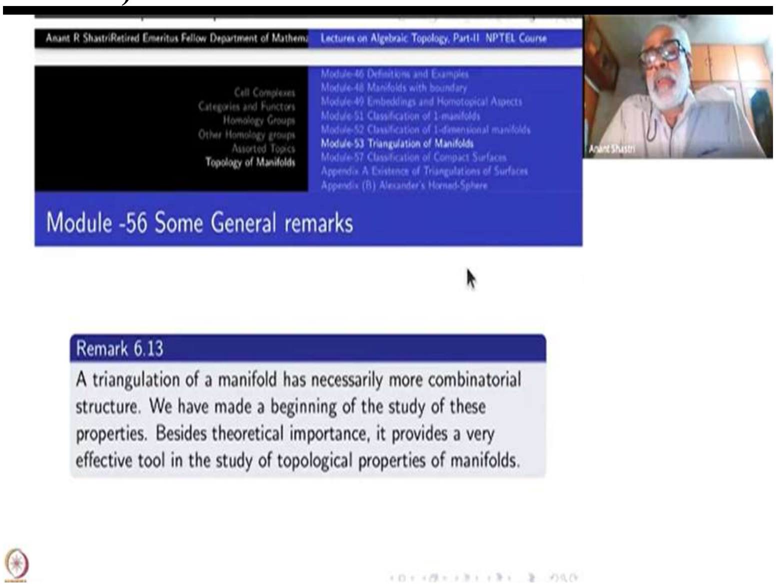
Introduction to Algebraic Topology (Part - II)

Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

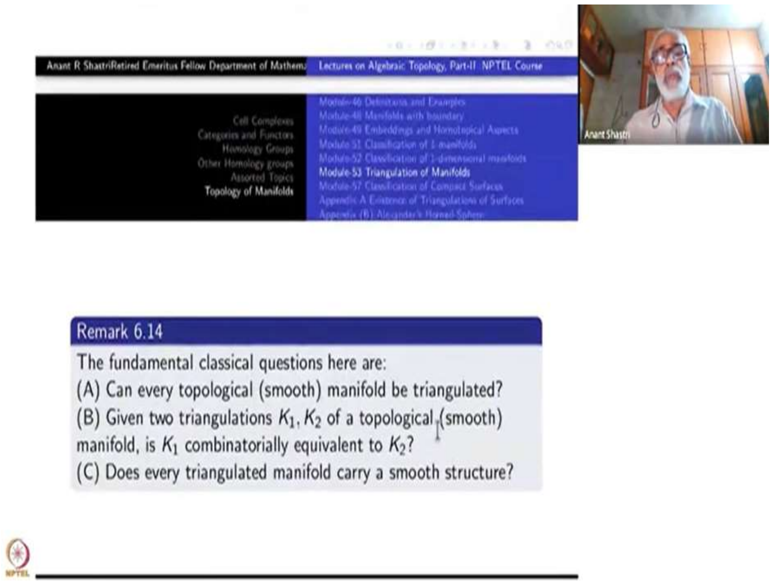
Lecture - 56
Some General Remarks

(Refer Slide Time: 00:12)



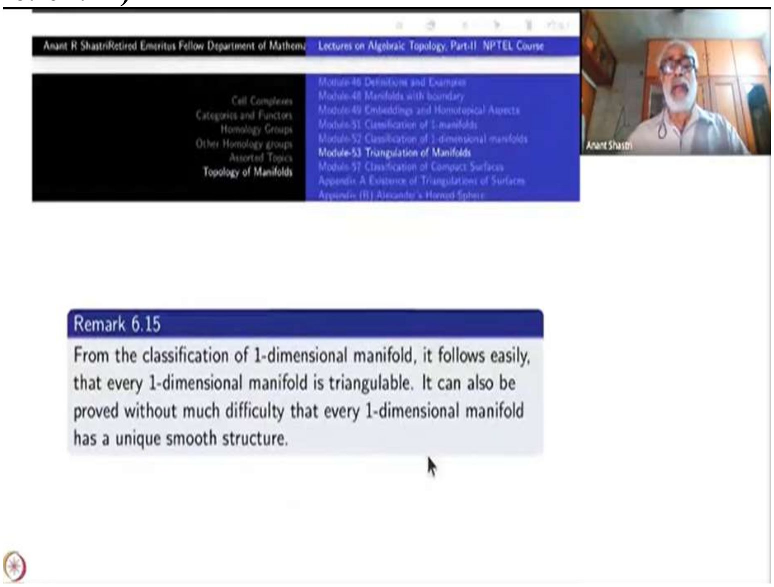
Before taking up the last topic namely classification of triangulated compact surfaces, we make a few general remarks and on triangulation in particular. A triangulation of a manifold is necessarily more convenient. The combinatorial structure gives more combinatorial information on the manifolds. We have made a small beginning of the study of this property. Besides theoretical importance, it provides a very good effective tool in the study of topological properties of manifolds. For example, I have told you that one can actually computerise the study of topology through simplicial complexes.

(Refer Slide Time: 01:25)



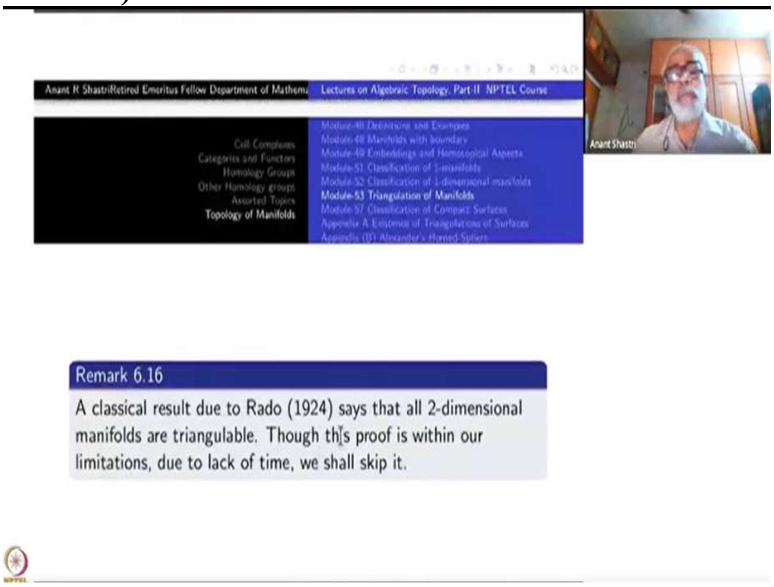
The fundamental classical questions here are: Can every topological manifold be triangulated? If you cannot do that, you can ask whether all differentiable manifolds can be triangulated. Next question is that given two triangulations K_1 and K_2 of a topological manifold, is K_1 combinatorially equivalent to K_2? Recall that by combinatorial equvivalence we mean that there are subdivisions K_1' of K_1 and K_2' of K_2 such that the two subdivisions are simplicially isomorphic. Third question is whether every triangulated manifold carries a smooth structure? So, these are the few standard questions.

(Refer Slide Time: 02:41)



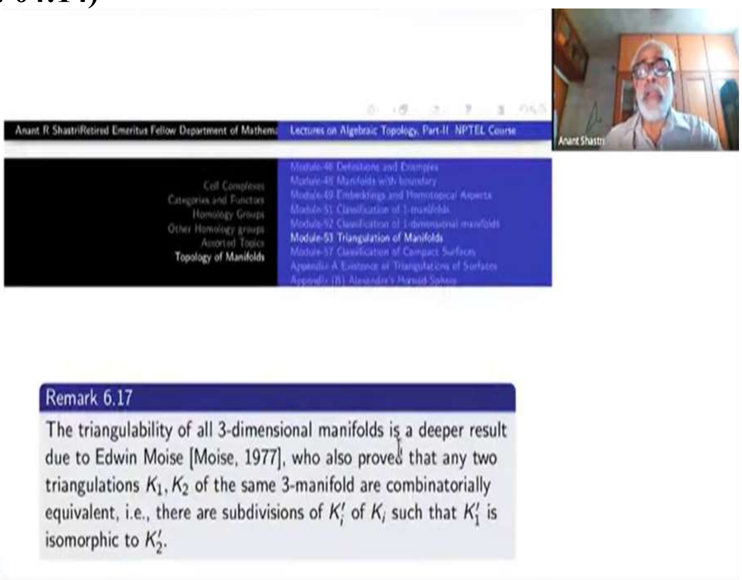
Let us see how much literature we know about these questions. From the classification of 1-dimensional manifolds, it easily follows that every 1-dimensional manifold is triangulable. Because the connected components are just open intervals, closed intervals or half closed intervals or a circles. Each of them you can triangulate, so 1-manifolds can be triangulated. It can also be proved that, each 1-dimensional topological manifold has a unique smooth structure okay up to diffeomorphism. If you go through the proof of the classification of 1-manifolds, only at a few steps, you will have to improve homeomorphism to diffeomorphism which will require a little more work, that is all.

(Refer Slide Time: 03:45)



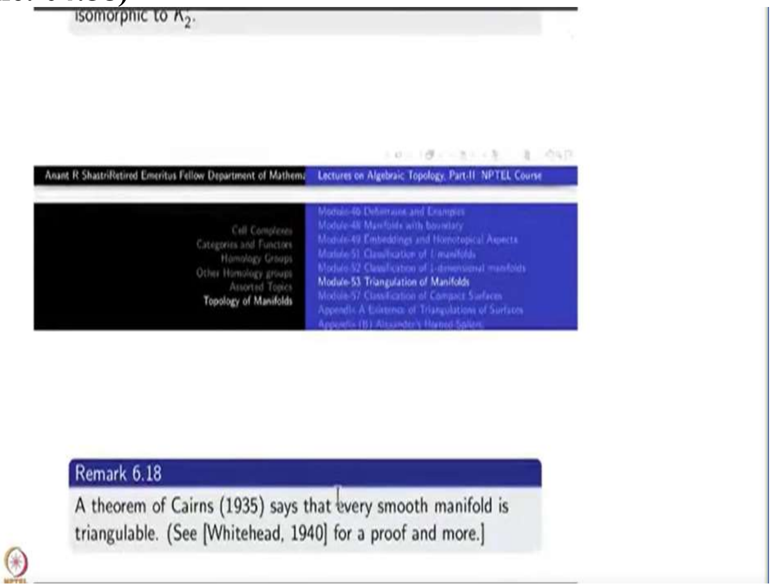
A classical result due to Rado, way back in 1924, almost a century back, says that all 2-dimensional manifolds are triangulable. Though this proof is within our limitations, due to lack of time, we shall skip it, okay?

(Refer Slide Time: 04:14)



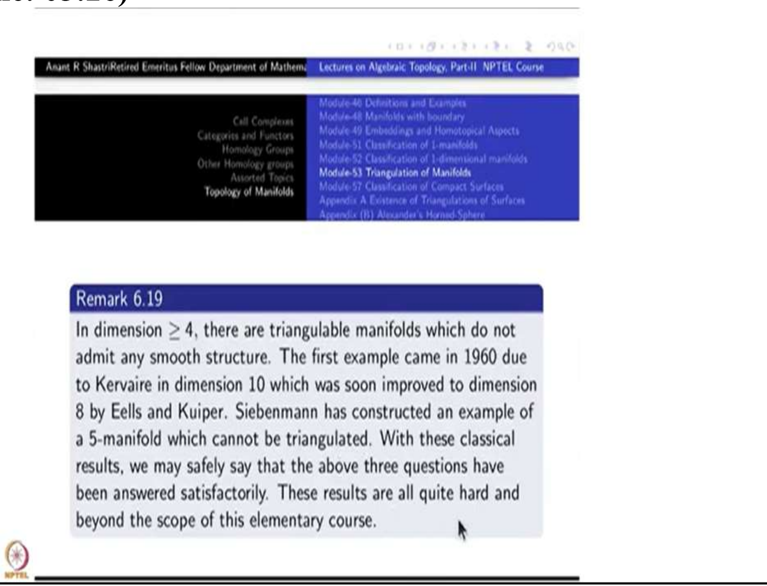
The triangulability of all 3-dimensional manifolds is a deeper result due to E. Moise who also proved that any two triangulations of the same 3-manifold are combinatorially equivalent. That means, as I have just told you, there are subdivisions K_i' of K_i such that K_1' and K_2' are isomorphic. By the way, this reference is for his book. His actual papers in which the original proofs are there, have appeared much before.

(Refer Slide Time: 04:58)



A theorem due to Cairns in 1935 says that every smooth manifold is triangulable. There is an improved version of this result in Whitehead[1940] which gives a neater proof and a stronger result. So, every smooth manifold is triangulable.

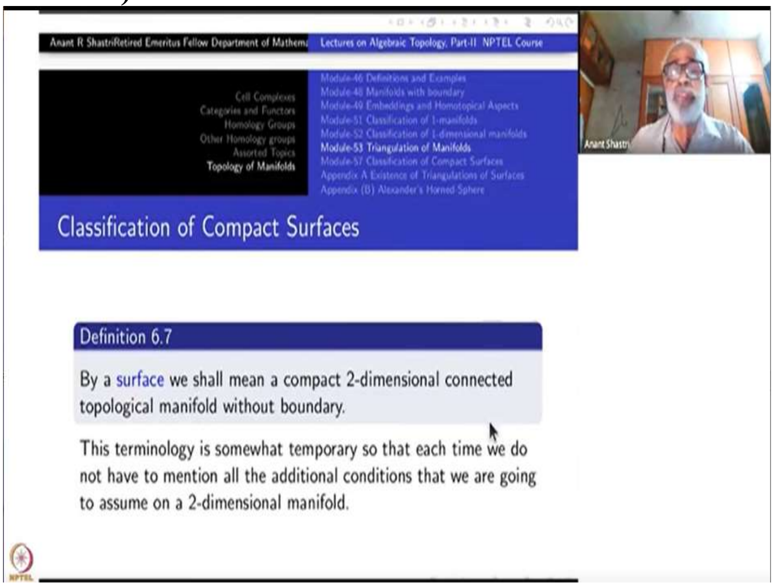
(Refer Slide Time: 05:26)



In dimension greater than or equal 4, there are triangulable manifolds which do not admit any smooth structure. Every smooth manifold is triangulable. But in the other direction, there are triangulable manifolds, which do not admit any smooth structure. The first example came in 1960, due to Kervaire and the example was in dimension 10. This was soon improved to dimension 8, by Eells and Kuiper. Then Siebenmann constructed an example of a 5-dimensional manifold which cannot be triangulated. With this classical result, we may safely say that above three questions have been answered satisfactorily.

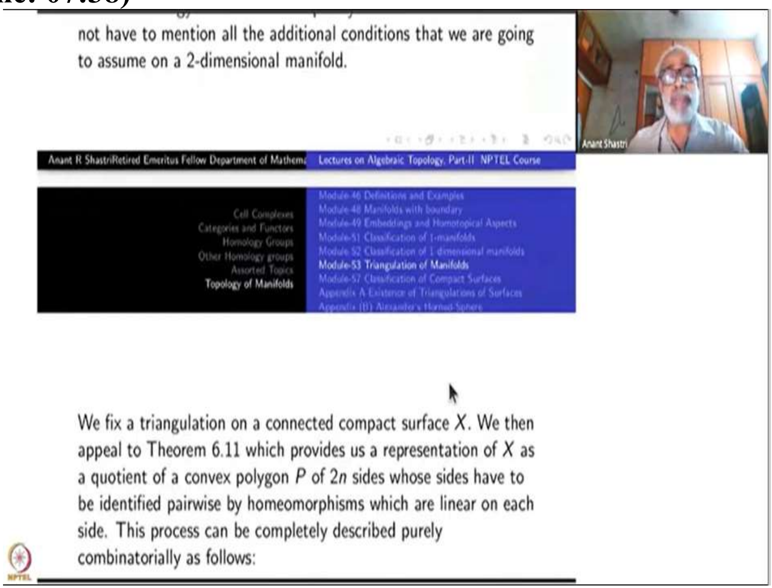
Why I am not talking about dimension 4? It is the craziest dimension among all of them okay? So, I will not speak about it here. These results are all quite hard and beyond the scope of this elementary course.

(Refer Slide Time: 06:58)



So, let us make a small beginning today of the study of compact surfaces, we do not intend to finish it, not even the laying down of a plan of action for the classification. First of all, I will use the word `surface' just to mean a compact 2-dimensional, connected, topological manifold without boundary. So, I will not keep on saying all these properties, just say a surface, okay? This is just temporary terminology just for another two-three lectures that is all. Okay?

(Refer Slide Time: 07:38)

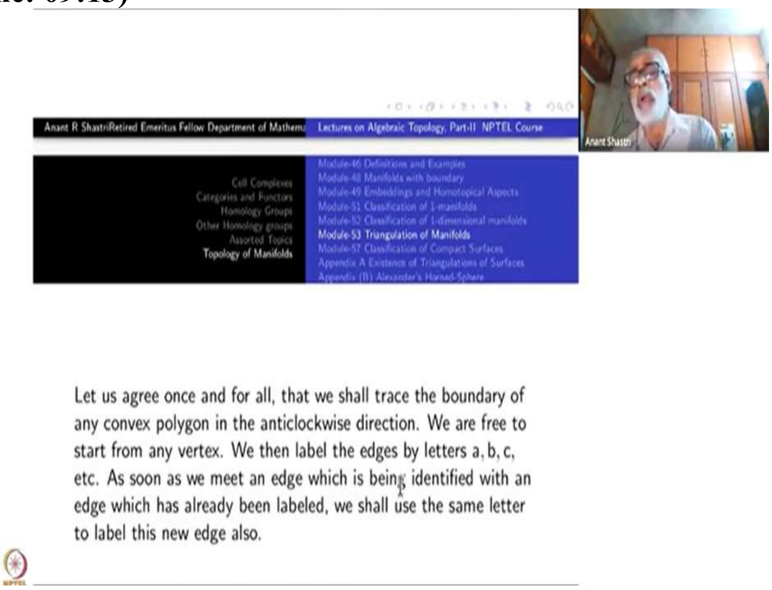


Next we fix a triangulation on a surface X. We then appeal to our theorem wherein we constructed a convex triangulated polygon P in $\mathbb{R}^2$, and a quotient map from P onto X, right?

So, that is a representation of X as a quotient of a convex polygon P with $2n$ sides. What are the identifications? Identifications are coming only from the boundary sides. (This was not explicitly proved but the proof of this is simpler than the proof of Poincare's result which we proved elaborately and the arguments their in proving that the interior points of all edges are nice will ditto in proving that the vertices are nice in this case.)

The boundary of P, you know, is the union of edges. They are paired out and then inside each pair, the identification is taking place from one edge to to the other edge in the pair through a linear homeomorphism. Okay?

(Refer Slide Time: 09:13)



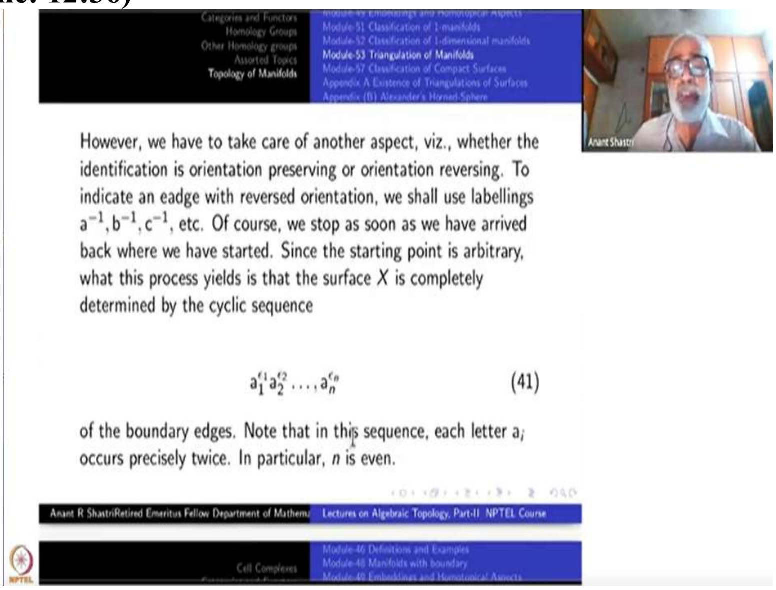
So, this process can be completely described now, purely combinatorically as if we do not have any topology there at all. What do we do? We will describe a surface by describing this quotient map, okay? So, what is the domain? It is a triangulated convex polygon in $\mathbb{R}^2$ with even number of sides, which are paired off. You take a linear isomorphism from one edge to the other edge in the pair. There are precisely two such isomorphisms there, just like the case of linear isomorphisms from a closed interval to a closed interval, okay? There are only two possibilities depending upon the two bijections of the boundary sets.

So, let us agree once and for all that we shall trace the boundary of any convex polygon in $\mathbb{R}^2$, in the anti-clockwise direction, okay? Like we trace a circle in 2-different ways anti-clockwise and clockwise, let us fix once for all the anti-clockwise direction. For actual tracing, we are free to start from any vertex that we do not fix that, okay? You can start from any vertex keep going to the next one and so on, labeling the edges by letters such as $a, b, c \ldots$ and so on. Okay?

As soon as we meet an edge which is being identified with an edge that we have already labelled, we shall not use a different letter to denote it. Instead we shall use the same letter. That means, edges belonging to the same pair are being labelled using the same letter. Suppose I have started with say, a, b and the next one is identified with a. Then I would not call that edge c, I will call it a. That takes case of the pairing data.

But now, I have one another stronger concern. In the orientation that we are taking namely following the anti-clockwise direction, we now look into the data whether these two edges are identified in the same orientation or not. Accordingly, instead of using the just the same letter to the second one of the pair, I will denote with the letter or its inverse, inverse being used if only if the ismomorphism is orientation reversing.

(Refer Slide Time: 12:36)

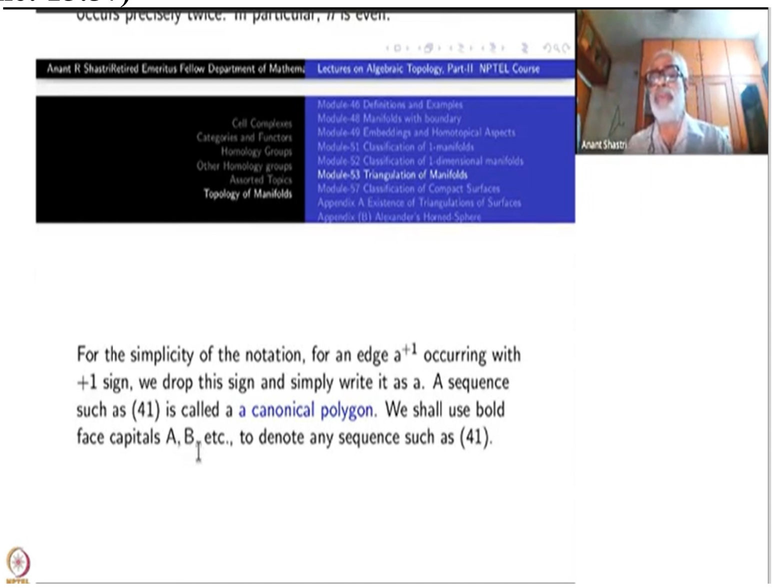


Also the orientations can be indicated by using arrow sign properly chosen on the edges while drawing a picture. The arrow will depend upon how we started with. For instance, suppose you start with an edge a and put the anticlockwise arrow on it, when come to another edge which is in the same pair with this edge a, you denote it with a of a^{-1} depending on whether the corresponding identification isomorphism is orientation preserving or reversing, and accordingly put the arrow also on this edge. Okay? Is that clear how we are going to label them? Alright as soon as we meet an edge, which is being identified with an earlier edge, which has already been labelled use the same letter or its inverse to label this new edge also.

Of course, we stop as soon as we have arrived back where we started. Thus we arrive at a finite sequence of even length, of letters, each letter occurring exactly twice and only one of them with a superscript -1. These superscripts are indicated with ϵ_i which is either $+1$ or -1. (We take the liberty not to write it at all if $\epsilon_i = +1$. Since the starting point is arbitrary, the sequence is well defined up to a cyclic permutation.)

So cyclic sequences look like $a_1^{\epsilon_1} a_2^{\epsilon_2} {}_n^{\epsilon_n}$, where n is an even integer greater than or equal to 4, $\epsilon_i = \pm 1$, but I will write a^{-1} simply as a.

(Refer Slide Time: 15:37)



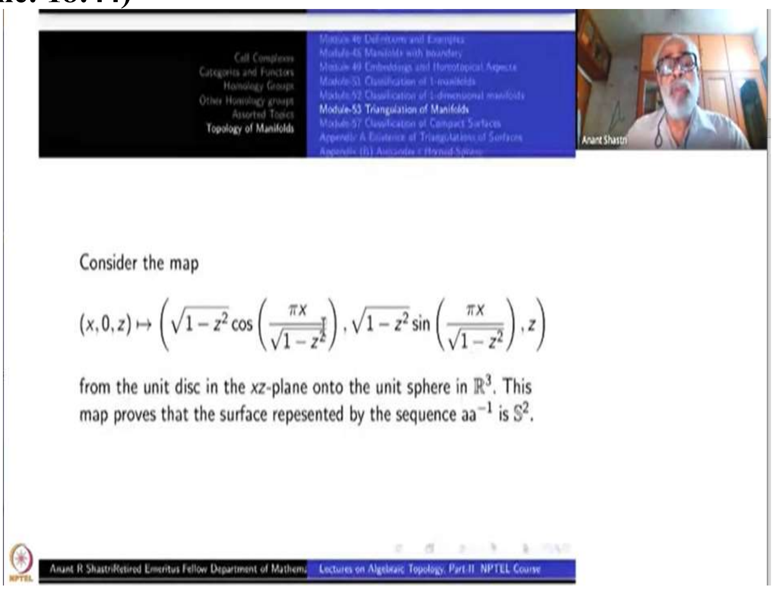
A sequence such as (47) is called a canonical polygon. Just to contrast it with an arbitrary finite sequence. Okay? Indeed, as soon as such a sequence is given, we take the regular $2n$-gon of side length one in $\mathbb{R}^2$ and label its vertices accordingly, which is well defined up to the choice of the starting point and hence well defined up to a rotation of the polygon. We may select any triangulation of the entire polygon without disturbing the edges on the boundary. Then the quotient space X obtained by edge identification as described by the sequence is determined up to a combinatorial equivalence and hence the underlying topological space X is completely determined by upto a homeomorphism.

We shall use boldface capitals A, B, C etc., to denote a part (a segment) of the sequence (47) whenever the part contains more than one term. For example, the sequence $aba^{-1}b^{-1}cd$ can be expressed as Acd where $A = aba^{-1}b^{-1}$. Upto cyclic permutation the same will be equal to cdA also.

(Refer Slide Time: 16:53)

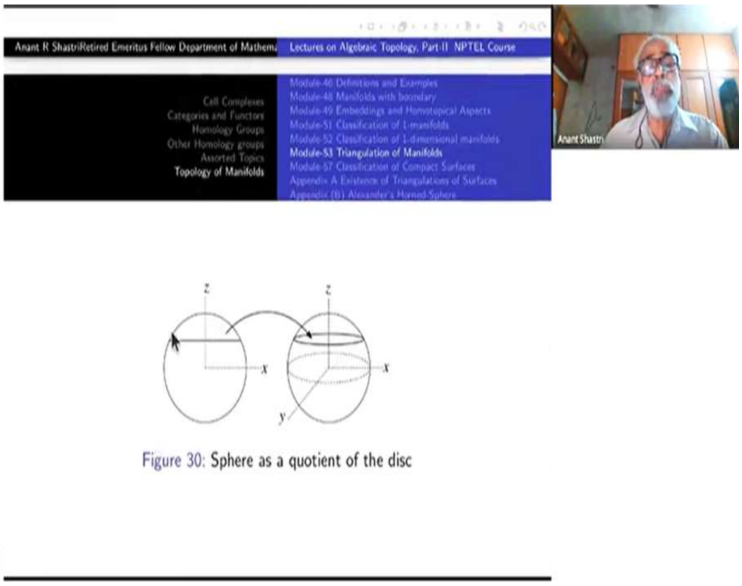
For completeness, we need to discuss sequences of length 0 and 2 as well. In these cases we do not get a convex polygon in $\mathbb{R}^2$. So what do we do? For $n = 0$, we just take P to be the emtpyset. For $n = 2$, will take the unit disc. Instead of a regular polygon, we just cut its boundary into two arcs, by taking the north pole and the south pole as vertices. There are only two cases for the canonical sequence for $n = 2$, viz., aa or a^{-1}.

(Refer Slide Time: 18:44)



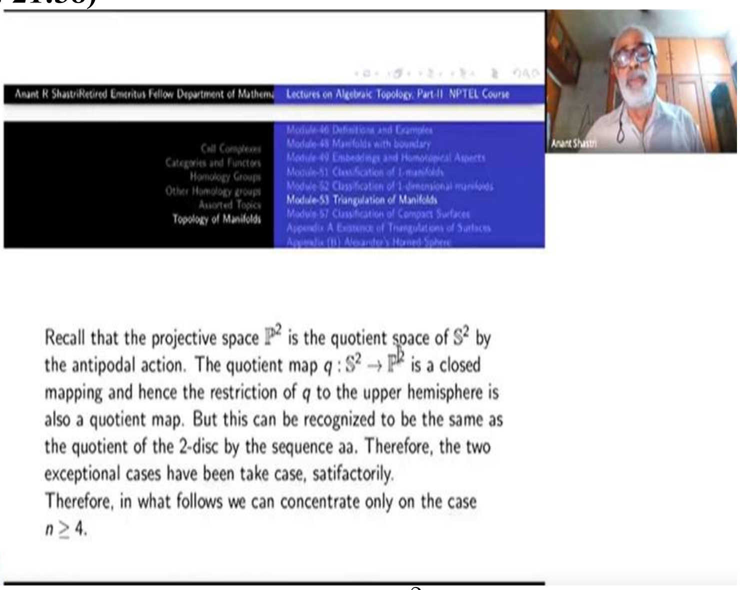
You consider the map from the unit disc in the xz-plane onto the unit sphere in $\mathbb{R}^3$ given by $(x, 0, z)$ mapsto .... as written in the slide, onto the unit sphere in $\mathbb{R}^3$. This map proves that the surface represented by the sequence aa^{-1} is homeomorphic to the 2-sphere. Okay? Look at any of these coordinate line segments as you move parallel to the x-axis up and down Okay? Under this I am they go to circles on the sphere which are intersection of the sphere with the coordinate planes parallel to xy-plane.

(Refer Slide Time: 19:42)



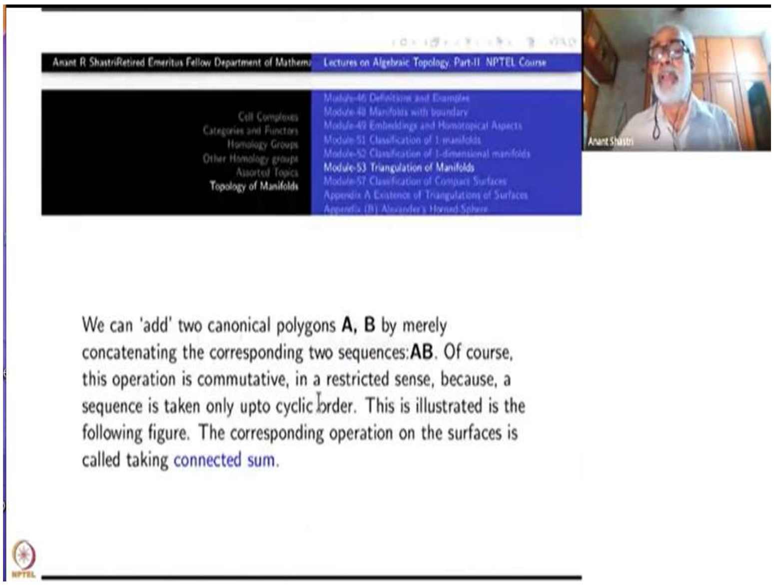
In the other case viz., aa, the identification is exactly same as the antipodal action x mapsto $-x$ on the boundary of the 2-disc. We know that this quotient space is the projective space P^2 of dimension 2.

(Refer Slide Time: 21:38)



(Recall that P^2 is, by definition, the quotient of $\mathbb{S}^2$ by the antipodal action. But you do not need the whole of $\mathbb{S}^2$, only take the upper hemi-sphere and perform the identification only on the equator.) So, that can be identified precisely by using the flat disk in $\mathbb{R}^2$ itself. And the sequence will be now aa. So, aa represents the projective space, aa^{-1} the sphere, okay? Thus we have started the classification and in the simplest cases viz, $n = 0$ and 2 we have completed it. Now let us concentrate on cases with n greater than or = 4 only, okay?

(Refer Slide Time: 22:31)



So, now I want to introduce a binary operation on the set of canonical polygons: Given two of them A, B, remember they are sequences of even length, $2n, 2m$ say, merely concatenate them to get AB of length $2n + 2m$. Okay? This operation is clearly associative and is commutative in a restricted sense, upto cyclic permutation:

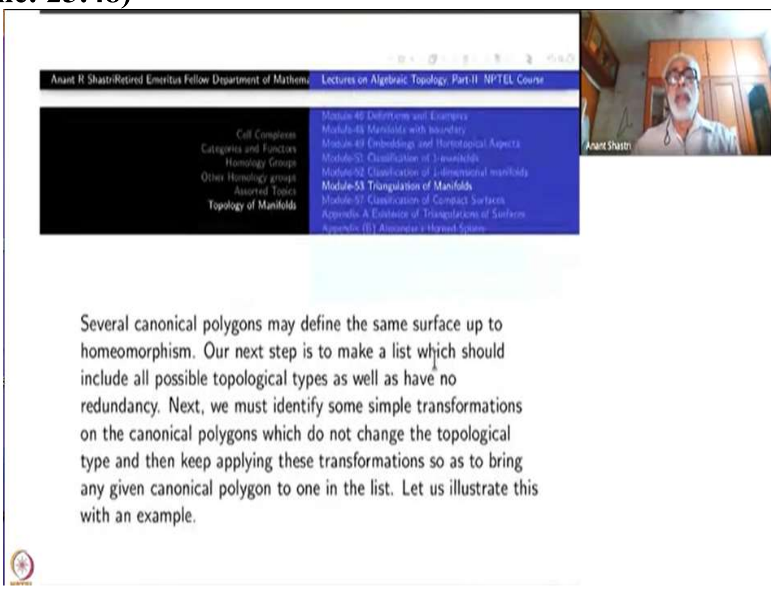
The problem is that it is not clear whether this operation is well defined on the class of canonical polygons up to cyclic permutation, viz., if you take a sequence A_1A_2, which is the same as A_2A_1 upto a cyclic permutation, then for any canonical polygon B, de we get A_1A_2B equal to A_2A_1B? Upto cyclic permutation? In general, the answer is No. However, there could be certain situations in which this holds. In order to understand this, we must appeal to the geometric operation of `connected sum' which actually has motivated this purely combinatorial binary operation.

In a simple geometric terms it means that you make a hole by removing a small disc in the interior of each of the two convex polygons and identify the two resulting circles by a homeomorphism, to get a new surface, which is called the connected sum of A and B. However, at this stage, we do not need to go deeper into this geometric aspect and just take the combinatorial definition as a definition of connected sum of two canonical polygons.

(Refer Slide Time: 25:20)

So I will give you an example, okay. So, here is my A, I do not know how many edges are there in it. I have drawn 3 edges here and three edges their which correspond the whole that you have made in the convex polygons corresponding to A, and B. The dotted parts represent the sequence A and B respectively. When you identify the three edges of the first one to those of the second, sequentially, you will get a larger polygon of size $2n + 2m$, which is the connected sum. okay?

(Refer Slide Time: 25:48)

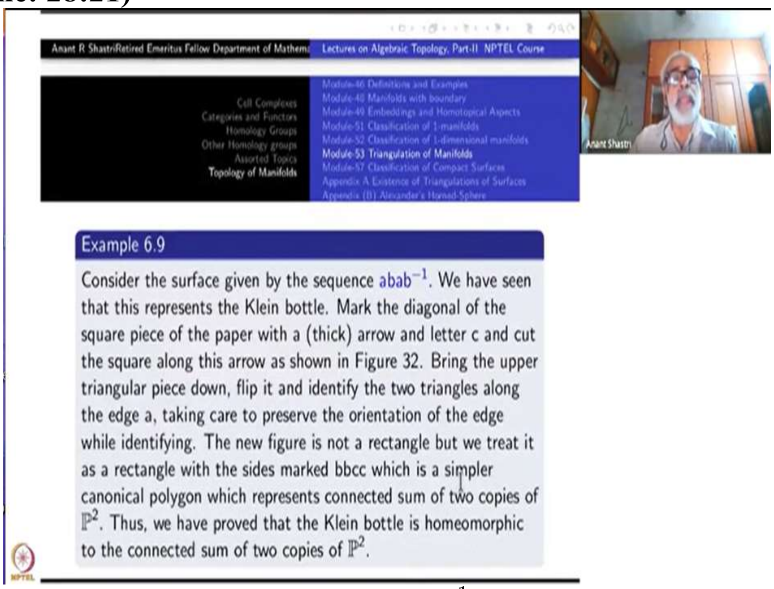


The crucial point now is that several canonical polygons may define the same surface up to homeomorphism. What we have so far is that every surface arises out of a canonical polygon. So there is a set theoretic surjective function from the set of all canonical polygons to the set of all homeomorphism classes of surfaces.

Our next step is to make a short list of canonical polygons, such that the above function is both surjective and injective, i.e., the list should include all possible topological types of surfaces and yet have no redundancy. Our list should not have A and A' which represent the same surface up to homeomorphism. Each member of the list should represent a distinct topological type. Once you have got such a list, you have achieved the classification of surfaces, okay? So, first we propose a list and then we go on to prove that it has the required property, Okay?

Let us now illustrate the point how two different canonical polygons may give the same surface. One easy way is that one polygon is obtained from the other by a cyclic permutation. For example, $abcdabcd$ is the same as $bcdabcda$. That is easy. The point is even cyclically different polygons may give the same surface.

(Refer Slide Time: 28:21)



So, consider surface given by the sequence $abab^{-1}$. Just recall this represents the familiar surface called Klein bottle. On a rectangle, oriented in the anticlockwise sense, you mark the four sides by letters. Here the horizontal sides are identified by taking both of them in the anticlockwise direction whereas the two vertical one are identifies with opposite direction. So, the sequence is $abab^{-1}$. That gives the Klein bottle okay.

Mark the diagonal with the letter c, and a thick arrow, (the direction does not matter), as shown in figure (35) here. Here I have chosen the diagonal from the starting point of a to the starting point of the next a, okay? (Or you could have taken the other diagonal, that will be from the end point of a to the end point of other a. That is what you have to remember in a more general situation.)

(Refer Slide Time: 29:56)

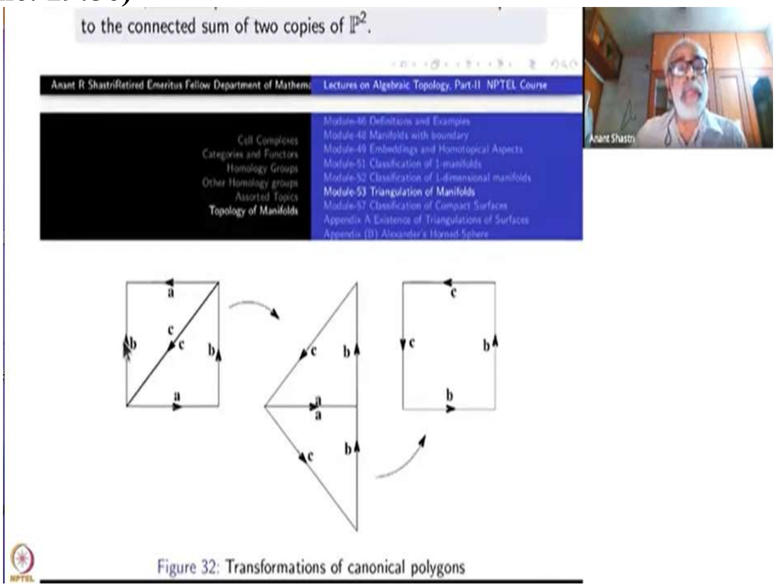


Figure 32: Transformations of canonical polygons

Now cut the rectangle along c, to get two triangle. The upper triangle you bring it down and flip it so that this edge marked with a becomes parallel with the corresponding edge in the lower triangle. So, now the bottom edge of one triangle and the top edge of the other are aligned in the correct direction. Identify the two triangles along these two edges.

The resulting figure does not look like a rectangle, is actually a triangle. But as a combinatorial object, it has 4 sides, and you can easily deform it inside $\mathbb{R}^2$ into a rectangle or a square. So, now the sequence is $bbcc$, which represents the same surface, a Klein bottle.

If I just start from here and cut this new square along the diagonal, running from the initial point of b to the initial point of c, what I get is two sequences bb and cc. Both sequences represent the projective space P^2. Therefore, the Klein bottle can be thought of as the connected sum of two copies of P^2.

So, this example is going to be used at the end of the proof of classification that is to come. The process itself becomes a technique called the cut-and-paste technique and is heavily used. Let us stop here today. So, tomorrow we will actually start the classification problem. Thank you.