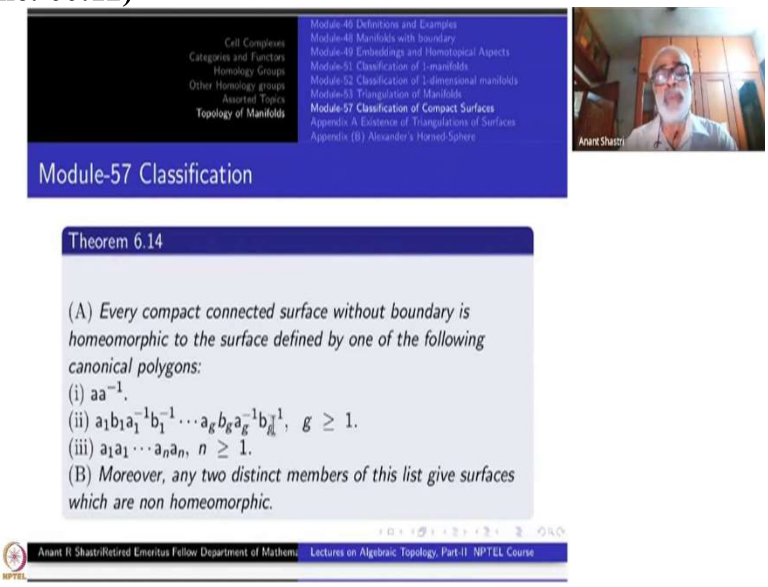
Introduction to Algebraic Topology (Part - II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 57
Classification of Compact Surface

(Refer Slide Time: 00:12)



So let us continue with the classification of surfaces. Here is the statement of the theorem. It has two parts Part A and Part B.

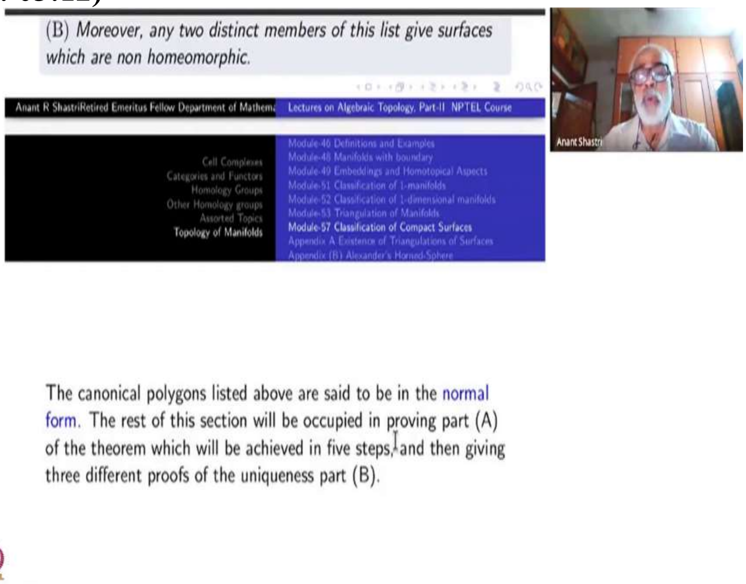
(A) Every compact connected 2-dimensional manifold without boundary (a surface) is homeomorphic to the surface defined by one of the following canonical polygons:

(i) aa^{-1};
(ii) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}$, where $g \geq 1$;
(iii) $a_1a_1a_2a_2a_3a_{3n}$, where $n \geq 1$. This is part (A) (To be precise, we must add the word `triangulated' also in the assumptions on the surface, since in the proof we are making this assumption. However, by Rado theorem which we have not proved of course, every surface is triangulated and hence in the statement, I have not put that hypothesis.)

(B) Second part says any two distinct members of the above list give you surfaces which are non homeomorphic.

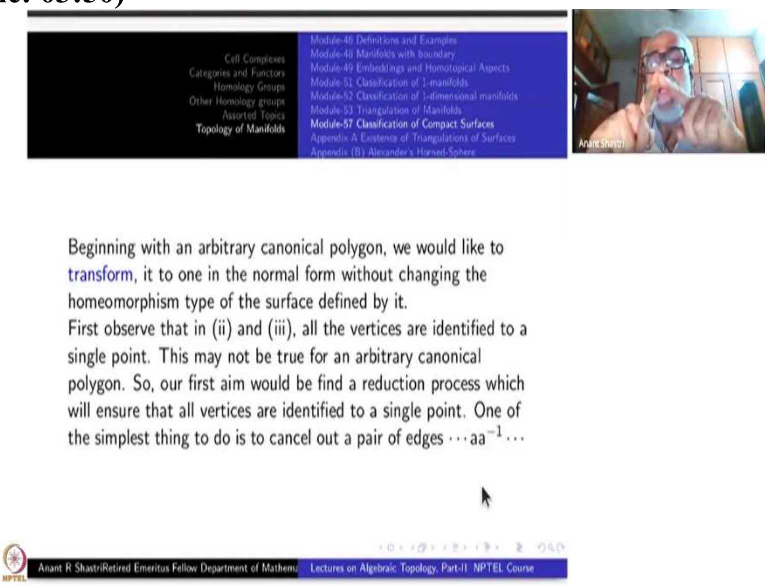
In (i) aa is a single distinct member. In (ii) there are infinitely many members okay? Indexed by $g \geq 1$. Similarly in (iii) there are infinitely many members okay? Each one of them is different from each one of the other from within the sub list as as as from members in the other two sub lists.

(Refer Slide Time: 03:12)



So this is the complete classification for surfaces without boundary. The canonical polygons listed above are said to be in the normal form, okay? When we normalize, uniqueness comes in. That is the whole idea. The rest of this section will be occupied in proving part (A) of the theorem which will be achieved in five steps. And then giving 3 different proofs of the uniqueness part viz., part (B).

(Refer Slide Time: 03:50)



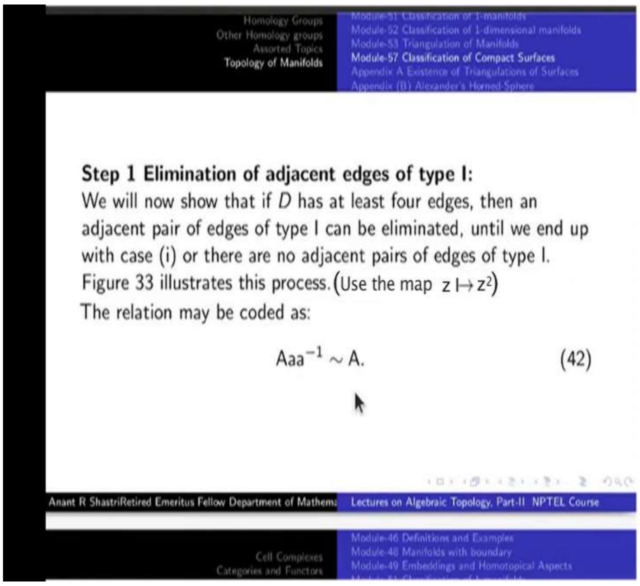
Beginning with an arbitrary canonical polygon, it may not be in the normal form, what we have to do is that we have to `reduce' it to a normal form and show that it is one of the members in the list. Clearly each member in the list is a canonical polygon. They are called canonical polygons in the normal form. So every canonical polygon is reduced or brought down to ... whatever we want to say ... we have to explain that. `Brought down' means what? Applying certain transformation process as we have illustrated in the discussion of an example last time by transforming $abab^{-1}$ in to $bbcc$. There may be some other transformations. They should not change the homeomorphism type of quotient space, we are allowed to do whatever we like, provided we remain in the same homeomorphism type of the quotient space. Okay? So that is the whole idea. Okay?

First observe that in (ii) and (iii) all the vertices are identified to a single point, you should know that. To see this it is enough to check that in the sequences $aba^{-1}b^{-1}$ (similarly in aa^{-1}), all the vertices get identified. However, in (i), we have aa^{-1} with two vertices which do not get identified.

(Refer Slide Time: 08:38)

In a given sequence, if both a letter and its inverse are present, we call such a pair of letters Type-I pair. Otherwise, i.e., a letter repeats itself with the inverse, then call it type-II okay? Also while representing a canonical polygon by a picture, instead writing the superscript over a letter, we shall use arrows which is easier to indicate the direction in which identifications are taking place. That is all. When you draw a picture it is just letters and arrows. Ok?

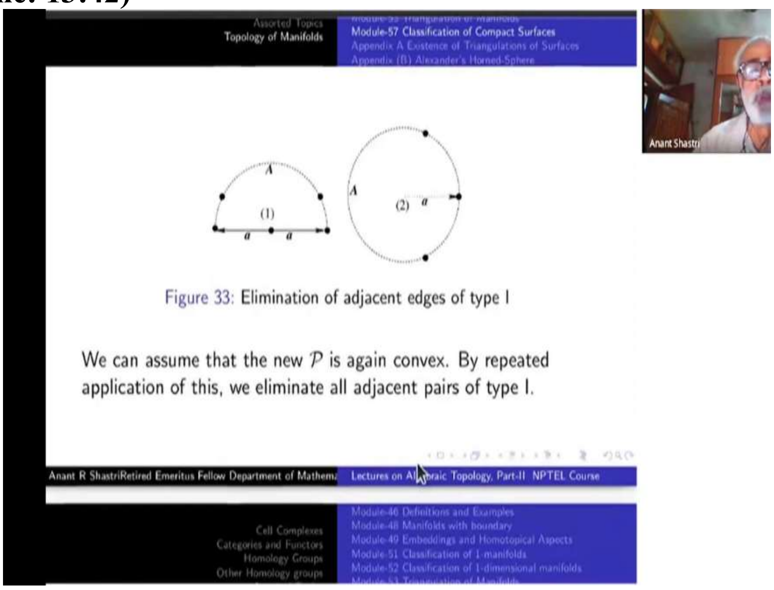
(Refer Slide Time: 09:35)



So this is the first thing we want to achieve our first aim would be to find a reduction process which will ensure that all vertices are identified to a single point, or we are in the list (i). Now we need a convenient terminology.

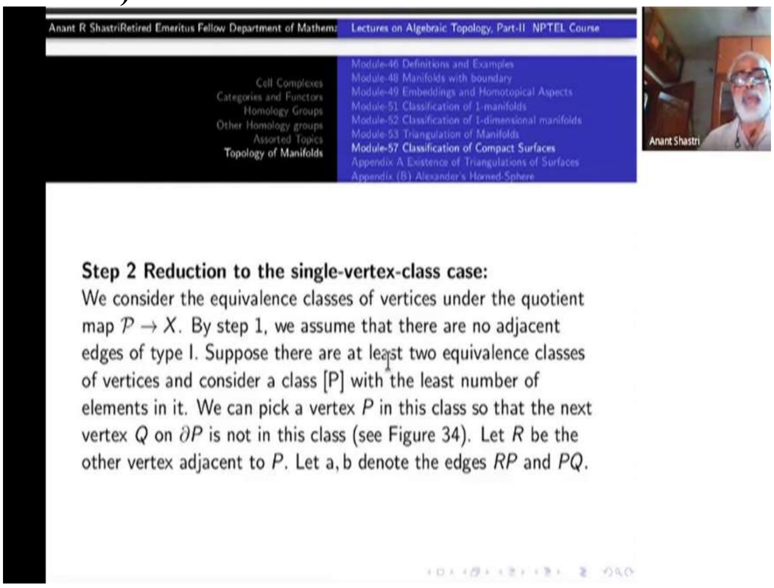
The simplest thing is to get rid of, to cancel out a pair of edges aa^{-1} occuring inside any sequence provided the sequence is of length at least 4, as follows: First write the sequence in the form $aa^{-1}A$ upto cyclic permutation, represent it on the boundary of a half disc as shown in the figure (36). Now use the doubling map or the folding map $re^{i\theta} \mapsto re^{2i\theta}$ onto the disc to see that the identification carried on the part aa^{-1} produces full disc with its boundary being marked by the sequence A. Therefore, the original surface is replaced by the sequence A itself. So, we code this step as follows $Aaa^{-1} \sim A$ (48).

(Refer Slide Time: 13:42)



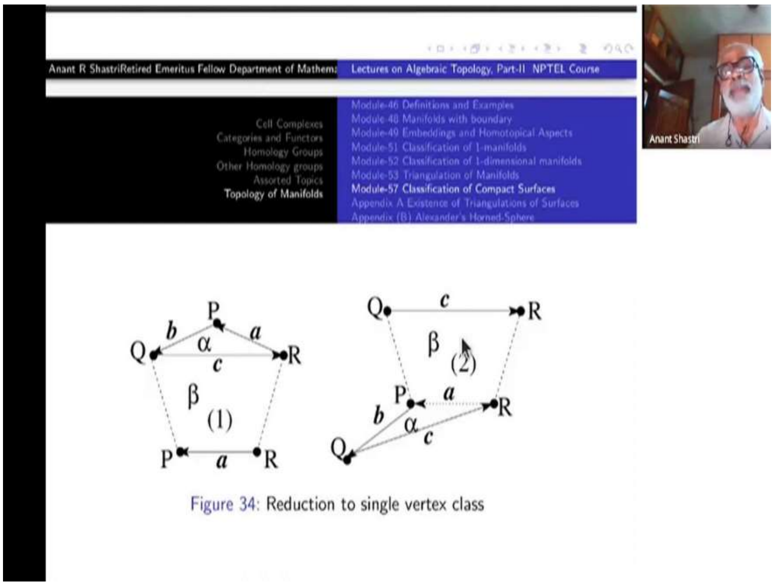
By repeated application of this step, we can assume that either the given sequence is as in (i) or there are no consecutive Type-I pairs in the sequence. (By the way, it is easy to see that this step can be computerized. Indeed, you may check that the entire reduction process that we are going to employ can be made algorithmic.) So let us assume that there are no adjacent Type-I pairs and the length of the sequence is at least 4.

(Refer Slide Time: 15:19)



We shall now introduce a process which will cut down the number of vertices in the quotient. i.e., the number of equivalence classes of vertices in $\mathcal{P}$. May be there are more than two classes. If there is only one class you are done. If there are more than one class consider a class $[P]$ which has smallest number of elements in it. Suppose one class has 6 elements and another class has 8 elements and so on. Look at the class that has the least number of elements. We pick up a vertex P in this class such that the next vertex Q in the sequence is not in this class. Since there are at least two classes of vertices this is possible. (After picking up a vertex in the class $[P]$, look at the next one (in the anticlockwise direction) if that is also in $[P]$, go further till you hit a vertex which is in a different class.)

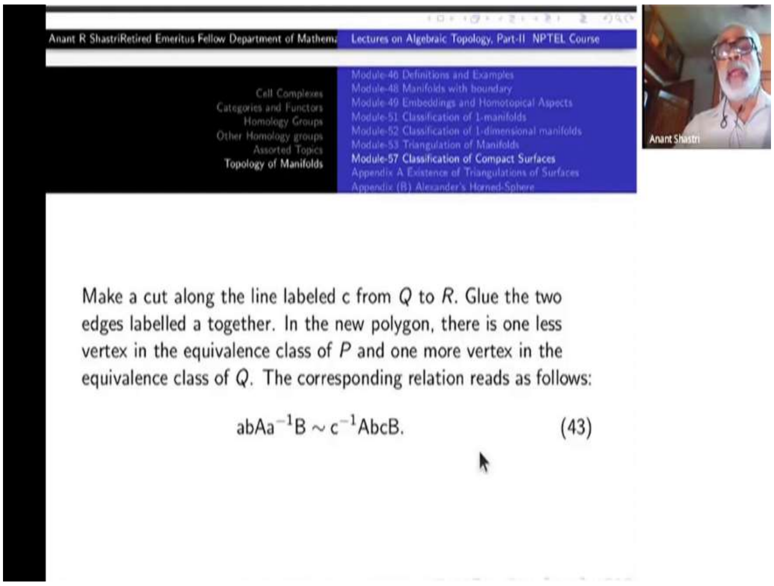
(Refer Slide Time: 17:28)



Let now R be the other vertex adjacent to P. P and Q are already chosen to be adjacent, so there is only one choice for R which will be behind P, okay? Now let a and b denote the edges RP and PQ respectively. See the diagram. We note that b cannot be equal to a because then P and Q would be in the same vertex class. Similarly, b cannot be equal to a^{-1} either, because we have assumed that there are no consecutive pairs of Type-I. Therefore our notation b for the edge PQ is justified.

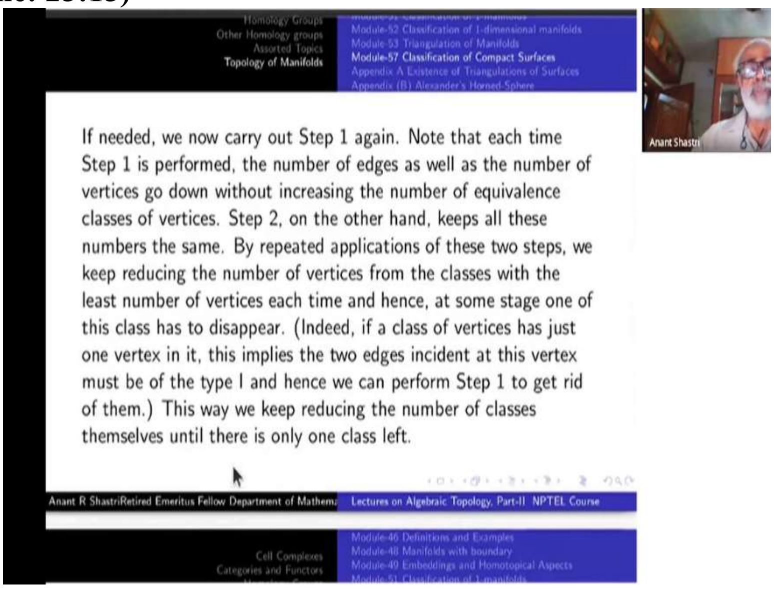
About the rest of the sequence all you know is that there must be an edge which is labeled a^{-1} or a. Accordingly, we may express the entire sequence in the form $abAa^{-1}B$ with A nonempty or $abAaB$, with B nonempty. (A may be also non empty but that is not necessary.) Make a cut R to Q. Cut out this triangle marked by alpha from the convex polygon. Denote the edge QR by c. You have have to put an arrow on this cut here whichever way you like. So I have put an arrow from Q to R okay? The arrow is here.

(Refer Slide Time: 24:28)



So when I cut out the triangle, the remaining convex polygon here is marked beta. I am bringing α down here and place it so that the two sides marked a on the two pieces are matching and then identify them any way. The resulting subspace of $\mathbb{R}^2$ need not be a convex set, but a polygon. You can deform it, straighten it out to become a convex polygon that part is no problem.

(Refer Slide Time: 25:15)

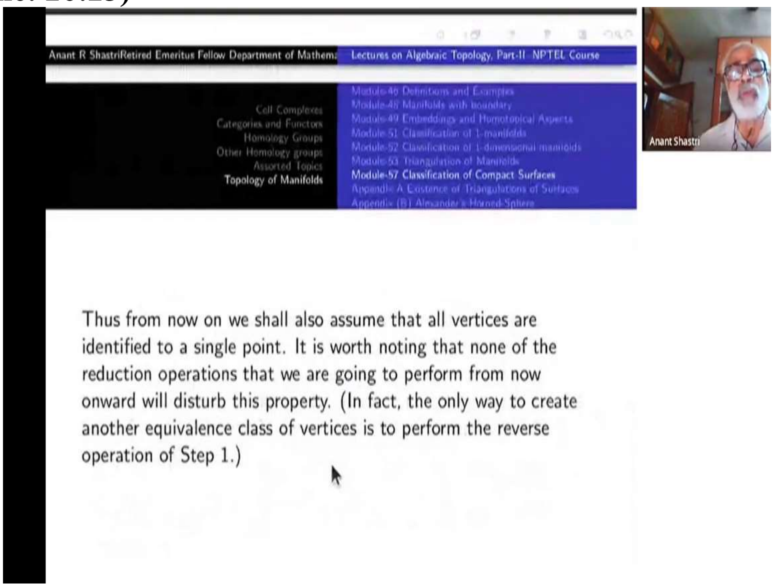


In the new sequence, the edge a disappears. What I want is that how many vertices are there in the class $[P]$ now? Two of the points marked by P have got identified. Therefore in the new class of $[P]$, there is one element less. Of course, you can also see that no new class of vertices has been introduced.

What may happen is now the new sequence may not satisfy the condition that there are no consecutive pairs of type-I. If so, perform the I-step as often as needed. This would further reduce the number of equivalence classes of vertices. Actually each time some class will disappear altogether.

Therefore, repeated application of these two steps will keep reducing the number of equivalence classes themselves till we hit upon the case (i) or there is only one equivalence class of vertices. So, this step can be coded as $abAa^{-1}B \sim c^{-1}AbcB$ and $abAaB \sim cAcb^{-1}B$. (49) (So you can computerize this operation also.)

(Refer Slide Time: 26:23)



From now onwards we will assume that our sequence has just one equivalence class of vertices. and operations that we consider will not disturb this property. Now there are some other kinds of operations we are bringing in okay? But they will never disturb this property. Means what? It will not introduce extra vertex classes. Alright, so, next time we shall do the final reduction and stop here today.