge and join them by a line segment and label it c. I am going to cut the polygon along c. The top portion here is denoted by α and the other one by β. The part β is kept as it is and the top portion α is turned around so that its edge marked a gets aligned with the edge of β marked a. Then identify these two edges with each other.

The resulting polygon as such may not be convex. That is not a problem at all. We can merely relocate the vertices so as to make it convex. The homeomorphism type of the quotient space does not change. All that we need is to look at the sequence defined by the edges in the boundary of the polygon. That is what we do. So, what has happened? The edges marked a have disappeared, these two have disappeared, but two new edges have come which are marked with c and c, okay? And they form an adjacent pair of type II. So, this is the step 3.

So, in effect, we have brought the non adjacent pair together. In doing this, other adjacent type II pairs are either in A or in B and hence they do not get disturbed. So, that is an important point here. Therefore, repeating this process again and again, all the edges of type II will become adjacent. That does not mean they are all together in one place in the sequence. Each of them will be an adjacent pair just means, the sequence may be of the form $xxAyyB$,... where A, B etc may or may not be empty. That is the achievement in step III.

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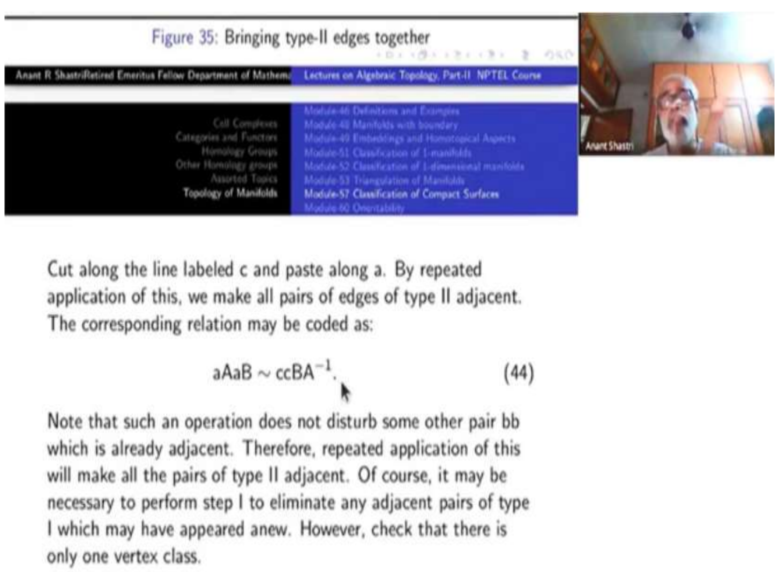

What is the corresponding transformation code? $aAaB$ is equivalent $ccBA^{-1}$ (50)

Okay? (Note that A^{-1} means if $A = a_1a_2 \ldots a_n$ then $A^{-1} = a_n^{-1}a_{n-1}^{-1} \ldots a_1^{-1}$. That is the inverse sequence. Okay?) Note that such an operation (50) does not disturb some of the other pairs bbm, cc etcetera which are present in A and already adjacent. Of course, it may be necessary to perform step I again, namely, a new adjacent pair of type I may occur which you may have to cancel out. okay? On the other hand, since there is only one vertex class to begin with it easily follows that there is only one vertex class after this step II as well and hence a sequence of type-I adjacent pair does not occur.

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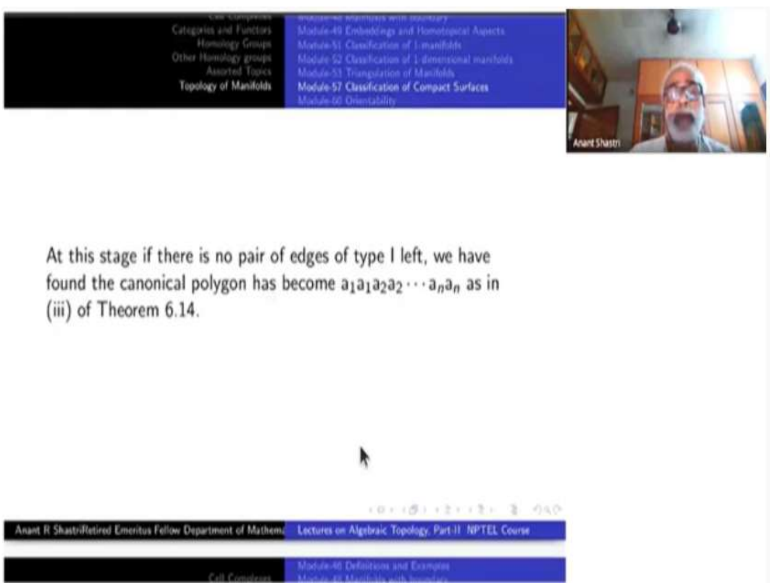

At this stage, let us take a look at the sequence. It may happen that there are no edge pairs of Type-I at all. That means that the sequence belongs to the sub list (iii) which is normal form. So, we have to look at the other possibility only, which means that there is at least one pair of type I which is not adjacent okay. So, what to do with that one? That is the point okay. So, this is the step named `handling an interlocked pair of edges of type I.

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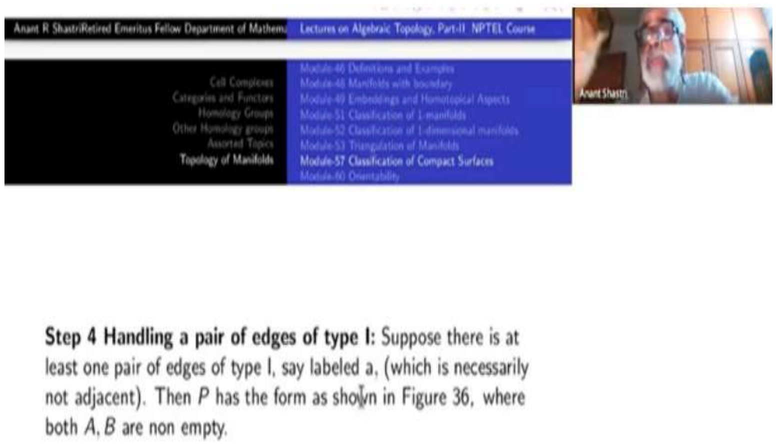

Suppose there is at least one pair of edges of type I say labeled a and then other one somewhere a^{-1}. Since adjacent pair of Type I are already eliminated, it follows that the sequence is of the form $aAa^{-1}B$ with A and B nonempty as shown in figure 39.

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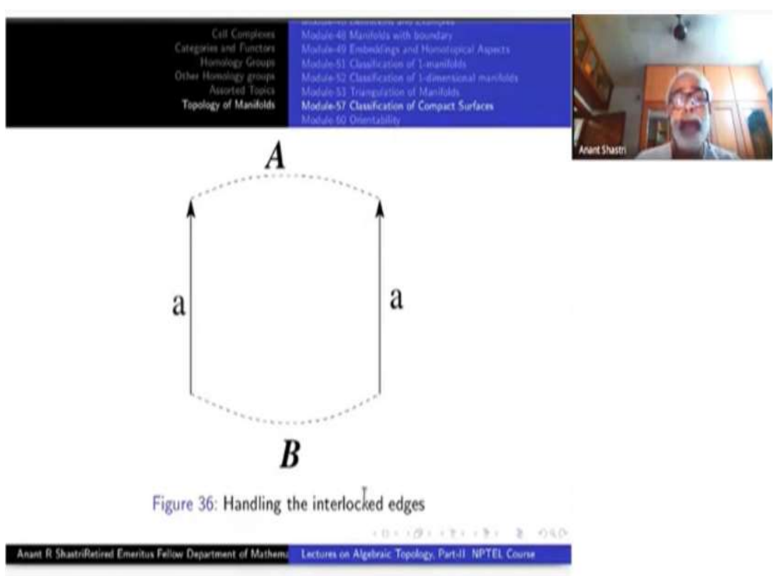

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We claim that there must be some edge in A which is identified with an edge in B. If not, that means, all edges inside A are identified with edges within A and the same will be true of edges in B also. It follows that there will be at least two equivalence classes of vertices in which case the sequence should have been as in (i) contradicting our assumption.

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Therefore there must be an edge b in A and an edge b^{-1} in B since Type II pairs have all become adjacent. So, we can rewrite the sequence in a better way as follows: $AbBa^{-1}Cb^{-1}D$. Notice that any or all of A, B, C, D may be empty. Therefore, what we get is a better picture of the sequence as shown in picture 1 of Fig 40. Thus we have actually another non adjacent pair of type I, the two of them interlocked.

Interlocked means what, starting with a in the sequence, before you come to a^{-1} the edge b will occur and vice versa. okay? Now, we want to transform this sequence in which the interlocked pairs will be brought together with out breaking any of the sequences A, B, C and D. Now it is better to assume that at least two of A, B, C, D are nonempty. Otherwise, cyclically, the sequence will be of the form $Aaba^{-1}b^{-1}$ and there is nothing to do, though we do not need to use this assumption.

So this time cut along the segment running from tip (end point) of first a to the tip of the other a, denote this segment by c, call the lower part of the polygon α and the upper part β. This β will be brought below (just slide it down), α so that the edges marked b in either of them are aligned. Then identify these two edges. The new edge-sequence will be what? We may just read it from wherever you like okay? say, $Bc^{-1}ADaca^{-1}C$. The edge c is new and is of type I and not adjacent. You redraw the polygon as a convex polygon, as shown in the third picture here.

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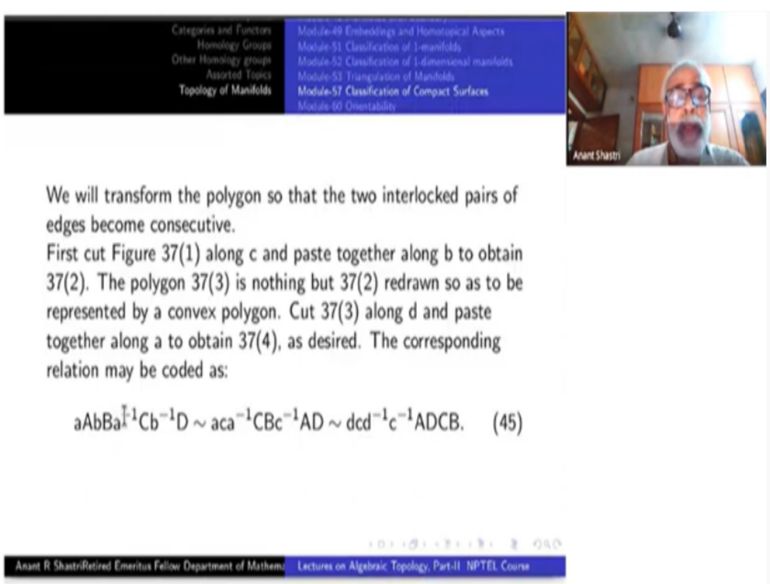

This time again, make the cut along the segment marked d running from tip of c to tip of the other c, and this time perform the identification along the two edges marked a. That means what? this portion marked α has to be brought here and turned around so that the edges marked by a are aligned together and then identified. In the resulting polygon, the edges marked a disappear and the new edges marked d appear and they are of Type I which is not adjacent. Indeed now you see that the two new pairs are together and the new sequence is $dcd^{-1}c^{-1}ADCB$.

Now you can look at the portion $ADCB$ and repeat this step if necessary, so as to bring yet another interlocked pair of edge together. It is important to note that at each step, the initial port of the sequence which consists of interlocked pairs together, will not be disturbed and the new interlocked pair just comes before that.

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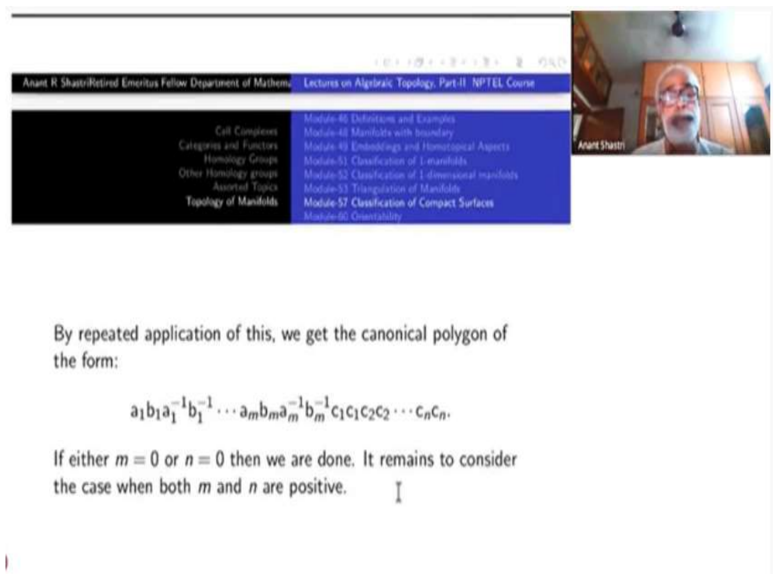

By repeated application of this step, we get the canonical polygon of the form $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}...a_mb_ma_m^{-1}b_m^{-1}$ followed by a sequence of type II pairs $c_1c_1c_2c_2....c_nc_n$. If either m is 0 or n is 0, then the entire sequence is either in the sub list (iii) or (ii), okay? That would be as per the statement of the classification. If both m and n are non zero, such a sequence is not a normal one.

So we have to do one more hat-trick here to convert such a sequence into a normal sequence Okay? So that will be the last step to go to viz., handling the case of mixed types. Okay?

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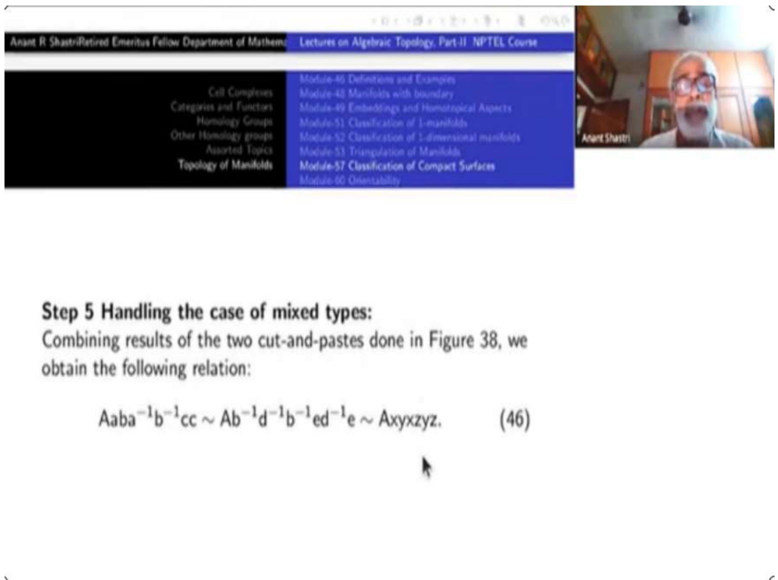

So now we will concentrate on the middle portion of this sequence viz., $\ldots a_m b_m a_m^{-1} b_m^{-1} c 1 c_1 \ldots$ one of the type I pair here together with a type II pair. If we under stand this case, viz., $m = 1$ and $n = 1$, then perhaps we may be able to handle the general case as well, okay? We will convert this portion into a sequence of entirely type II edges. That is the whole idea.

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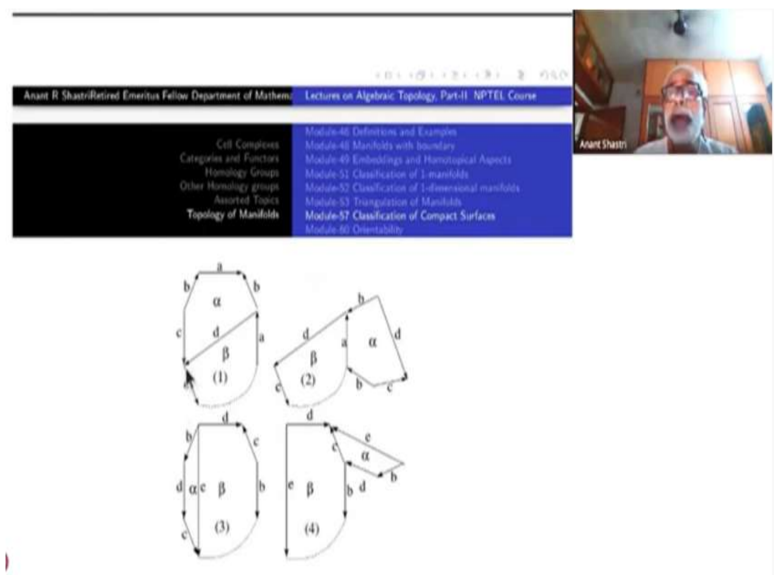

Rewriting the sequence cyclically in the form $aba^{-1}b^{-1}ccA$, perform a cut and paste as shown in the figure (41), the final result can be relabeled as $xyxzyzA$, which is quite near to what we want but not exactly. That is only the first step. Let us see how this is done. Okay?

Look at the segment from the tip a to the tip of c. So, you have to make this guess: where to perform the first cut. This is not at all obvious okay. This cut is refuting our intuition of not spoiling some thing which is already nice. You may be worried because you have done so much of effort to bring them together and now you are asking to cut it, okay?

So, you have to do that cut here okay? And bring the two parts together so that the edges marked a are aligned and perform the identification. At this stage the edges marked a disappear and new edge pair marked d appear, which is of type-I.

The new sequence is $b^{-1}cd^{-1}bdcA$. A is undisturbed. Now, one more cut you have to do. Look at the segment running from starting point of b to the end of c. Mark it by e and make the cut along this e. Bring the two pieces together, this time, such that the edges marked with c are aligned. This requires that one of the pieces has to be flipped also not just sliding and rotating. As usual, the edge pair c disappears and we get a new edge pair marked e, which is of type II. The old pair marked d will become a pair of Type II as well. So, the new sequence can be read as $b^{-1}d^{-1}b^{-1}ed^{-1}eA$.

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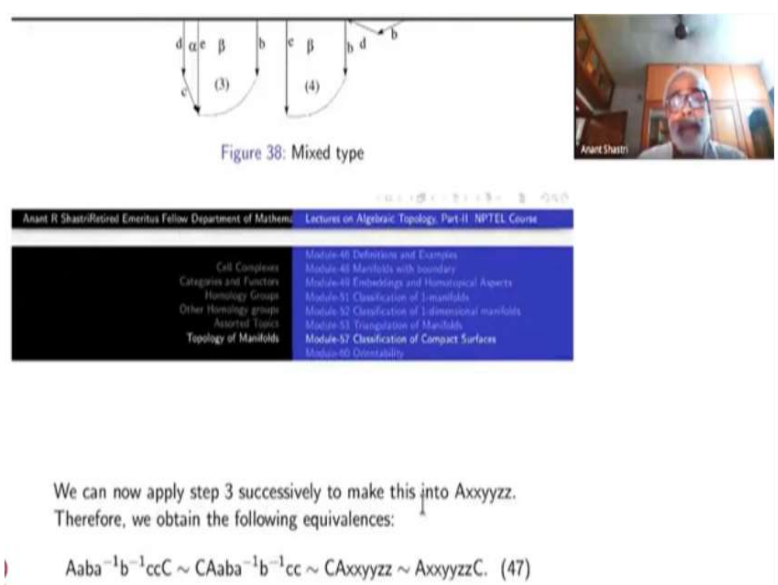

We can now apply step 3 successively to make this into $xxyyzzA$. In summary this step provides us a method to convert $aba^{-1}b^{-1}cc$ into $xxyyzz$.

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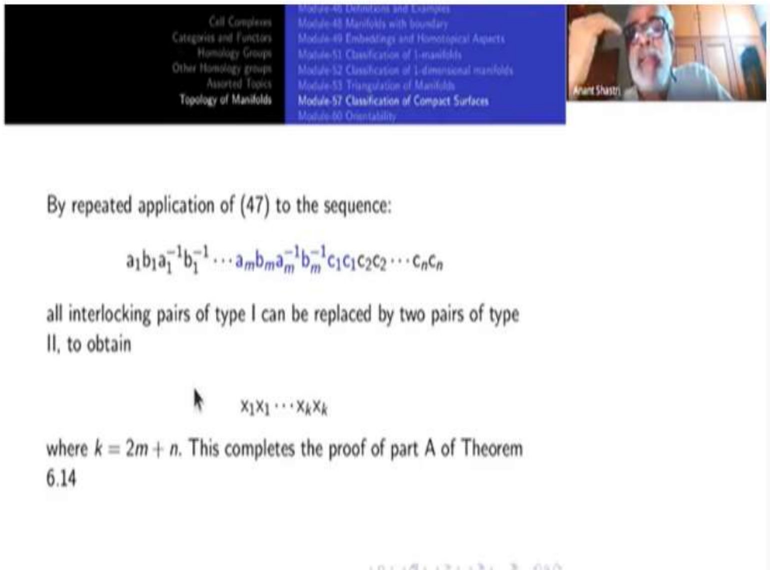

Repeated application of this step yields the required result. Namely, let us denote the two normal sequences in (ii) and (iii) by A_g and B_n. Then we have for g and $n \geq 1$, $A_gB_m \sim A_{g-1}aba^{-1}b^{-1}ccB_{m-1} \sim aba^{-1}b^{-1}ccB_{m-1}A_{g-1}$. Further, which is $\sim xxyyzzB_{m-1}A_{g-1} \sim A_{g-1}xxyyzzB_{m-1} \sim A_{g-1}B_{m+2}$. Repeated application of this yields that $A_gB_m \sim B_{m+2g}$ which is in the normal form and belongs to sub list (iii). So, this completes the proof for part A of the theorem 6.4.

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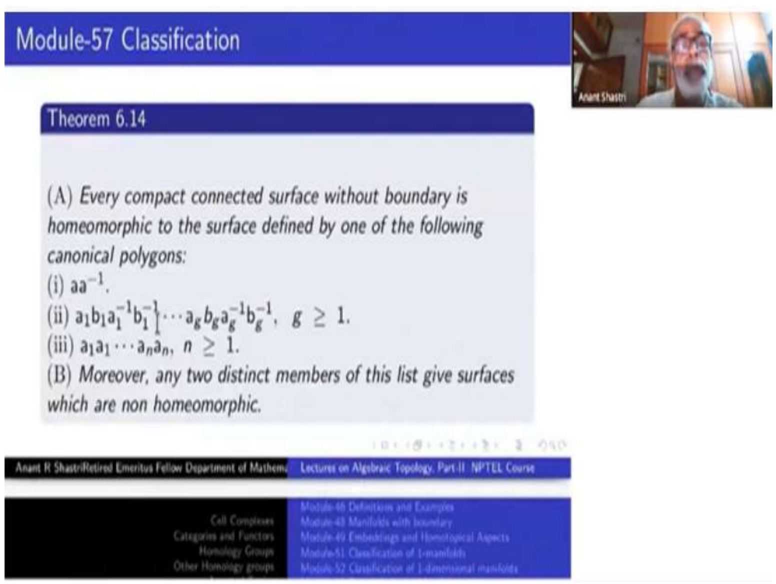

The first member (i) represents the sphere $\mathbb{S}^2$. List (ii) gives all those called orientable surfaces of genus g. $g = 0$ also makes sense but that in the list (i). So we take $g > 0$. $g = 1$ is the torus. Similarly, (iii) gives all the non orientable surfaces. Part (B) of the theorem asserts that there are no repetitions in the entire list.

We give two different proofs of this using whatever we have done so far. Okay? So today, we will stop here. Thank you.