Introduction to Algebraic Topology (Part - II)
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Lecture - 05
Topological Properties

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So, having seen a number of examples of CW complexes we return to the more elaborate study of topological properties of CW complexes.

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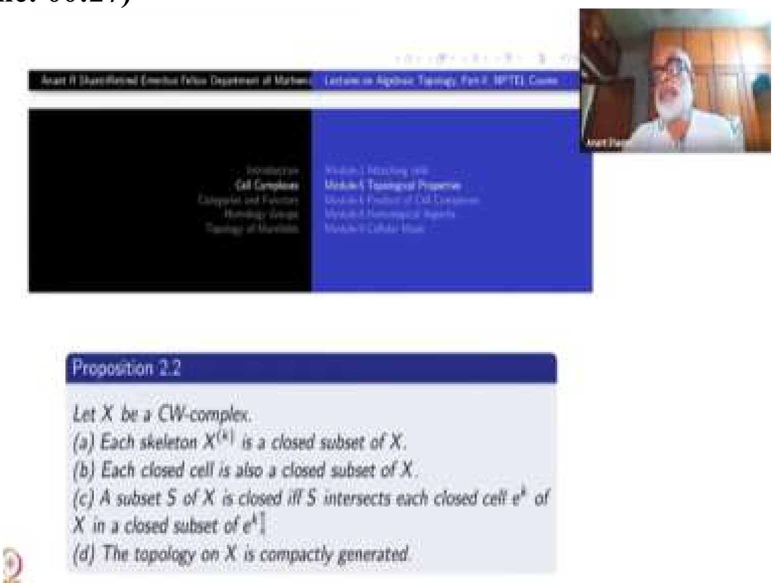

To start with a CW complex X then $X^{(k)}$ is the k-th skeleton of X, namely, union of all cells of dimension less than or equal to k, right? Each skeleton $X^{(k)}$ is a closed subset of X. This is the first statement.

The second statement is: each closed cell is also a closed subset of X. A closed cell is not just a name, but it refers to the closeness of the underlying set in X.

A subset S of X is closed if and only if S intersects each closed cell e^k of X in a closed subset of e^k. In particular, this means that the topology on X is compactly generated. So, these are the 4 observations you want to make about the topology of CW complex okay? Remember a subset is closed in X if and only if it intersects each skeleton in a closed set that is the definition of the topology on X, right? If X itself is actually a skeleton then we do not need this definition because each skeleton $X^{(k)}$ is obtained by attaching k-cells to $X^{(k-1)}$, and therefore, the topology is well defined there as a quotient topology of a certain object. Alright. So, this definition this statement statement (c) or (d) are needed only when dimension of X is infinite.

So, first of all each $X^{(k)}$ is a closed subset of X. How do you see that? Look at $X^{(k)}$ intersection with $X^{(\ell)}$. If $k < \ell$, then this is just $X^{(k)}$. If $k \geq l$ it is $X^{(\ell)}$. So, either it is $X^{(\ell)}$ or $X^{(k)}$, they are closed subsets is what I have to show, right? Now, if $X^{(k)}$ is contained in $X^{(\ell)}$ i.e, $\ell \geq k$, we know that $X^{(\ell)}$ is obtained by attaching cells to $X^{(k)}$ of dimension $k+1, k+2$ and so on upto ℓ. When you attach cells to a space Y to get X, we have seen that Y is a closed subset of X, okay? So, successively $X^{(k)}$ will be closed subset of $X^{(k+1)}$ closed in $X^{(k+2)}$ and so on, it follows that $X^{(k)}$ is closed in $X^{(\ell)}$, if $\ell \geq k$. So, you can interchange k and ℓ roles to conclude that the intersection of $X^{(k)}$ with every other $X^{(\ell)}$ is closed in $X^{(\ell)}$. Therefore, $X^{(k)}$ will be closed in X.

Next one. Each closed cell is a closed subset of X. To see that it is enough to show that each k-cell is closed in $X^{(k)}$, because we have already proved that each $X^{(k)}$ is closed in X. This will be done by induction on k.

For $k = 0$, if $X^{(0)}$ is a discrete space and hence each 0-cell is closed. Moreover, $X^{(0)}$ is Hausdorff. Inductively, by cor 1.1, it follows that each $X^{(k)}$ is Hausdorff. Since e^k is compact it follows that it is closed in $X^{(k)}$.

The next thing: Take a subset S of X. One part is obvious. If it is closed in X intersection with each cell e is closed in X. Now, we have to look at the converse. Given S intersection e is closed in e for all cells e. We want to show that $S \cap X^{(k)}$ is closed in $X^{(k)}$ for every k. This again is done by induction. For $k = 0$ since $X^{(0)}$ is discrete, there is nothing to prove. Having proved that $S \cap X^{(k)}$ is closed, it follows from Lemma 1.1 (c) that $S \cap X^{(k+1)}$ is closed in $X^{(k+1)}$.

Finally to prove (d): What is the meaning of compactly generated the same thing as instead of e^k, I have to put a compact set for every compact set of X. Take a set S, it is closed in X iff $S \cap K$ is closed in K for every compact subset K of X. Again we have to show only the `if' part. Conversely, since each closed cell e is compact it follows that $S \cap e$ is closed in e. Hence, by the previous part it follws that S is closed in X.

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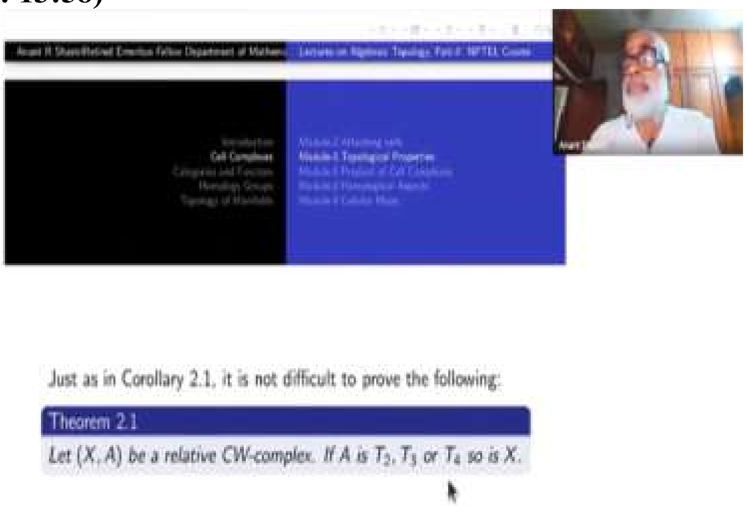

So I will elaborate on that one a little bit later. Right now I say only this much. If you have worked out related exercises on attaching cells, then the following theorem will be obvious. Namely, if Y is Hausdorff/regular/ or normal, then X is Hausdorff/ regular/or normal. That is what I have given in next slide. Now inductively, you can prove that if A is a Hausdorff space or a regular space or a normal space then so is X, where (X, A) is a relative CW complex. In particular, every CW complex is a T_4-space. So, this is left as an exercise for you now, having solved that exercise this should be easier.

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The next one is you should pay attention to this one. It is a very, very important result on CW complexes, all the time used, okay? So, what is it? It just says that every CW complex is a disjoint union of open cells. Do not erroneously conclude that this implies it is disconnected. `Open cell' is just a name--- they are not open subsets of X, not necessarily, some of them maybe some of them are not indeed they are open cells if they are maximal, that are no cells of higher dimension will intersect the interior. Then only they are open in X. An open cell, by definition, is just the interior of $\mathbb{D}^k$ whatever under the image of the quotient map occurring in the attaching k-cells. It is open inside the closed cell, that is all. So, what we have here is: take all the interior of the cells that will be the union of the entire CW complex.

We have seen similar result in case of simplicial complexes. It is exactly same here and in fact, much simpler. What is it? First of all, by the very definition of CW complex, you can write X as $X^{(0)} \cup X^{(1)} \setminus X^{(0)} \cup X^{(2)} \cup X^{(1)}$ and so on, a disjoint union. In this notation I have taken $X^{(-1)}$ as the empty set to start with. $X^{(0)}$ is the union of 0-cells and 0-cells are both open and closed. So, all these points belong $X^{(0)}$ which is disjoint union right.? So, we start with all the 0-cells, they are open cells Okay? Now, look at $X^{(1)} \setminus X^{(0)}$ what is it? There are open 1-cells and they are disjoint. Any two distinct open 1-cells are always disjoint. Okay? So, more generally, $X^{(k)} \setminus X^{(k-1)}$ is a disjoint union of open k-cells. Take any k-cell, its boundary is contained inside $X^{(k-1)}$ and you are throwing away that. What you are left with is its interior okay.

So, this is a totally obvious statement, but this is very useful one.

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Now, let us see how it is useful. This lemma says take a subset S that contains at the most one point from each open cell, i.e., S intersection with an open cell is a singleton or an emptyset. So, this is the condition on S. Such a set is closed automatically and is a discrete subset of X, okay? If we think a little bit, it is obvious, but now, let me write down the full proof of this one, okay? There are books and books, which give you long, long proofs of this.

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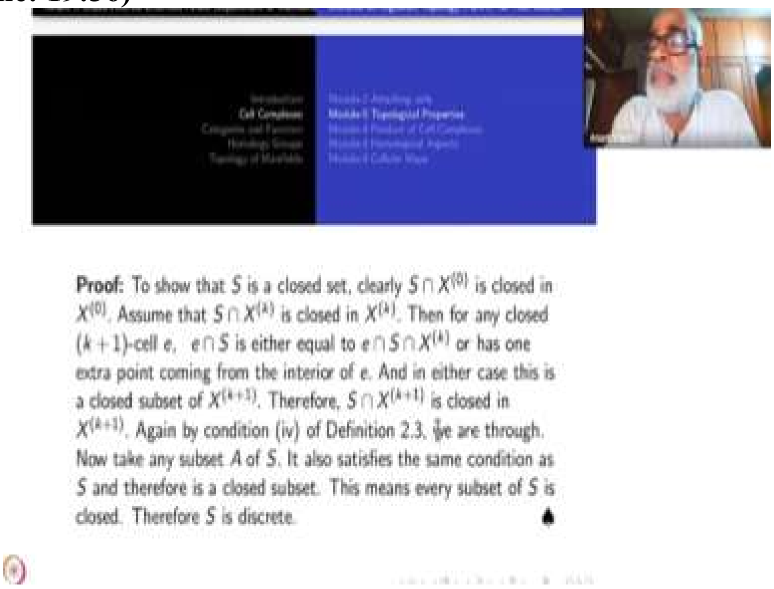

So, for myself, I learnt this proof only much later. and then I found out that there are books which give this proof also.

To show that S is a closed set what you have to do? Intersect with each $X^{(k)}$ and show that it is closed inside $X^{(k)}$. Start with $X^{(0)}$, $X(0)$ is a discrete space every subset is closed that is fine. Now assume that $S \cap X^{(k)}$ is closed in $X^{(k)}$, I have to show that $S \cap X^{(k+1)}$ is closed in $X^{(k+1)}$, right? For that what I have to show? By our lemma, it is enough to show that for any closed $(k+1)$-cell e, $e \cap S$ is closed in e. This set has at most one extra isolated point in the interior of e, other than the set $e \cap X^{(k)}$, which is any way a closed subset of $X^{(k)}$. Since a singleton is closed in the interior of e, we are done.

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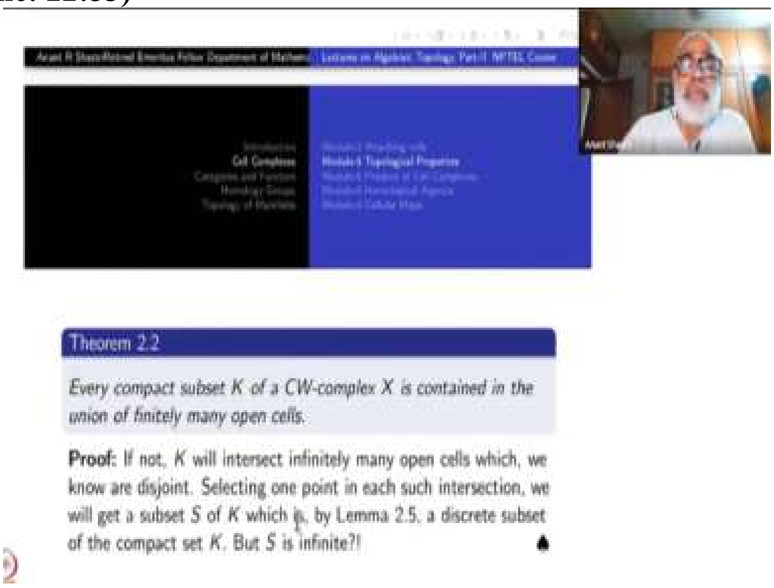

We can use this one to prove certain properties that I was telling you about finitely many cells and so on. That comes now very easily okay? Take any compact subset K of a CW complex X; it is contained in the union of finitely many open cells. The funny thing here is that these open cells may not form an open cover for K, because they may not be open subsets of X. Yet the conclusion is that K is contained in a finite number of open cells. This is an easy consequence of the previous lemma okay? Since X itself the union of open cells, every subset is is contained in the union of open cells right? You start with a compact set K. Suppose you need infinitely many open cells to cover it. That means what? That means there will be infinitely many points infinitely many open cells, which will intersect K. Pick up 1 point from each of these open cells, only one point which is common with K and for a subset A. Then what will the property of this subset A? This subset A has the same property as in the previous lemma. But it is a subset of K. The previous lemma says that such a subset is discrete, but A is a subset of K, K is compact. A discrete subset of a compact set is finite. So, A could not have infinitely many points like that. Okay?

So, I repeat if the claim not true what happens? K will intersect infinitely open cells which we know are disjoint. Therefore, I can select 1 point in each such intersection to get a subset A which is infinite and discrete subset of K. That is not possible okay?

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So, this is an easy consequence. Now the closure of every cell, we know, is compact. Therefore, it meets only finitely many open cells because every compact set intersects only finitely many open cells. This property that closure of a cell meeting finitely open cells was called `closure finite' property. Okay?

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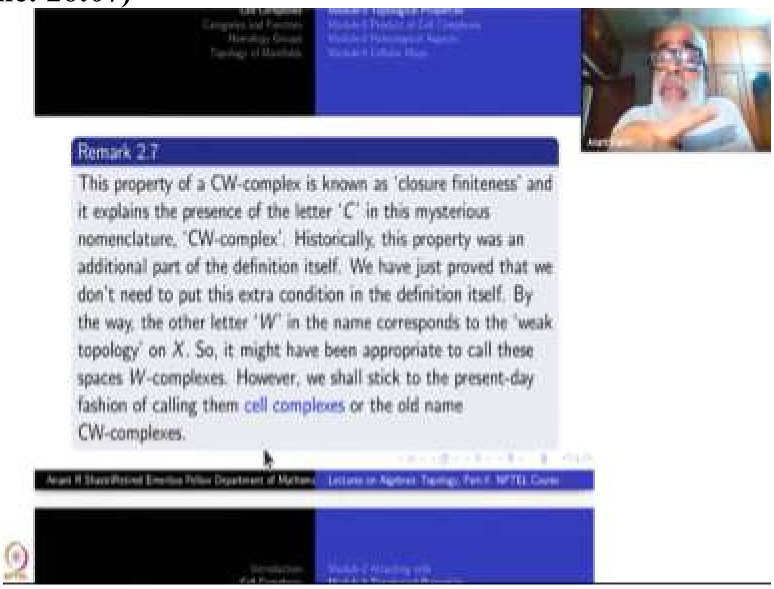

So, this is the remark I wanted to make. Now I am going to elaborate it. I was all the time referring to the property of a CW-complex known as closure finiteness Okay? that means what? Closure of every cell will meet only finitely many open cells okay? So, this explains the presence of the letter C, in CW-complex. `C' corresponds to the closure finiteness in this mysterious nomenclature of CW-complex. CW complex was the name given by J. H. C Whitehead, the creator of this concept. Historically this property was put as an additional condition of the definition itself by J. H. C Whitehead. His definition of a CW complex was very, very long. He had put many of the standard properties which we are discussing now as a consequence in the definition itself. Okay? So, this was one of the properties, closure finiteness which we have just proved. We do not need to put this extra condition in the definition.

By the way, there is this letter W. What is this W corresponding to in the name CW? C corresponds to closure finite, what is W? W corresponds to the weak topology, the compactly generated topology okay? That is also called weak topology okay? W corresponds to the weak topology and so. There is no need for the letter C we have seen that. But the weak topology part is necessary. Therefore, we could just delete C and kept only W and call them W-complexes. But nobody has made that kind of suggestion so far. I am alone perhaps. So, let us not introduce a new name, let us not create a new nomenclature here. Many authors just call them cell complexes. So, that is a good name okay. But I would say that you better stick to the name CW-complex, because there are many notions of cell complexes, each author has his own definition and sometimes they just differ slightly like someone may not put Hausdorffness condition, someone may put something else and so on and the structure also somewhat different. So, let us not bother about these new terminology. If you read old papers, then what you will get is this kind of definition.

So, the closure finiteness is a consequence, alright? Actually, the fundamental thing is here is the quotient space structure in the definition of attaching cells which gives you inductively, a sequence of topological spaces, $X^{(0)}, X^{(1)}, X^{(2)}, X^{(3),\cdots}$ one contained in the other. X itself is the union as a set and the topology is the one infinity that can be X itself has the topology or induced by these inclusion maps. You may not directly say that it is compactly generated, because before talking about compactly generated, you have to have a topology on X and consider the family of compact subsets of it. So, all these problems are neatly resolved in the definition of a CW complex by the author. He designed CW complex you know. Though he had the correct ideas, many things were not available to him. So, many of these things were developed much later. So, this was just a great invention by J. H. C Whitehead, after all.

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Let me now go ahead with the study of topology of CW complexes, in the spirit of our earlier lemma, of extending the neighbourhoods, which can be deformed and so on. Remember that lemma or the proposition? Okay. So, I am going to make another such lemma here which will be useful later in the study of CW complexes. What does it say?

Start with a CW complex. For each n, let V_n be a neighbourhood of $X^{(n)}$ inside $X^{(n+1)}$ okay? Let x be a point in the interior of, say, some k-cell σ and W_k be an open neighbourhood of $x \in \sigma$. Note that W_k need not be open in X but just open in σ. Then there exists an open neighbourhood W of $x \in X$ such that $W \cap X^{(k)}$ is W_k and $W \cap X^{(n)}$ is contained in V_n for all n.

So, this W_k which is an open neighbourhood of $x \in \sigma$ gets extended to W. Extended means what? Intersection of W with $X^{(k)}$ is exactly W_k. Also it is controlled in the sense that intersection with $X^{(n)}$ is contained inside V_n. It is not becoming too big. This W is a neighbourhood of $x \in X$ itself. W_k in open in $X^{(k)}$ and may not be open inside X, Okay? It is like you have a point on the line and the neighbourhood of that point on the line is like an interval, but now the interval is contained is $\mathbb{R}^2$, that interval is not open inside $\mathbb{R}^2$, but you can extend it to an open square and that is precisely what we have done in the first lemma.

We took an arbitrary subset A of the boundary of $\mathbb{D}^n$ then we extend it to a neighbourhood of A inside the disk, right? And this elementary construction is going to help us in all this.

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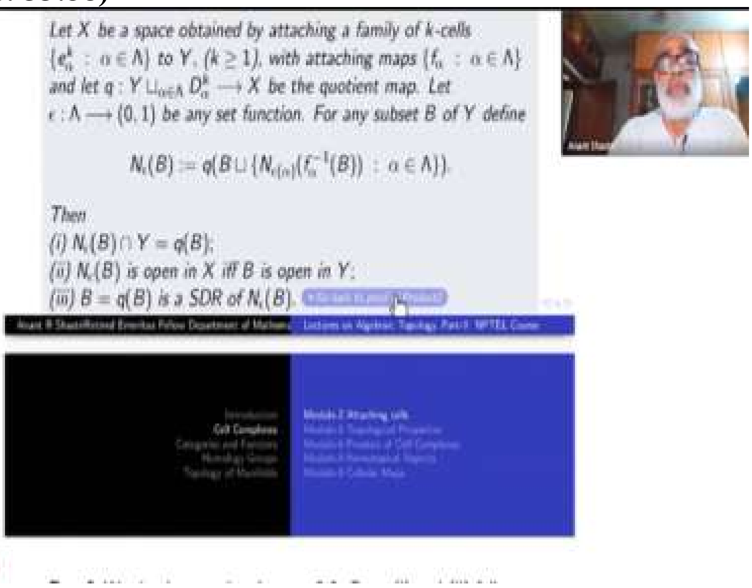

So, this is Proposition 1.1 that we had for attaching cells Okay? Start with a subset B inside Y. This B gets thickened to $N_\epsilon(B)$, okay? (this ϵ could be chosen at your will) and that is an open subset of X iff B is open in Y, okay? And if you intersect it with Y, then it is just B itself. Moreover, B is strong deformation retraction of $N_\epsilon(B)$. This last part is an extra thing, we do not need this one right now, Okay? All I want at present is the extension of an open set in Y to X. So, we have to do an induction, going from $X^{(k)}$ to $X^{(k+1)}$ and then to $X^{(k+2)}$ for which we are going to use this proposition, okay?

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So, by this from the proposition: for each $(k+1)$-cell $\tau, \tau \cap V_{(k+1)}$ is an open subset in τ containing the entire boundary of τ because V_n's are neighbourhoods of $X^{(n)}$ for all n, so they contain the boundary of any simpler $(n+1)$-simplex which is a subset of $X^{(n)}$. When you take the inverse image under the attaching map, it will contain the entire boundary sphere. However, I am not going above to the disjoint union at all via these attaching maps etc. I am just working in X itself, so, the notations become simpler here okay. So, for each $(k+1)$-cell τ, $\tau \cap V_{(k+1)}$ is an open subset of τ containing the entire boundary which is actually a subset of $X^{(k)}$. Since the boundary $\tau.$ is compact. you can choose $\epsilon(\tau)$ positive, without any problem, such that $N_\epsilon(\tau.)$ is contained inside $\tau \cap V_{(k+1)}$. This happens for every τ therefore, you can take B_τ, which is equal to a smaller set, namely, $N_\epsilon(\tau. \cap W_k)$, that will be a subset of $\tau \cap V_{(k+1)}$. Okay? Therefore, N_ϵ of this intersection will be an open subset in $X^{(k+1)}$. This is the import of this lemma. Okay?

So take now take $W_{(k+1)}$ to be the union of all these B_τ's as done in the proposition. You should take union of all these B_τ's, that is going to be an open neighbourhood of this x because x is there that will be an open subset of $X^{(k+1)}$, it is an open neighbourhood of x. Okay? And its intersection with $X^{(k)}$ is precisely W_k. And $W_{(k+1)}$ is contained in $V_{(k+1)}$, The inductive step is over. Okay?

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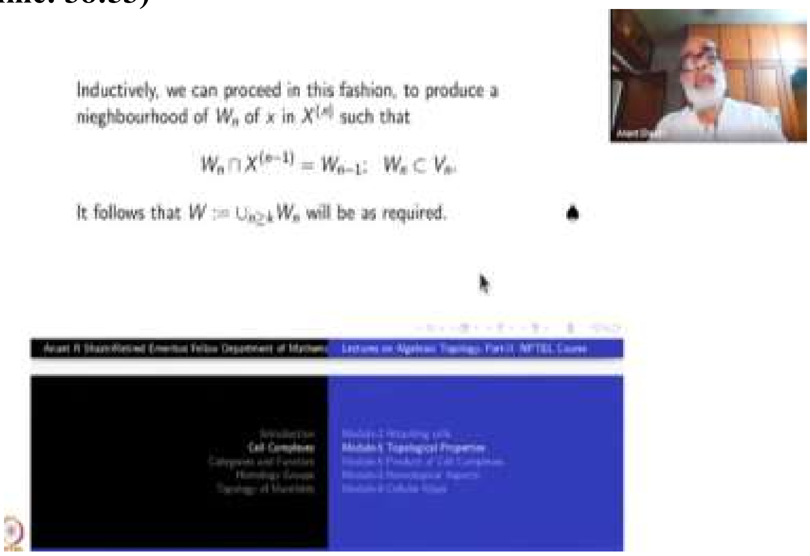

Now what do you do? Just take W equal to union of all the W_n's with $n \geq k$. We start with W_k because x happens to be inside one of the k-cells. If x is a vertex, i.e., a 0-cell, you would have taken all these things and greater equal to 0. Okay, Why this is open? Intersection with each $X^{(n)}$ is W_n which is open in $X^{(n)}$. Why $W \cap X^{(n)}$ is contained in V_n, because W_n is contained in V_n. Okay?

So, I call this a control extension. So, we will use this one much later of course, but we will use that alright. So, we have done quite a bit of topology, but there are many more things to be done. One of the most important thing is that in CW complexes, the weak topology is very, very useful in constructing continuous functions on CW complexes (this was the motivation) and in verifying that some function is continuous. Defining a continuous function as well as verifying the continuity of a given function on a CW complex structure, the weak topology is very useful whereas if on an arbitrary space this will be hard. In particular, we will be able to see that there are lots of continuous function. This will do next time. So today this is over, thank you.