Introduction to Algebraic Topology (Part-II)
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Lecture - 55
One Result Due to Poincare and Another Due to Munkres

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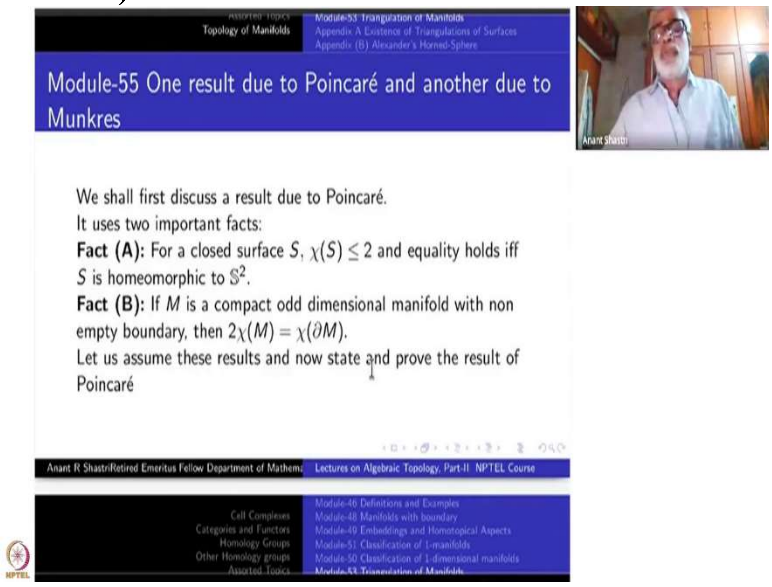

As promised last time today let us discuss one interesting result due to Poincare and another due to Munkres. First let us take Poincare's result. So, we will require two facts which we have not proved so far for lack of time. The first fact is that for any connected closed surface (closed surface means a compact surface without boundary) the Euler characteristic is always less than or equal to 2 and the equality holds if and only if S is homeomorphic to the 2-sphere.

The second fact is that if M is a compact odd dimensional manifold with non empty boundary, then its Euler characteristic is half of the Euler characteristic of the boundary, i.e., $2\chi(M) = \chi(\partial M)$. The second fact is an easy consequence of the following fact which itself is not an easy result, namely, every odd dimensional closed manifold has Euler characteristic 0. From that you can deduce this one, by our technique of doubling the manifold. So, that is the hint. I will leave the rest of it to you as an exercise.

A proof of the first fact you will see in the last part of these lectures during classification of surfaces. So, right now, we have not proved it. We will assume it.

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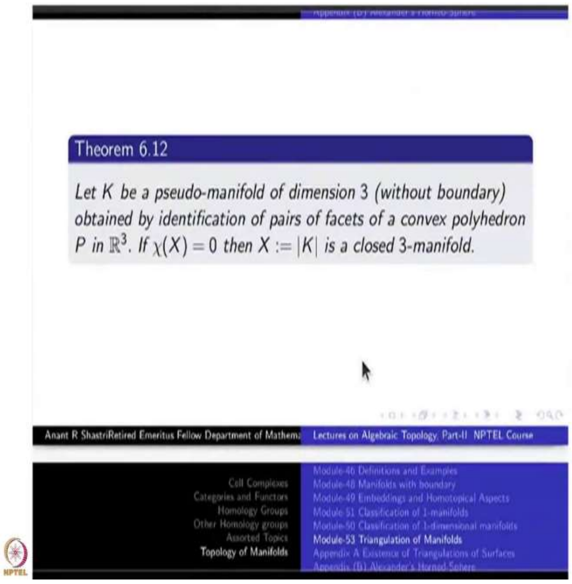

Now let us look at Poincare's result. Take K to be a simplicial complex which is a pseudo manifold of dimension 3 without boundary. Suppose it is obtained by identification of pairs of facets of a convex polyhedron P inside $\mathbb{R}^3$, just like we have done in the previous theorem. If the Euler characteristic of K is 0, then $X = |K|$ is a closed 3-manifold.

If it is a closed 3-manifold, then we know that its Euler characteristics must be 0. That is the general result. So, Poincare comes up with this great result which is a kind of converse. Of course we have to use that it is a quotient of a convex polyhedron by facet identifications on the boundary. So that is the hypothesis. Not all pseudo 3-manifolds are quotients of convex polyhedron wherein the identifications are precisely the facet identifications and nothing extra. They are of course quotients of Convex polyhedrons, but there may be further identifications. That is what we have proved in the earlier theorem. After proving it we have asked this question and now Poincare has an answer here, a positive answer. The proof is not very difficult. Let us go through this one.

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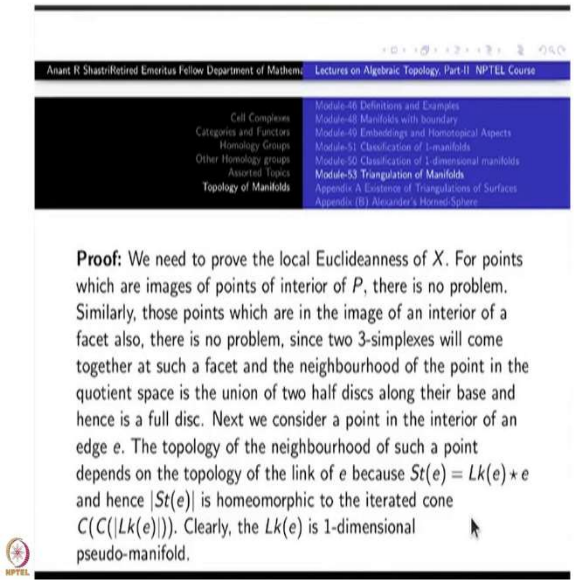

We need to prove the local Euclideanness of $X = |K|$. There is nothing else to be done since it already compact Hausdorff. For points which are images of points in the interior of P, we can take the quotient map Θ restricted to the interior of P to give a coordinate chart, because Θ is a homeomorphism onto an open subset of X. Therefore, there is no problem.

Similarly, for those points which are in the image of the relative interior of a 2-simplex, what happens? In a pseudo 3-manifold whenever you have a 2-simplex, it is a facet of exactly two of the 3-simplexes, one on this side one on the other side. So you can take the union of these two 3-simplexes. One half open disc on this side and another half open disc on the other side, together they will form a neighbourhood homeomorphic to $\mathbb{R}^3$. So at such points also, there is no problem.

So, what are the points that are left out? Points on the edges, both interior of an edge as well as finally all the vertex points. These are the points which have to be carefully studied? What happens to the neighbourhood of these points. So, consider a point in the interior of an edge e. The topology of the neighborhood of such a point depends on the topology of the link of e, because the closed star of e is equal to the $Lk(e)$ joined with the edge e. Any edge is homeomorphic to a closed interval. So, there I know what is the topology? There is no problem. So, what is the $Lk(e)$. This is the mystery, I do not know how it looks like. If this is also like a ball then you would have completed the proof? So, if I address myself to find out the topology of $Lk(e)$ then I know what is going to happen to $st(e)$. $st(e)$ being the join of $Lk(e)$ with a closed interval, is homeomorphic to the iterated cone, Cone of the cone of $Lk(e)$.

Clearly the $Lk(e)$ is one a dimensional pseudo-manifold. Why? Because K is a 3-dimensional psuedo-manifold, dimension of the $Lk(e)$ plus dimension of $e + 1$ must be equal to 3. That is the equation for dimensions. $1 + 1 + 1 = 3$. So, that is why dimension of $Lk(e)$ is 1.

It is not very difficult to see that link of any simplex in a pseudo-manifold is also a pseudo-manifold. We have already classified one-dimensional manifolds.

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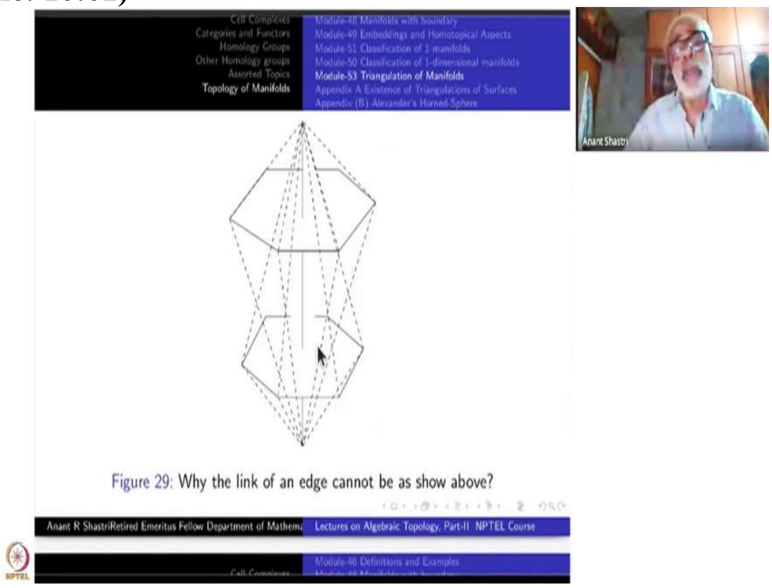

If it is connected then we know it is a circle. A cone over it is already a 2-disc and the cone over that will be a 3-disc. So, you have found a neighborhood which is a 3-disc for every point in the interior of an edge e.

The only problem now is to show that $Lk(e)$ is connected, which may not be the case in general. So, here is a picture. You want to say that this will not happen. Here the middle line segment is our 1-simplex e. Can it happen that all these 3-simplexes in K share e as a common edge and they forming two families, members of one around one vertex and the other around other vertex of e. (Our picture is in $\mathbb{R}^3$ wherein it is not possible for other obvious reasons, but K may not a subspaces of $\mathbb{R}^3$.) So, here $Lk(e)$ is a disjoint union of two circles, one circle here one circle there.

So, why this picture is wrong, why this picture does not occur? That is what we have to show by pure argument. The only extra condition is that $\chi(X)$ is 0. But we shall not need it here.

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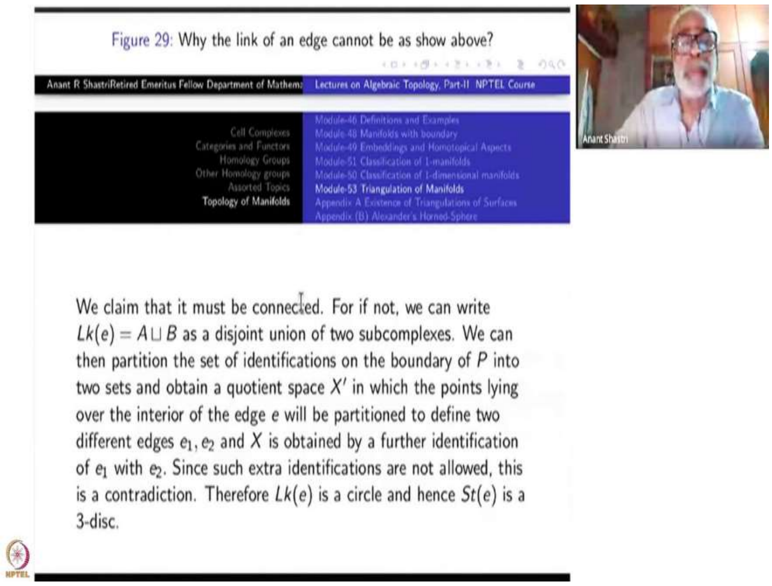

So, we claim that $Lk(e)$ must be connected. (So, here it is a topological argument not using any picture.) If not what happens? We can write $Lk(e)$ as a disjoint union of two subcomplexes A and B (like in the picture). We can then partition the set of identifications on the boundary of P into two disjoint sets accordingly. Because each edge e' in $Lk(e)$ here will determine, along with this edge e, a unique 3-simplex in K, viz., $e' * e$. So, consider the A' (respectively B') collection of all 3-simplexes in P which are mapped onto a 3-simplex of the form $e' * e$, where e' belongs to A (respectively B). A' and B' are some collection of 3-simplexes in P which share a facet in the boundary of P. Together they cover all 3-simplexes such that their image under Θ contains e. You see that facet identifications are occuring with these two families. Therefore, no 2-simplex in the first A' will have anything to do with a 2-simplex in the second family B'.

But there is this 1-simplex common to the image of all of them. how did that happen? Remember, when you are identifying two 2-faces, automatically certain edges of the faces will get identified. But there is no additional edge to edge identification, extra identifications are not allowed. (This is a very important point which not many people may understand this. When you identify a triangle with another triangle via a linear map, automatically the boundary edges of this triangle and the boundaries of that triangle are also identified in a corresponding order. After this if you say one ore edge is identified with one other edge and so on, that is not allowed.

Therefore, will give you a contradiction. In the quotient there will be at least two distinct edges e_1 and e_2 but they are both mapped to the to same edge e here. So, how did that happen? So, this is why the link of e must be connected 1-dimensional pseudo manifold and therefore, it is a circle. A circle star e is homeomorphic to the cone over the cone of the circle which is a 3-disc.

So, that will take care of all the points in the interior of all the edges. Finally, we are left with vertices in boundary of P. Why there are neighborhoods of vertices which are homeomorphic to $\mathbb{R}^3$ or homeomorphic to an open disc in $\mathbb{R}^3$? This is what we have to see. This is where the last condition $\chi(X) = 0$ comes in to play.

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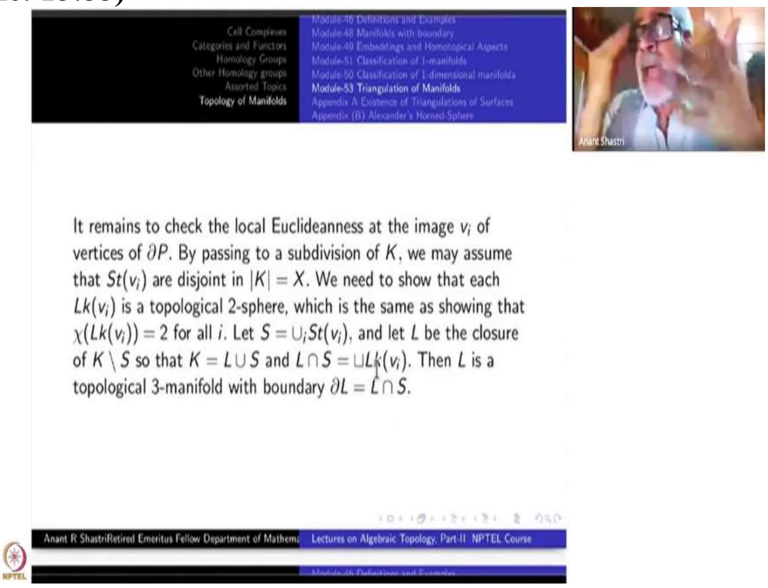

It remains to check the local Euclideanness at the image v_i of vertices of the boundary of P. We work with a subdivision K' of K, say $K' = sd^2(K)$, where in $St_{K'}(v_i)$ are disjoint disjoint from each other. It is enough to show that each open star $st_{K'}(v_i)$ is homeomorphic to an open disc. Equivalently, it is enough to show that $Lk_{K'}(v_i)$ are all homeomorphic to the 2-sphere. Note that we are not taking all vertices of K' here, v_i are vertices of the original K only. So, all that I have to prove is that each $Lk_{K'}(v_i)$ is a topological 2-sphere which is the same as showing that $\chi(Lk_{K'}(v_i)) = 2$ for all i.

This is where I am using the fact (B). What I do? Let S equal to union of all the closed stars of the vertices v_i, and let L equal to the closure $K \setminus S$. Each closed star is a closed subset and S being a finite union of these, S is also closed. Take $K \setminus S$. That is the open set, take its closure, so that K is actually $L \cup S$ and what is $L \cap S$? It is precisely the union of all the boundaries of $St(v_i)$ which is nothing but a disjoint union of $Lk_{K'}(v_i)$. Since we have shown that K is 3-manifold away from all the vertices v_i, it follows that L is 3-manifold with boundary equal to $L \cap S$. And now I use the fact (B). What does it say? χ of boundary of L is equal to twice the $\chi(L)$?

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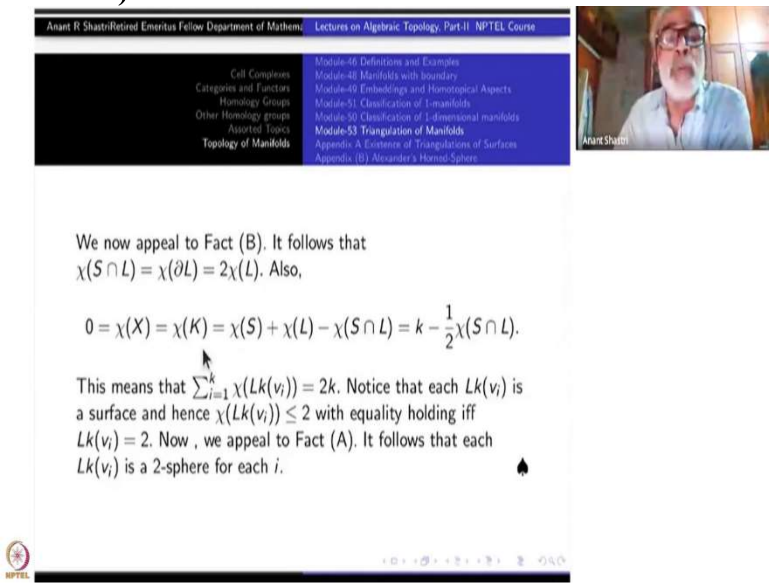

But what is boundary of L? It is $S \cap L$. Since $\chi(X)$ is 0 by the hypothesis. For any simplicial complex you have sen the addition formula for Euler characteristic. $\chi(K)$ is equal to $\chi(L) + \chi(S) - \chi(L \cap S)$, where L and S are subcomplexes such that $K = L \cup S$. Therefore, $0 = \chi(L) + \chi(S) - \chi(S \cap L) = \chi(S \cap L)/2 + \chi(S) - \chi(S \cap L)$. Therefore, $2\chi(S) = \chi(L \cap S)$.

What is S? S is a disjoint union of star of v_i's. The problem is to prove that each of them is 3-disc, that is our problem. But each of them is contractible, being a cone. For a contractible space, Euler characteristic is 1. But there are how many of them? Let us say that k is number of vertices in K. Euler characteristic of a disjoint union is the sum of the Euler characteristics. Therefore, $\chi(S) = k$. So $\chi(L \cap S) = 2k$.

Since $L \cap S$ is the boundary of 3-manifold, it the disjoint union closed surfaces $Lk_{K'}(v_i)$ and there are k of them. For each of them $\chi(Lk(v_i))$ is less than or equal to 2. But the sum total is $2k$. So, therefore, each of them must be equal to 2. It follows that $Lk(v_i)$ is a topological 2-sphere, for each i.

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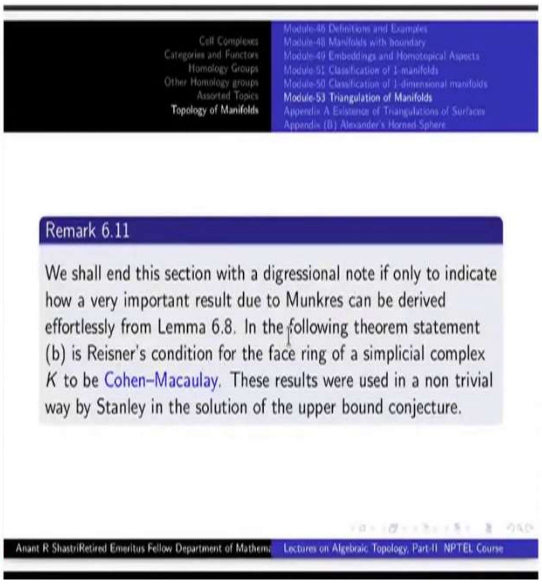

So, let us now take a result due to Munkres which is actually used in a big theorem by Stanley, through the works of another important person called Reisner. Reisner gave combinatorial condition for the face ring to be a Cohen-Macaulay face ring of a simplicial complex. Munkres result uses to prove that the Reisner condition is satisfied in a particular way.

So, in that way, Munkres result is useful in combinatorics. Ultimately all these things were used in proving a big conjecture which is called the upper bound conjecture, by Stanley. So, we are not going to do anything about the face rings here. But, we will do the topological aspect of that. Namely, Munkres result which an important result on its own. I want to show that just because something is important does not mean the proofs are difficult. I want to give you an easy proof of this one. Here part of it we have already done.

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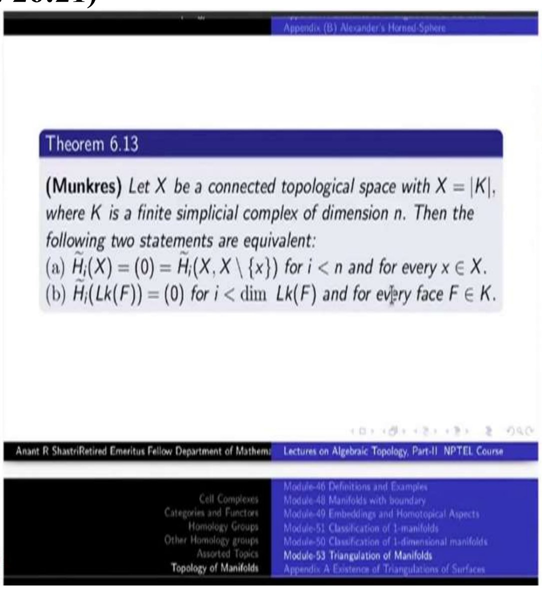

So, what is the theorem of Munkres? It says, take a connected topological space which is triangulated, $X = |K|$. I assuming that K is a finite simplicial complex of dimension n. Then the following two statements are equivalent. These are some local-global homological conditions:

(a) $\tilde{H}_i(X) = 0$ (this is a global condition) and $H_i(X, X \setminus \{x\})$ is 0 for every for $i < n$ and for every $x \in X$. So, this is true for $i < n$. Both $H_i(X, X \setminus \{x\})$ and $\tilde{H}_i(X)$ are 0. It is the first condition.
(b) Second condition is $\tilde{H}_i$ of link of F is 0 for i less than dimension of the link of F and for every simplex of K. The reduced homology of the link vanishes below its dimension.

And these 2 conditions are equivalent is the statement of theorem. Since we have done a little bit of the study of these things already, it will be very easy for you to follow this proof.
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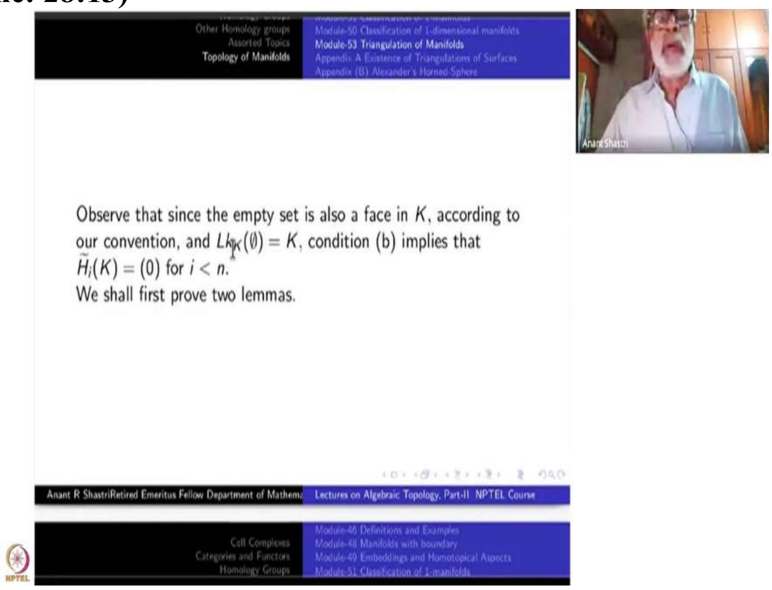

Observe that since the empty set is also a face, it is allowed to take $F =$ in the statement (ii). According to our convention as well as according to Munkres' convention the link in K of the empty set the whole of K.This follows by the definition as well. Therefore, condition (b) implies that $\tilde{H}_i(K)$ is 0 for all $i < n$.

So, this gives you the first part of (a). For the rest of the proof you have to wait. We shall first prove two lemmas. In fact, I am breaking down the proof into simpler statements here, instead of complicating the whole proof by trying it in one go.
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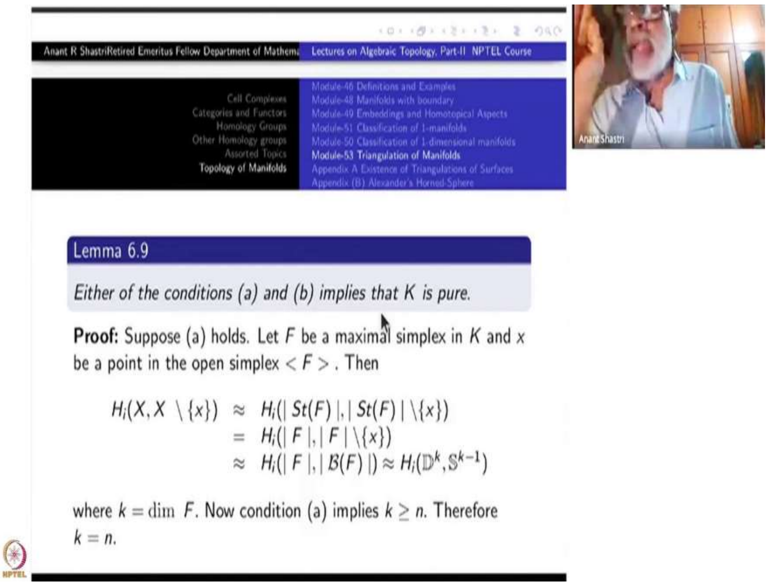

Take just these conditions, condition (a) or condition (b). Each one of them implies that K is a pure simplicial complex. We have seen that for a triangulation of a topological manifold. The proofs here are similar.

Suppose (a) holds. Take a maximal simplex inside K. I have to show that dimension of that simplex is equal to n. That is what I have to show for purity of K. Take F to be a maximal simplex. Take a point x in the interior of F. Then we have seen that by excision, $H_i(X, X \setminus \{x\})$ is isomorphic to $H_i(\mathbb{D}^k, \mathbb{S}^{k-1})$. These steps you have seen before. Here k is the dimension of F. The excision works because for any maximal simplex F, the open simplex F is an open subset of $|K|$. Now (a) implies that this $k \geq n$, because H_k of this one is infinite cyclic. But k cannot be bigger than than n and so, it must be equal to n. Now, suppose (b) holds.

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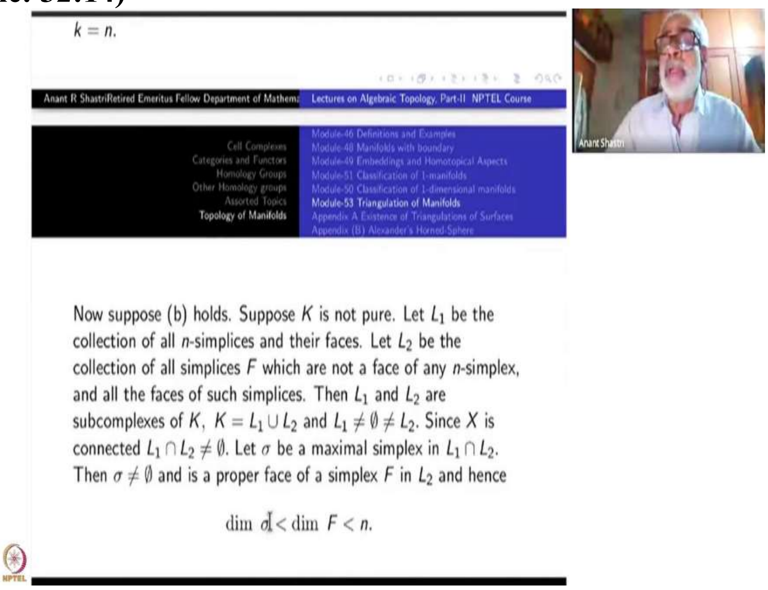

Now, this time the proof is a little more elaborate and we are again repeating the old arguments here. Suppose K is not pure. Let L_1 be the collection of all n-simplexes and their faces. If it is not pure that means there are simplexes in K which are not faces of any n-simplex and therefore L_1 is a proper subcomplex.

So, let L_2 be the collection all simplexes in K which are not a face of any n-simplex and all faces of such simplexes. Then $K = L_1 \cup L_2$. Both L_1 and L_2 are non-empty (by our assumptions that dimension of K is n and K is not pure) subcomplexes.

Since X is connected it follows that $L_1 \cap L_2$ must be non-empty. All these arguments you have seen earlier also. Take a maximal simplex σ inside the subcomplex $L_1 \cap L_2$. Of course σ is non-empty and it is a proper face of a simplex F in L_2. Why? Because if it is a maximal in L_2 also, then it will not be in L_1. Hence dimension of σ is less than dimension of F, dimension of F itself is less than n because F is inside L_2. So, the codimension of σ in K, which is n minus dimension of σ is at least 2.

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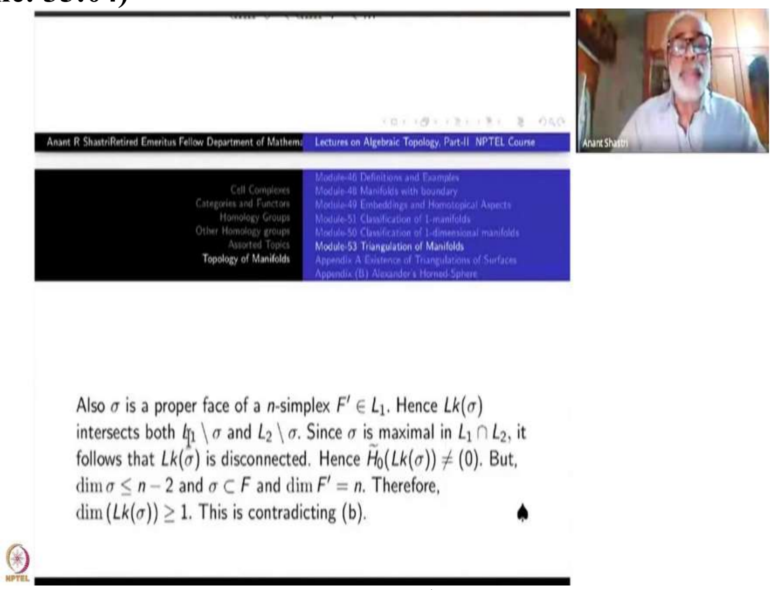

Also, sigma is the proper face of a n-simplex F' in L_1, because by definition of L_1, every simplex inside L_1 will be contained in a larger simplex which is of n-dimension. Hence link of σ, $Lk_K(\sigma)$ intersects both L_1 and L_2. (Now we come to a part which is different from the earlier proof where we used the connectivity of K. So, here we are doing something different.)

Since σ is maximal in $L_1 \cap L_2$ it follows that $Lk(\sigma)$ intersection with $L_1 \cap L_2$ is empty. Therefore link of σ is disconnected. Therefore, $\tilde{H}_0$ of link of σ is not 0. (If it is 0, it would have been connected.) On the other hand, we have seen that σ is of dimension at most $n-2$. And σ is inside F' which is of dimension n. Therefore, dimension of the link of σ is at least 1. Below that, the reduced homology must be 0 according to (b). That is a contradiction to condition (b).

So, this proves our lemma namely, both conditions imply K is pure. After this lemma, the proof of Munkres' result is very easy.

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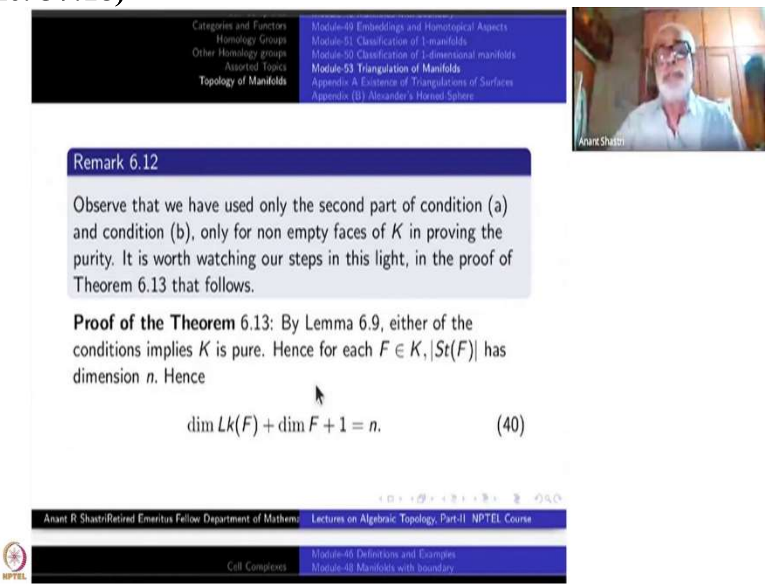

All that your observe is that we have used only the second part of condition (a) or condition b for non-empty faces in proving the purity. The condition (b) for an empty face implies the first part of (a). I have already told you this one. So, now proof of the theorem. By the lemma either of the conditions imply K is pure. Hence, for each F belonging to $K, St(F)$ has dimension n, no matter what simplex F you take, a vertex, an edge or a n-simplex.

Therefore, dimension of $Lk(F)$ plus dimension of F plus 1 is equal to n, which is the dimension of the star, because star is always link starrd with the simplex. So, that is dimension equation, we have been using this one again and again.

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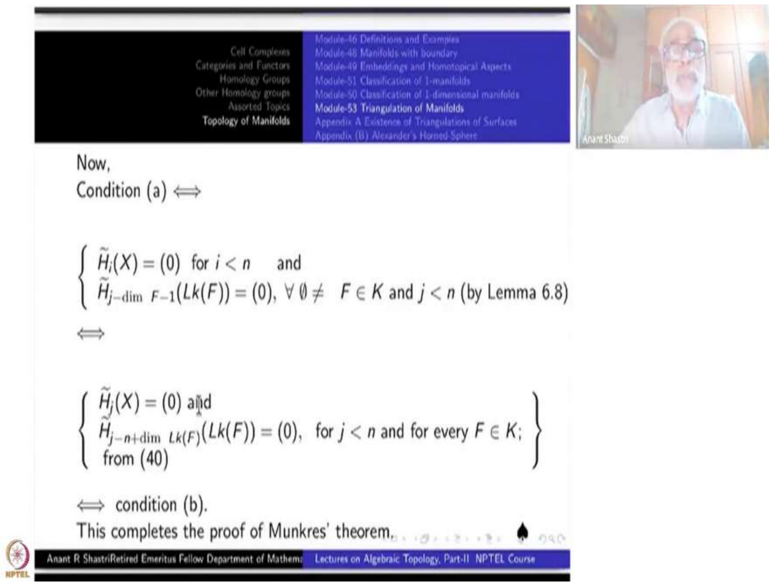

Now, start with condition (a), I want to prove that it is equivalent to condition (b). There are only these two steps now, you have to understand and that is the proof. Let us see condition a implies and implied by, is equivalent to to say that $\tilde{H}_i(X) = 0$ for $i < n$ (it is the first part and (the second part is) $H_i(X, X \setminus \{x\}) = 0$. So, what we have seen that this latter group is equal to $\tilde{H}_{j-\dim\ F-1}$(link of F) by taking a point x in the interior of F. This is true for every non empty face F of K and for all $j < n$.

This is the same as saying $\tilde{H}_{j-n+\dim\ (\text{link of } F}(Lk(F))$ is 0 for $j < n$ and for every face F of K. We have just seen the formula of dimension of link, I am putting that here. Dimension of the link is n minus dimension of F minus 1. How I got it? I got from (40) here. So, from here to here to here you have come, but this is nothing but condition (b). So, Munkres' theorem is proved.

Next time I shall make some general remarks about triangulations. Where to find more material, what are the general results known and so on with no proofs. After that we will start classification of triangulated surfaces. Thank you.