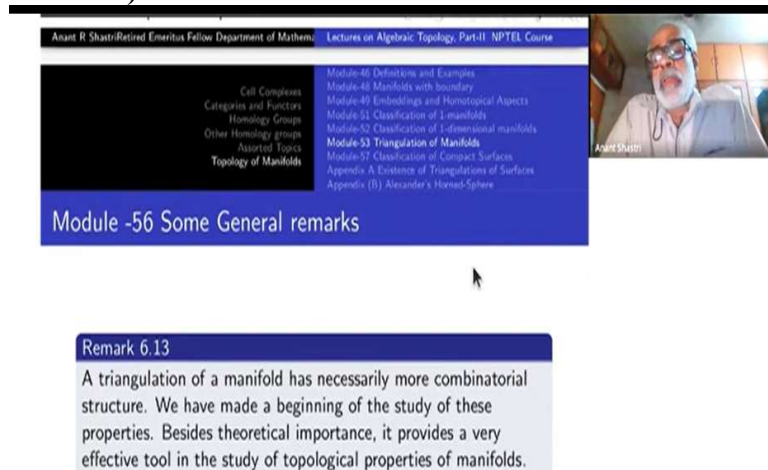


Introduction to Algebraic Topology (Part - II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 56
Some General Remarks

(Refer Slide Time: 00:12)




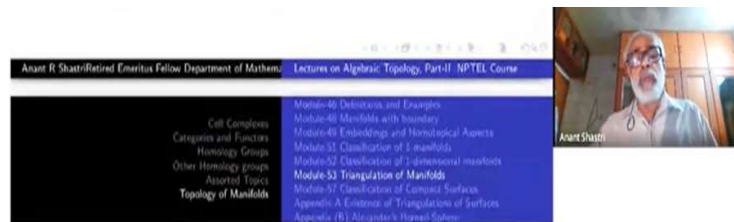
Cell Complexes	Module-46: Definitions and Examples
Categories and Functors	Module-48: Manifolds with boundary
Homology Groups	Module-49: Embeddings and Homotopical Aspects
Other Homology groups	Module-51: Classification of 1-manifolds
Assorted Topics	Module-52: Classification of 1-dimensional manifolds
Topology of Manifolds	Module-53: Triangulation of Manifolds
	Module-57: Classification of Compact Surfaces
	Appendix: A: Existence of Triangulations of Surfaces
	Appendix: (B) Alexander's Horned-Sphere

Module -56 Some General remarks

Remark 6.13
A triangulation of a manifold has necessarily more combinatorial structure. We have made a beginning of the study of these properties. Besides theoretical importance, it provides a very effective tool in the study of topological properties of manifolds.

Before taking up the last topic namely classification of triangulated compact surfaces, we make a few general remarks and on triangulation in particular. A triangulation of a manifold is necessarily more convenient. The combinatorial structure gives more combinatorial information on the manifolds. We have made a small beginning of the study of this property. Besides theoretical importance, it provides a very good effective tool in the study of topological properties of manifolds. For example, I have told you that one can actually computerise the study of topology through simplicial complexes.

(Refer Slide Time: 01:25)


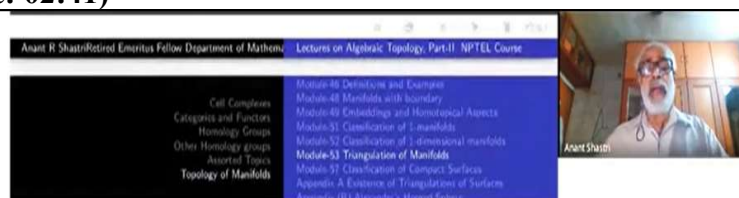
Remark 6.14

The fundamental classical questions here are:

- (A) Can every topological (smooth) manifold be triangulated?
- (B) Given two triangulations K_1, K_2 of a topological (smooth) manifold, is K_1 combinatorially equivalent to K_2 ?
- (C) Does every triangulated manifold carry a smooth structure?

The fundamental classical questions here are: Can every topological manifold be triangulated? If you cannot do that, you can ask whether all differentiable manifolds can be triangulated. Next question is that given two triangulations K_1 and K_2 of a topological manifold, is K_1 combinatorially equivalent to K_2 ? Recall that by combinatorial equivalence we mean that there are subdivisions K'_1 of K_1 and K'_2 of K_2 such that the two subdivisions are simplicially isomorphic. Third question is whether every triangulated manifold carries a smooth structure? So, these are the few standard questions.

(Refer Slide Time: 02:41)

Remark 6.15

From the classification of 1-dimensional manifold, it follows easily, that every 1-dimensional manifold is triangulable. It can also be proved without much difficulty that every 1-dimensional manifold has a unique smooth structure.

Let us see how much literature we know about these questions. From the classification of 1-dimensional manifolds, it easily follows that every 1-dimensional manifold is triangulable. Because the connected components are just open intervals, closed intervals or half closed intervals or a circles. Each of them you can triangulate, so 1-manifolds can be triangulated. It can also be proved that, each 1-dimensional topological manifold has a unique smooth

structure okay up to diffeomorphism. If you go through the proof of the classification of 1-manifolds, only at a few steps, you will have to improve homeomorphism to diffeomorphism which will require a little more work, that is all.

(Refer Slide Time: 03:45)

Remark 6.16

A classical result due to Rado (1924) says that all 2-dimensional manifolds are triangulable. Though this proof is within our limitations, due to lack of time, we shall skip it.



A classical result due to Rado, way back in 1924, almost a century back, says that all 2-dimensional manifolds are triangulable. Though this proof is within our limitations, due to lack of time, we shall skip it, okay?

(Refer Slide Time: 04:14)

Remark 6.17

The triangulability of all 3-dimensional manifolds is a deeper result due to Edwin Moise [Moise, 1977], who also proved that any two triangulations K_1, K_2 of the same 3-manifold are combinatorially equivalent, i.e., there are subdivisions K'_i of K_i such that K'_1 is isomorphic to K'_2 .



The triangulability of all 3-dimensional manifolds is a deeper result due to E. Moise who also proved that any two triangulations of the same 3-manifold are combinatorially equivalent. That means, as I have just told you, there are subdivisions K'_i of K_i such that K'_1 and K'_2 are isomorphic. By the way, this reference is for his book. His actual papers in which the original proofs are there, have appeared much before.

(Refer Slide Time: 04:58)

isomorphic to \mathbb{R}^2 .

Anant R Shastri/Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II: NPTEL Course

Cell Complexes	Module-46: Definitions and Examples
Categories and Functors	Module-48: Manifolds with boundary
Homology Groups	Module-49: Embeddings and Homotopical Aspects
Other Homology groups	Module-51: Classification of 1-manifolds
Assorted Topics	Module-52: Classification of 1-dimensional manifolds
Topology of Manifolds	Module-53: Triangulation of Manifolds
	Module-57: Classification of Compact Surfaces
	Appendix A: Existence of Triangulations of Surfaces
	Appendix (B): Alexander's Horned Sphere

Remark 6.18
A theorem of Cairns (1935) says that every smooth manifold is triangulable. (See [Whitehead, 1940] for a proof and more.)

A theorem due to Cairns in 1935 says that every smooth manifold is triangulable. There is an improved version of this result in Whitehead[1940] which gives a neater proof and a stronger result. So, every smooth manifold is triangulable.

(Refer Slide Time: 05:26)

Anant R Shastri/Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II: NPTEL Course

Cell Complexes	Module-46: Definitions and Examples
Categories and Functors	Module-48: Manifolds with boundary
Homology Groups	Module-49: Embeddings and Homotopical Aspects
Other Homology groups	Module-51: Classification of 1-manifolds
Assorted Topics	Module-52: Classification of 1-dimensional manifolds
Topology of Manifolds	Module-53: Triangulation of Manifolds
	Module-57: Classification of Compact Surfaces
	Appendix A: Existence of Triangulations of Surfaces
	Appendix (B): Alexander's Horned Sphere

Remark 6.19
In dimension ≥ 4 , there are triangulable manifolds which do not admit any smooth structure. The first example came in 1960 due to Kervaire in dimension 10 which was soon improved to dimension 8 by Eells and Kuiper. Siebenmann has constructed an example of a 5-manifold which cannot be triangulated. With these classical results, we may safely say that the above three questions have been answered satisfactorily. These results are all quite hard and beyond the scope of this elementary course.

In dimension greater than or equal 4, there are triangulable manifolds which do not admit any smooth structure. Every smooth manifold is triangulable. But in the other direction, there are triangulable manifolds, which do not admit any smooth structure. The first example came in 1960, due to Kervaire and the example was in dimension 10. This was soon improved to dimension 8, by Eells and Kuiper. Then Siebenmann constructed an example of a 5-dimensional manifold which cannot be triangulated. With this classical result, we may safely say that above three questions have been answered satisfactorily.

Why I am not talking about dimension 4? It is the craziest dimension among all of them okay? So, I will not speak about it here. These results are all quite hard and beyond the scope of this elementary course.

(Refer Slide Time: 06:58)

Anant R Shastri/Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II NPTEL Course

Cell Complexes	Module-46: Definitions and Examples
Categories and Functors	Module-48: Manifolds with boundary
Homology Groups	Module-49: Embeddings and Homotopical Aspects
Other Homology groups	Module-51: Classification of 1-manifolds
Assorted Topics	Module-52: Classification of 1-dimensional manifolds
Topology of Manifolds	Module-53: Triangulation of Manifolds
	Module-57: Classification of Compact Surfaces
	Appendix A: Existence of Triangulations of Surfaces
	Appendix (B) Alexander's Horned Sphere

Classification of Compact Surfaces

Definition 6.7

By a **surface** we shall mean a compact 2-dimensional connected topological manifold without boundary.

This terminology is somewhat temporary so that each time we do not have to mention all the additional conditions that we are going to assume on a 2-dimensional manifold.

So, let us make a small beginning today of the study of compact surfaces, we do not intend to finish it, not even the laying down of a plan of action for the classification. First of all, I will use the word 'surface' just to mean a compact 2-dimensional, connected, topological manifold without boundary. So, I will not keep on saying all these properties, just say a surface, okay? This is just temporary terminology just for another two-three lectures that is all. Okay?

(Refer Slide Time: 07:38)

Anant R Shastri/Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II NPTEL Course

Cell Complexes	Module-46: Definitions and Examples
Categories and Functors	Module-48: Manifolds with boundary
Homology Groups	Module-49: Embeddings and Homotopical Aspects
Other Homology groups	Module-51: Classification of 1-manifolds
Assorted Topics	Module-52: Classification of 1-dimensional manifolds
Topology of Manifolds	Module-53: Triangulation of Manifolds
	Module-57: Classification of Compact Surfaces
	Appendix A: Existence of Triangulations of Surfaces
	Appendix (B) Alexander's Horned Sphere

not have to mention all the additional conditions that we are going to assume on a 2-dimensional manifold.

We fix a triangulation on a connected compact surface X . We then appeal to Theorem 6.11 which provides us a representation of X as a quotient of a convex polygon P of $2n$ sides whose sides have to be identified pairwise by homeomorphisms which are linear on each side. This process can be completely described purely combinatorially as follows:

Next we fix a triangulation on a surface X . We then appeal to our theorem wherein we constructed a convex triangulated polygon P in \mathbb{R}^2 , and a quotient map from P onto X , right?

So, that is a representation of X as a quotient of a convex polygon P with $2n$ sides. What are the identifications? Identifications are coming only from the boundary sides. (This was not explicitly proved but the proof of this is simpler than the proof of Poincare's result which we proved elaborately and the arguments their in proving that the interior points of all edges are nice will ditto in proving that the vertices are nice in this case.)

The boundary of P , you know, is the union of edges. They are paired out and then inside each pair, the identification is taking place from one edge to to the other edge in the pair through a linear homeomorphism. Okay?

(Refer Slide Time: 09:13)



Let us agree once and for all, that we shall trace the boundary of any convex polygon in the anticlockwise direction. We are free to start from any vertex. We then label the edges by letters a, b, c, etc. As soon as we meet an edge which is being identified with an edge which has already been labeled, we shall use the same letter to label this new edge also.



So, this process can be completely described now, purely combinatorically as if we do not have any topology there at all. What do we do? We will describe a surface by describing this quotient map, okay? So, what is the domain? It is a triangulated convex polygon in \mathbb{R}^2 with even number of sides, which are paired off. You take a linear isomorphism from one edge to the other edge in the pair. There are precisely two such isomorphisms there, just like the case of linear isomorphisms from a closed interval to a closed interval, okay? There are only two possibilities depending upon the two bijections of the boundary sets.

So, let us agree once and for all that we shall trace the boundary of any convex polygon in \mathbb{R}^2 , in the anti-clockwise direction, okay? Like we trace a circle in 2-different ways anti-clockwise and clockwise, let us fix once for all the anti-clockwise direction. For actual tracing, we are free to start from any vertex that we do not fix that, okay? You can start from

any vertex keep going to the next one and so on, labeling the edges by letters such as $a, b, c \dots$ and so on. Okay?

As soon as we meet an edge which is being identified with an edge that we have already labelled, we shall not use a different letter to denote it. Instead we shall use the same letter. That means, edges belonging to the same pair are being labelled using the same letter. Suppose I have started with say, a, b and the next one is identified with a . Then I would not call that edge c , I will call it a . That takes care of the pairing data.

But now, I have one another stronger concern. In the orientation that we are taking namely following the anti-clockwise direction, we now look into the data whether these two edges are identified in the same orientation or not. Accordingly, instead of using the just the same letter to the second one of the pair, I will denote with the letter or its inverse, inverse being used if only if the isomorphism is orientation reversing.

(Refer Slide Time: 12:36)

Categories and Functors
Homology Groups
Other Homology groups
Assorted Topics
Topology of Manifolds

Isomorphisms & Homeomorphisms and Homeomorphism
Module-51: Classification of 1-manifolds
Module-52: Classification of 1-dimensional manifolds
Module-53: Triangulation of Manifolds
Module-57: Classification of Compact Surfaces
Appendix A: Existence of Triangulations of Surfaces
Appendix (B): Alexander's Horned Sphere

However, we have to take care of another aspect, viz., whether the identification is orientation preserving or orientation reversing. To indicate an edge with reversed orientation, we shall use labellings a^{-1}, b^{-1}, c^{-1} , etc. Of course, we stop as soon as we have arrived back where we have started. Since the starting point is arbitrary, what this process yields is that the surface X is completely determined by the cyclic sequence

$$a_1^{e_1} a_2^{e_2} \dots a_n^{e_n} \quad (41)$$

of the boundary edges. Note that in this sequence, each letter a_i occurs precisely twice. In particular, n is even.

Anant B Shrivastava Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II: NPTEL Course

Cell Complexes
Module-46: Definitions and Examples
Module-47: Manifolds with boundary
Module-48: Embeddings and Homeomorphisms of Surfaces

Also the orientations can be indicated by using arrow sign properly chosen on the edges while drawing a picture. The arrow will depend upon how we started with. For instance, suppose you start with an edge a and put the anticlockwise arrow on it, when come to another edge which is in the same pair with this edge a , you denote it with a or a^{-1} depending on whether the corresponding identification isomorphism is orientation preserving or reversing, and accordingly put the arrow also on this edge. Okay? Is that clear how we are going to label them? Alright as soon as we meet an edge, which is being identified with an earlier edge, which has already been labelled use the same letter or its inverse to label this new edge also.

Of course, we stop as soon as we have arrived back where we started. Thus we arrive at a finite sequence of even length, of letters, each letter occurring exactly twice and only one of them with a superscript -1 . These superscripts are indicated with ϵ_i which is either $+1$ or -1 . (We take the liberty not to write it at all if $\epsilon_i = +1$. Since the starting point is arbitrary, the sequence is well defined up to a cyclic permutation.)

So cyclic sequences look like $a_1^{\epsilon_1} a_2^{\epsilon_2} \dots a_n^{\epsilon_n}$, where n is an even integer greater than or equal to 4, $\epsilon_i = \pm 1$, but I will write a^{-1} simply as a .

(Refer Slide Time: 15:37)

occurs precisely twice. In particular, n is even.

For the simplicity of the notation, for an edge a^{+1} occurring with $+1$ sign, we drop this sign and simply write it as a . A sequence such as (41) is called a **canonical polygon**. We shall use bold face capitals A, B , etc., to denote any sequence such as (41).

A sequence such as (47) is called a canonical polygon. Just to contrast it with an arbitrary finite sequence. Okay? Indeed, as soon as such a sequence is given, we take the regular $2n$ -gon of side length one in \mathbb{R}^2 and label its vertices accordingly, which is well defined up to the choice of the starting point and hence well defined up to a rotation of the polygon. We may select any triangulation of the entire polygon without disturbing the edges on the boundary. Then the quotient space X obtained by edge identification as described by the sequence is determined up to a combinatorial equivalence and hence the underlying topological space X is completely determined by upto a homeomorphism.

We shall use boldface capitals A, B, C etc., to denote a part (a segment) of the sequence (47) whenever the part contains more than one term. For example, the sequence $aba^{-1}b^{-1}cd$ can be expressed as Acd where $A = aba^{-1}b^{-1}$. Upto cyclic permutation the same will be equal to cdA also.

(Refer Slide Time: 16:53)


Anant R Shastri/Retired Emeritus Fellow Department of Mathem.	Lectures on Algebraic Topology, Part-II, NPTEL Course
Cell Complexes	Module-46: Definitions and Examples
Categories and Functors	Module-48: Manifolds with boundary
Homology Groups	Module-49: Embeddings and Homotopical Aspects
Other Homology groups	Module-51: Classification of 1-manifolds
Assorted Topics	Module-52: Classification of 1-dimensional manifolds
Topology of Manifolds	Module-53: Triangulation of Manifolds
	Module-57: Classification of Compact Surfaces
	Appendix A: Existence of Triangulations of Surfaces
	Appendix (B): Alexander's Horned Sphere

For $n \geq 4$, we can represent a canonical polygon by a regular convex polygon \mathcal{P} in \mathbb{R}^2 with n sides with its sides appropriately labeled. Observe that we allow the exceptional case when $n = 2$ also. In this case, we do not get a convex polygon in \mathbb{R}^2 . However, in this case, we take \mathcal{P} to be the unit disc with its boundary being divided into two edges, the sequence itself being aa^{-1} or aa .



For completeness, we need to discuss sequences of length 0 and 2 as well. In these cases we do not get a convex polygon in \mathbb{R}^2 . So what do we do? For $n = 0$, we just take P to be the empty set. For $n = 2$, we will take the unit disc. Instead of a regular polygon, we just cut its boundary into two arcs, by taking the north pole and the south pole as vertices. There are only two cases for the canonical sequence for $n = 2$, viz., aa or aa^{-1} .

(Refer Slide Time: 18:44)

Cell Complexes	Module-46: Definitions and Examples	
Categories and Functors	Module-48: Manifolds with boundary	
Homology Groups	Module-49: Embeddings and Homotopical Aspects	
Other Homology groups	Module-51: Classification of 1-manifolds	
Assorted Topics	Module-52: Classification of 1-dimensional manifolds	
Topology of Manifolds	Module-53: Triangulation of Manifolds	
	Module-57: Classification of Compact Surfaces	
	Appendix A: Existence of Triangulations of Surfaces	
	Appendix (B): Alexander's Horned Sphere	

Consider the map


$$(x, 0, z) \mapsto \left(\sqrt{1-z^2} \cos\left(\frac{\pi x}{\sqrt{1-z^2}}\right), \sqrt{1-z^2} \sin\left(\frac{\pi x}{\sqrt{1-z^2}}\right), z \right)$$

from the unit disc in the xz -plane onto the unit sphere in \mathbb{R}^3 . This map proves that the surface represented by the sequence aa^{-1} is S^2 .

You consider the map from the unit disc in the xz -plane onto the unit sphere in \mathbb{R}^3 given by $(x, 0, z)$ mapsto as written in the slide, onto the unit sphere in \mathbb{R}^3 . This map proves that the surface represented by the sequence aa^{-1} is homeomorphic to the 2-sphere. Okay? Look at any of these coordinate line segments as you move parallel to the x -axis up and down Okay? Under this I am they go to circles on the sphere which are intersection of the sphere with the coordinate planes parallel to xy -plane.

(Refer Slide Time: 19:42)

Anant R Shastri/Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II NPTEL Course



Cell Complexes Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-46: Definitions and Examples Module-46: Manifolds with boundary Module-49: Embeddings and Homotopical Aspects Module-51: Classification of 1-manifolds Module-52: Classification of 1-dimensional manifolds Module-53: Triangulation of Manifolds Module-57: Classification of Compact Surfaces Appendix A: Existence of Triangulations of Surfaces Appendix (B): Alexander's Horned Sphere
--	---

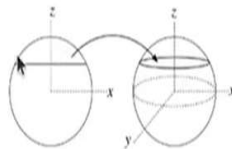


Figure 30: Sphere as a quotient of the disc



In the other case viz., aa , the identification is exactly same as the antipodal action $x \mapsto -x$ on the boundary of the 2-disc. We know that this quotient space is the projective space P^2 of dimension 2.

(Refer Slide Time: 21:38)

Anant R Shastri/Retired Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II NPTEL Course



Cell Complexes Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-46: Definitions and Examples Module-46: Manifolds with boundary Module-49: Embeddings and Homotopical Aspects Module-51: Classification of 1-manifolds Module-52: Classification of 1-dimensional manifolds Module-53: Triangulation of Manifolds Module-57: Classification of Compact Surfaces Appendix A: Existence of Triangulations of Surfaces Appendix (B): Alexander's Horned Sphere
--	---

Recall that the projective space \mathbb{P}^2 is the quotient space of S^2 by the antipodal action. The quotient map $q : S^2 \rightarrow \mathbb{P}^2$ is a closed mapping and hence the restriction of q to the upper hemisphere is also a quotient map. But this can be recognized to be the same as the quotient of the 2-disc by the sequence aa . Therefore, the two exceptional cases have been taken care, satisfactorily. Therefore, in what follows we can concentrate only on the case $n \geq 4$.



(Recall that P^2 is, by definition, the quotient of S^2 by the antipodal action. But you do not need the whole of S^2 , only take the upper hemi-sphere and perform the identification only on the equator.) So, that can be identified precisely by using the flat disk in \mathbb{R}^2 itself. And the sequence will be now aa . So, aa represents the projective space, aa^{-1} the sphere, okay? Thus we have started the classification and in the simplest cases viz, $n = 0$ and 2 we have completed it. Now let us concentrate on cases with n greater than or $= 4$ only, okay?

(Refer Slide Time: 22:31)

Anant R. Shastri Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part-II: NPTEL Course

Cell Complexes	Manifolds: Definitions and Examples
Categories and Functors	Manifolds with boundary
Homology Groups	Embeddings and Homotopy Aspects
Other Homology groups	Classification of 1-manifolds
Algebraic Topology	Classification of 2-manifolds
Topology of Manifolds	Triangulation of Manifolds
	Classification of Compact Surfaces
	Existence of Triangulations of Surfaces
	Alexander's Horned Sphere

We can 'add' two canonical polygons A, B by merely concatenating the corresponding two sequences AB . Of course, this operation is commutative, in a restricted sense, because, a sequence is taken only upto cyclic order. This is illustrated in the following figure. The corresponding operation on the surfaces is called taking **connected sum**.

NPTEL

Anant Shastri

So, now I want to introduce a binary operation on the set of canonical polygons: Given two of them A, B , remember they are sequences of even length, $2n, 2m$ say, merely concatenate them to get AB of length $2n + 2m$. Okay? This operation is clearly associative and is commutative in a restricted sense, upto cyclic permutation:


The problem is that it is not clear whether this operation is well defined on the class of canonical polygons up to cyclic permutation, viz., if you take a sequence $A_1 A_2$, which is the same as $A_2 A_1$ upto a cyclic permutation, then for any canonical polygon B , do we get $A_1 A_2 B$ equal to $A_2 A_1 B$? Upto cyclic permutation? In general, the answer is No. However, there could be certain situations in which this holds. In order to understand this, we must appeal to the geometric operation of 'connected sum' which actually has motivated this purely combinatorial binary operation.

In simple geometric terms it means that you make a hole by removing a small disc in the interior of each of the two convex polygons and identify the two resulting circles by a homeomorphism, to get a new surface, which is called the connected sum of A and B . However, at this stage, we do not need to go deeper into this geometric aspect and just take the combinatorial definition as a definition of connected sum of two canonical polygons.

(Refer Slide Time: 25:20)

Anant R Shastri-Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part-II NPTEL Course

Cell Complexes Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-46: Definitions and Examples Module-48: Manifolds with boundary Module-49: Embeddings and Homotopical Aspects Module-51: Classification of 1-manifolds Module-52: Classification of 2-dimensional manifolds Module-53: Triangulation of Manifolds Module-55: Classification of Compact Surfaces Appendix A: Existence of Triangulations of Surfaces Appendix (B) Alexander's Horned Sphere
--	--



Anant Shastri

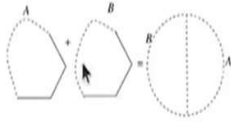



Figure 31: $(2n\text{-gon}) \text{ plus } (2m\text{-gon}) \text{ equals } 2(m+n)\text{-gon}$

So I will give you an example, okay. So, here is my A , I do not know how many edges are there in it. I have drawn 3 edges here and three edges their which correspond the whole that you have made in the convex polygons corresponding to A , and B . The dotted parts represent the sequence A and B respectively. When you identify the three edges of the first one to those of the second, sequentially, you will get a larger polygon of size $2n + 2m$, which is the connected sum. okay?

(Refer Slide Time: 25:48)

Anant R Shastri-Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part-II NPTEL Course

Cell Complexes Categories and Functors Homology Groups Other Homology groups Assorted Topics Topology of Manifolds	Module-46: Definitions and Examples Module-48: Manifolds with boundary Module-49: Embeddings and Homotopical Aspects Module-51: Classification of 1-manifolds Module-52: Classification of 2-dimensional manifolds Module-53: Triangulation of Manifolds Module-55: Classification of Compact Surfaces Appendix A: Existence of Triangulations of Surfaces Appendix (B) Alexander's Horned Sphere
--	--



Anant Shastri

Several canonical polygons may define the same surface up to homeomorphism. Our next step is to make a list which should include all possible topological types as well as have no redundancy. Next, we must identify some simple transformations on the canonical polygons which do not change the topological type and then keep applying these transformations so as to bring any given canonical polygon to one in the list. Let us illustrate this with an example.

The crucial point now is that several canonical polygons may define the same surface up to homeomorphism. What we have so far is that every surface arises out of a canonical polygon. So there is a set theoretic surjective function from the set of all canonical polygons to the set of all homeomorphism classes of surfaces.

Our next step is to make a short list of canonical polygons, such that the above function is both surjective and injective, i.e., the list should include all possible topological types of surfaces and yet have no redundancy. Our list should not have A and A' which represent the same surface up to homeomorphism. Each member of the list should represent a distinct topological type. Once you have got such a list, you have achieved the classification of surfaces, okay? So, first we propose a list and then we go on to prove that it has the required property, Okay?

Let us now illustrate the point how two different canonical polygons may give the same surface. One easy way is that one polygon is obtained from the other by a cyclic permutation. For example, $abcdabcd$ is the same as $bcdabceda$. That is easy. The point is even cyclically different polygons may give the same surface.

(Refer Slide Time: 28:21)

The screenshot shows a video lecture interface. At the top, it says 'Anant R. Shastri, Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course'. Below this is a table of contents with the following items: Cell Complexes, Categories and Functors, Homology Groups, Other Homology Groups, Assorted Topics, Topology of Manifolds, Module 46: Definitions and Examples, Module 47: Manifolds with boundary, Module 48: Embeddings and Homomorphisms, Module 51: Classification of 1-manifolds, Module 52: Classification of 1-dimensional manifolds, Module 53: Triangulation of Manifolds, Module 57: Classification of Compact Surfaces, Appendix A: Existence of Triangulations of Surfaces, and Appendix (B): Alexander's Horned Sphere. The main content area is titled 'Example 6.9' and contains the following text: 'Consider the surface given by the sequence $abab^{-1}$. We have seen that this represents the Klein bottle. Mark the diagonal of the square piece of the paper with a (thick) arrow and letter c and cut the square along this arrow as shown in Figure 32. Bring the upper triangular piece down, flip it and identify the two triangles along the edge a , taking care to preserve the orientation of the edge while identifying. The new figure is not a rectangle but we treat it as a rectangle with the sides marked $bbcc$ which is a simpler canonical polygon which represents connected sum of two copies of \mathbb{P}^2 . Thus, we have proved that the Klein bottle is homeomorphic to the connected sum of two copies of \mathbb{P}^2 .' A small NPTEL logo is visible in the bottom left corner of the slide.

So, consider surface given by the sequence $abab^{-1}$. Just recall this represents the familiar surface called Klein bottle. On a rectangle, oriented in the anticlockwise sense, you mark the four sides by letters. Here the horizontal sides are identified by taking both of them in the anticlockwise direction whereas the two vertical one are identifies with opposite direction. So, the sequence is $abab^{-1}$. That gives the Klein bottle okay.

Mark the diagonal with the letter c , and a thick arrow, (the direction does not matter), as shown in figure (35) here. Here I have chosen the diagonal from the starting point of a to the starting point of the next a , okay? (Or you could have taken the other diagonal, that will be

from the end point of a to the end point of other a . That is what you have to remember in a more general situation.)

(Refer Slide Time: 29:56)

to the connected sum of two copies of \mathbb{P}^2 .

Anant R Shastri, Emeritus Fellow Department of Mathem. Lectures on Algebraic Topology, Part-II, NPTEL Course

Cell Complexes
Categories and Functors
Homology Groups
Other Homology groups
Assorted Topics
Topology of Manifolds

Module-46: Definitions and Examples
Module-46: Manifolds with boundary
Module-48: Embeddings and Homotopy Aspects
Module-51: Classification of 2-manifolds
Module-52: Classification of 3-dimensional manifolds
Module-53: Triangulation of Manifolds
Module-57: Classification of Compact Surfaces
Appendix A: Existence of Triangulations of Surfaces
Appendix 10: Alexander's Horned Sphere

Anant Shastri

Figure 32: Transformations of canonical polygons

Now cut the rectangle along c , to get two triangle. The upper triangle you bring it down and flip it so that this edge marked with a becomes parallel with the corresponding edge in the lower triangle. So, now the bottom edge of one triangle and the top edge of the other are aligned in the correct direction. Identify the two triangles along these two edges.

The resulting figure does not look like a rectangle, is actually a triangle. But as a combinatorial object, it has 4 sides, and you can easily deform it inside \mathbb{R}^2 into a rectangle or a square. So, now the sequence is $bbcc$, which represents the same surface, a Klein bottle.

If I just start from here and cut this new square along the diagonal, running from the initial point of b to the initial point of c , what I get is two sequences bb and cc . Both sequences represent the projective space P^2 . Therefore, the Klein bottle can be thought of as the connected sum of two copies of P^2 .

So, this example is going to be used at the end of the proof of classification that is to come. The process itself becomes a technique called the cut-and-paste technique and is heavily used. Let us stop here today. So, tomorrow we will actually start the classification problem. Thank you.