Introduction to Algebraic Topology (Part-II)
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Lecture - 54
Pseudo - Manifolds

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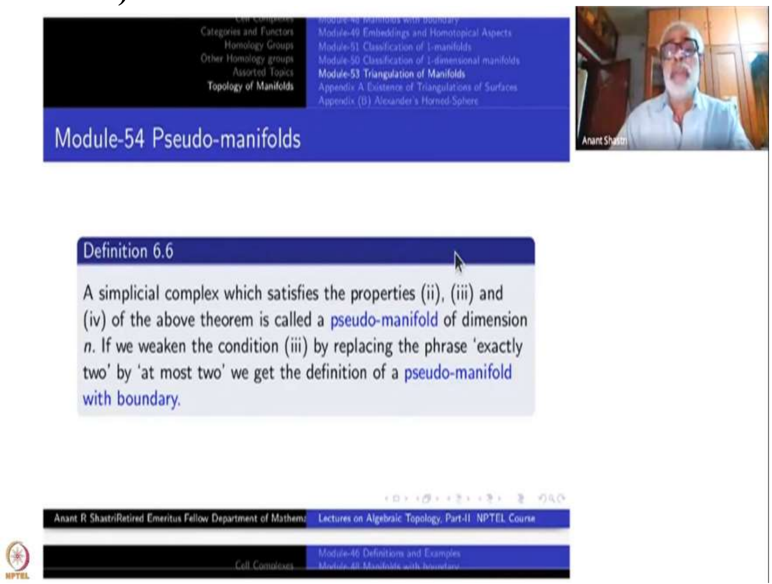

Having studied some local homological properties of a simplicial complex and then having studied some local homological properties of a triangulated manifold, we were lead to make the following definition now. A simplicial complex which satisfies properties (ii), (iii) and (iv) of Theorem 6.10 is called a pseudo-manifold of dimension n. So, this was part of the statement of a theorem that we proved last time. Let us go through that theorem first. So, these was the statement.

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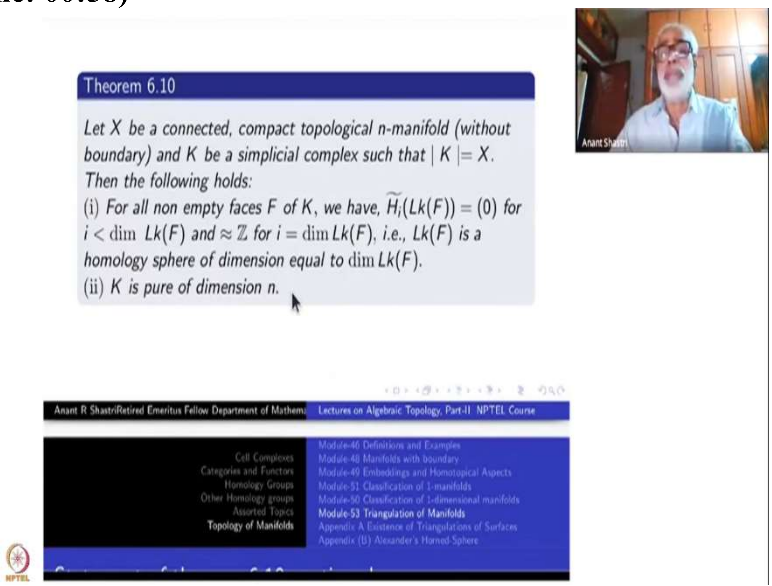

Take connected compact topological manifold without boundary and let K be a simplicial complex such that $|K| = X$. In other words, X is a connected compact topological n-dimensional manifold and is triangulated. Then the following holds:

(i) For all non-empty faces of K, we have $\tilde{H}_i(Lk(F)) = 0$, for i less than the dimension of $Lk(F)$, and is isomorphic to $\mathbb{Z}$ for i equal to dimension of $Lk(F)$. This can be restated, in nut-shell as follows: link of F looks like a homology sphere.

(ii) The second statement is that K is pure of dimension n.

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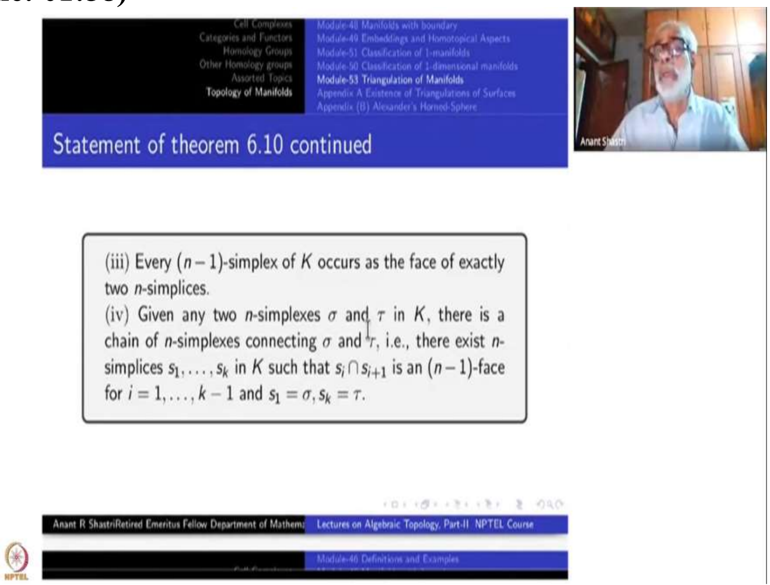

(iii) Third statement is that every $(n-1)$-face of K occurs as the face of exactly two n-simplices.

(iv) The fourth condition says that from any n-simplex to another n-simplex in K, we can go via a path of n-simplexes.

By a path of n-simplexes we mean a sequence of n-simplexes, $s_1, \ldots, s_k$ such that the intersection of s_i with s_{i+1} is exactly an $(n-1)$-face. You start with $s_1 = \sigma$ and end up with $s_k = \tau$, then this a path from σ to τ.

So, instead of two of course the pure means that there will be at least one. So, if there is only one then that kind of simplexes will become boundary part so then you will get manifold with its boundary.

This condition especially along with of course (ii) and (iii) themselves become important now. X may not be topological manifold now, forget about condition (i) also. Assume only that K is a simplicial complex satisfying these there conditions. Forget about compactness also. Anyway, connectivity comes automatically from (ii) and (iv). Make these conditions as axioms for the definition of a pseudo manifold.

In condition (iii), if you replace the phrase `exactly two' by `at most two', then you get the definition pseudo-manifold with boundary, This means that an $(n-1)$-simplex may have only one n-simplex containing it, since condition (ii) says that there is at least one. Such $(n-1)$-simplexes will constitute a subcomplex called the boundary of K.

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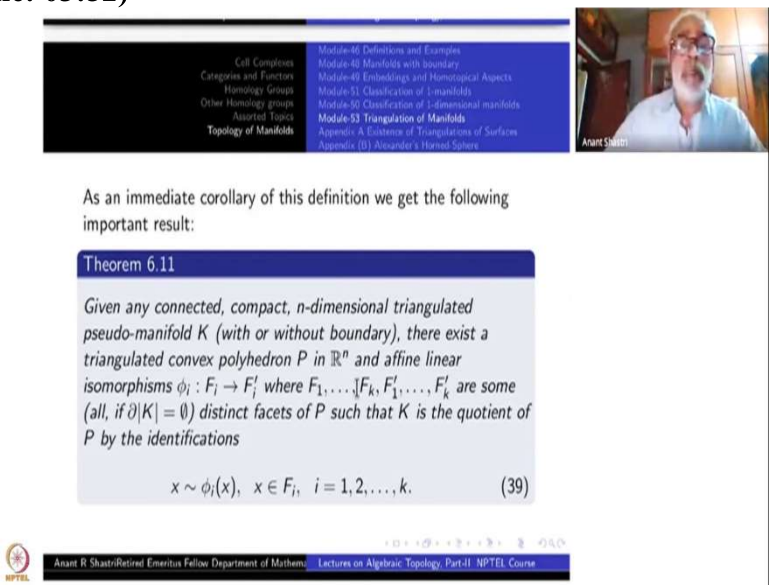

So, now, having made a definition we have the following corollary, stated as a theorem 6.11. Given any compact pseudo-manifold of dimension n, with or without boundary, there exists a triangulated convex polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ a surjective linear map Θ from $\mathcal{P}$ to K which is an isomorphism restricted each simplex in $\mathcal{P}$. Moreover, there are affine linear isomorphism ϕ_i from F_i to F_i', where $F_1, F_2, \ldots, F_k$ and $F_1', F_2', \ldots, F_k'$ are some distinct facets of $\mathcal{P}$ such that $\Theta(x) = \Theta(\phi(x))$ for all $x \in F_i$ and for all $i = 1, 2, \ldots, k$.

(Reviewer's note: It should be noted that the statement of the theorem 6.11 as appears in the slide is somewhat incorrect and incomplete. It also follows from the above statement that the boundary of K is empty iff the boundary of $\mathcal{P}$ is equal to the union of of all F_i' and F_j''s.)

This result is quite simple minded and is the starting point of our classification for two dimensional manifolds. Indeed, this is the way how Poincare had perceived a whole for classifying three dimensional manifolds. Unfortunately, even after several years, almost a whole century of trials by various authors to complete his programme, it has failed in some sense. But, for $n = 2$, we going to use it in the classification.

Though in higher dimension it has failed to yield proof of classification, this result gives quite a lot of information on the topology of a triangulated manifold. So let us go through the proof of this one today.

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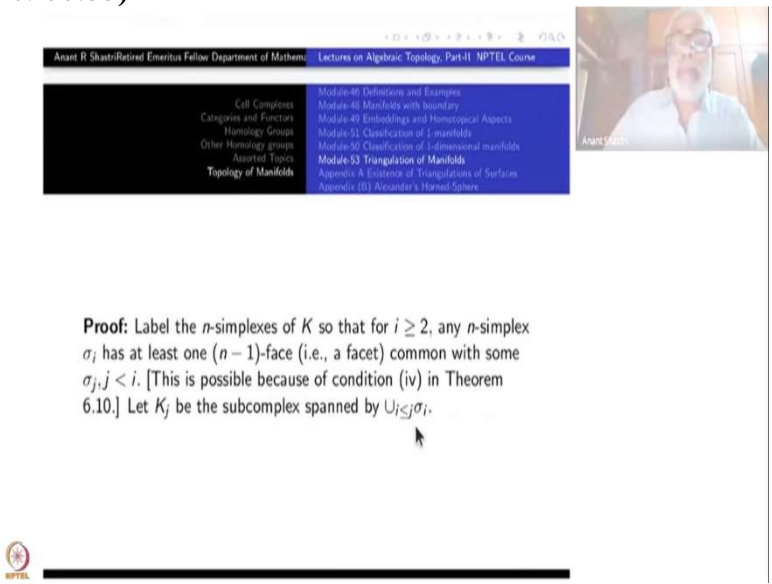

We start by labeling the n-simplexes of K. If there is only one n-simplex, there is nothing to prove. Just call it σ_1. Put $K_1 = \sigma_1$. So, we assume that there are more than one n-simplex. Choose σ_2 to be any one of the remaining simplexes such that σ_2 shares one of its $(n-1)$-facet with σ_1.

Having labeled $i-1$ of them, $\sigma_1, \ldots, \sigma_{i-1}$, put K_{i-1} equal to the union of σ_j for $j < i$, and choose σ_i to be yet another n-simplex which shares at least one of its facets with σ_j for $j < i$.

And this is possible because of condition (iv) in theorem 6.10. For if there are more that $i-1$ n-simplexes in K, (i,e, K_{i-1} is not equal to K), then fix one of them say τ in in K_{i-1} and choose a path of n-simplexes from σ_1 to τ. The first n-simplex which is not equal to any of σ_j for $j < i$ will qualify to become σ_i.

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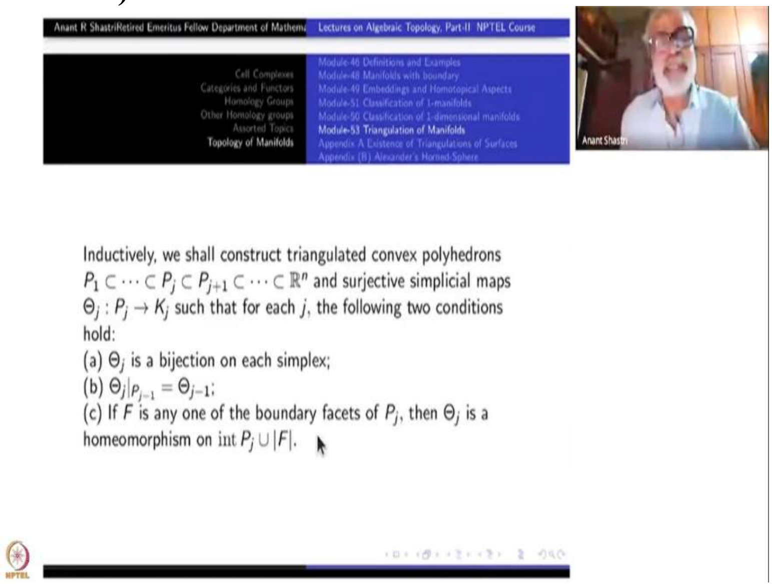

Now what I am going to construct triangulated convex polyhedrons $\mathcal{P}_1$ contained in $\mathcal{P}_2 \subseteq \ldots \mathcal{P}_j \subseteq \ldots$ and surjective simplicial maps Θ_j from $\mathcal{P}_j$ to K_j such that for each j the following 3 conditions hold.

(a) Θ_j is a bijection on each simplexes of $\mathcal{P}_j$.

(b) Θ_j restricted to $\mathcal{P}_{j-1}$ is Θ_{j-1}. That means, each successive Θ_j is an extension of the previous ones.

(c) the third condition is that if F is any one of the boundary facets of $\mathcal{P}_j$, then Θ_j is a homeomorphism restricted to the interior of $\mathcal{P}_j \cup F$. If you take the whole of $\mathcal{P}_j$ along with all of its boundary, Θ_j may not be injective. In other words, injectivity fails at each stage, only on the boundary of $\mathcal{P}_j$ if at all.

For example, when $j = 1, \Theta_1 \mathcal{P}_1$ to K_1 is actually an isomorphism, where $\mathcal{P}_1$ can be chosen to be the standard n-simplex in $\mathbb{R}^n$, convex hull of $\{0, e_1, \ldots, e_n\}$.

So, this condition looks like a weak condition but this is all we can ensure. As we keep going up inductively, this will help us to prove the next step. By the way, if I just say that in the interior of $\mathcal{P}_j$, it is injection that is not enough.

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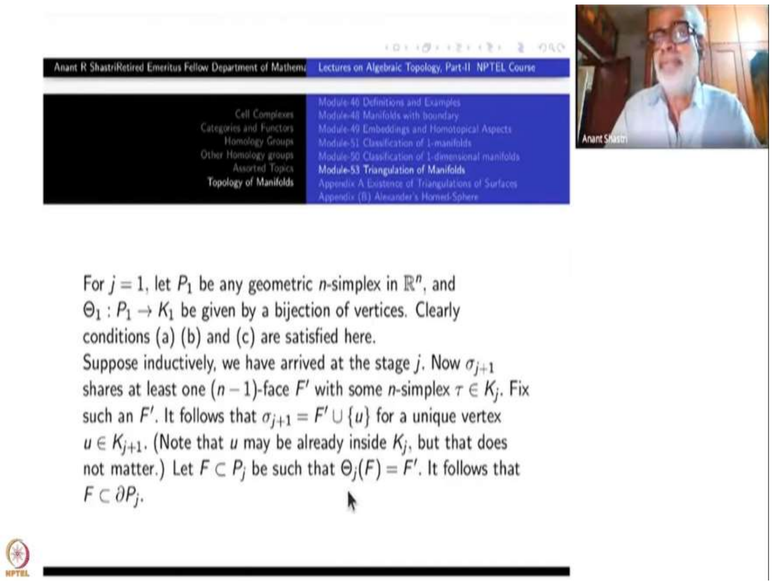

So, let us see how we are going to do this inductively. We are interested in the size of the convex polyhedrons of course. So for $j = 1$, I am free to choose any $n + 1$ points in $\mathbb{R}^n$ and their convex hull as $\mathcal{P}_1$, which is automatically isomorphic to any n-simplex.

So, start with Θ_1 from $\mathcal{P}_1$ to K_1 given by a bijection of vertices extended linearly. Clearly condition (a), (b) (c) are satisfied. Suppose inductively we have arrived at the stage j of the construction with $\mathcal{P}_1$ contained in $\mathcal{P}_2 \subset \cdots \subset \mathcal{P}_j$, and Θ_j from $\mathcal{P}_j$ to K_j satisfying (a), (b) (c).

Now look at σ_{j+1}. By the very choice of this labeling, it will share at least one $(n-1)$-facet with some simplex τ inside K_j. (It may share more of than one also.) Fix one such facet F'. It follows that σ_{j+1} is the union of F' with a unique vertex u belonging to K_{j+1}, where recall that $K_{j+1} = K_j \cup \sigma j + 1$. So, it will be inside σK_{j+1}. But what may happen is that this extra vertex may be already inside K_j. It is can happen. But we do not have any objection for that.

Now let F contained $\mathcal{P}_j$ be a facet which mapped onto F' by Θ_j. Such an F exists because Θ_j is surjective. It follows that F must be also in the boundary of $\mathcal{P}_j$ because below, F' is in the boundary of K_j.

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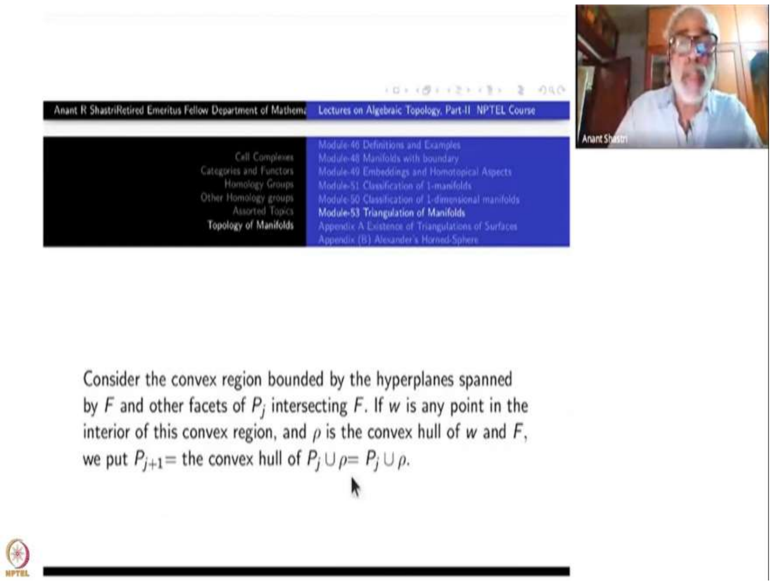

Now, I want to construct P_{j+1}. I have a convex polyhedron $\mathcal{P}_j$ and I have located a facet F on the boundary. So, consider the convex region bounded by the hyperplanes (i.e. affine linear subspaces in $\mathbb{R}^n$) spanned by F and each facet in the boundary of $\mathcal{P}_j$ which intersect F, and which lies outside $\mathcal{P}_j$, (which is actually a bounded one also). Take w to be any point in the interior of this convex region. Let ρ be the convex hull of $F \cup \{w\}$, which is nothing but an n -simplex now. Put $\mathcal{P}_{j+1}$ = convex hull of $\mathcal{P}_j \cup \rho$. Indeed, $\mathcal{P}_{j+1}$ is just the union of {$\mathcal{P}_j$ with ρ.

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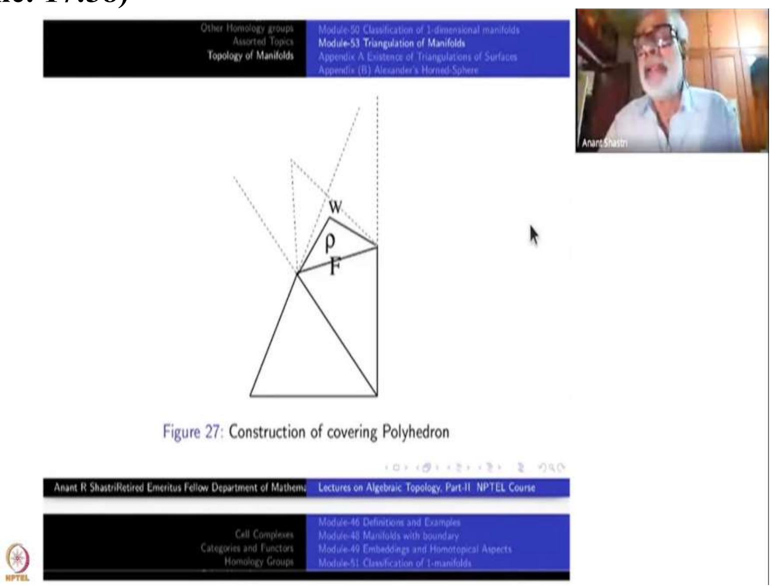

So, look at this picture it will tell you the story. I have constructed up to here and you are located a facet F in $\mathcal{P}_j$ and look at this plane, ..., this plane all that at all of them intersecting this F, so there will be a convex region bounded by all of them. and outside of this convex region. Choose some point w like this then take the convex hull of F and w. That is my ρ. This is a n-simplex. Here in the picture $n = 2$. Automatically union of $\mathcal{P}_j$ and ρ will be a convex region. If you choose w in the boundary of ρ even then the convexity of the union holds but then the facets of the union will not be appropriate. If you choose w outside rho then the union may easily fail to be convex. So, that is the criterion for choosing this w.

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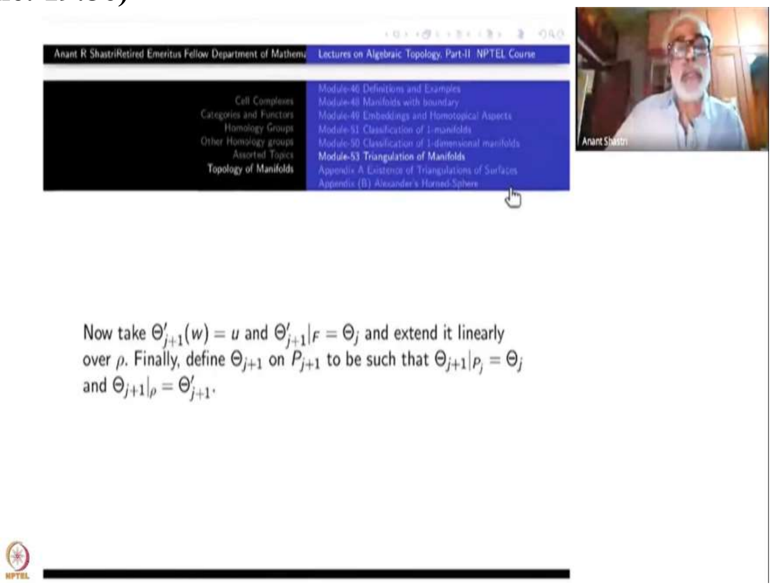

Now, define Θ' from ρ to σ_{j+1} by taking Θ' on F to be Θ_j restricted to F and $\Theta'(w) = u$ and extend linearly over the whole of ρ. Finally define Θ_{j+1} on $\mathcal{P}_{j+1}$ to be Θ_j on $\mathcal{P}_j$ and Θ' on ρ.

So, the construction is over, but we have to prove that all these three conditions now. First two are obvious. But condition (c) condition I have to prove, assuming it to true upto j, I have to prove it for $j + 1$. Then the construction will be over.

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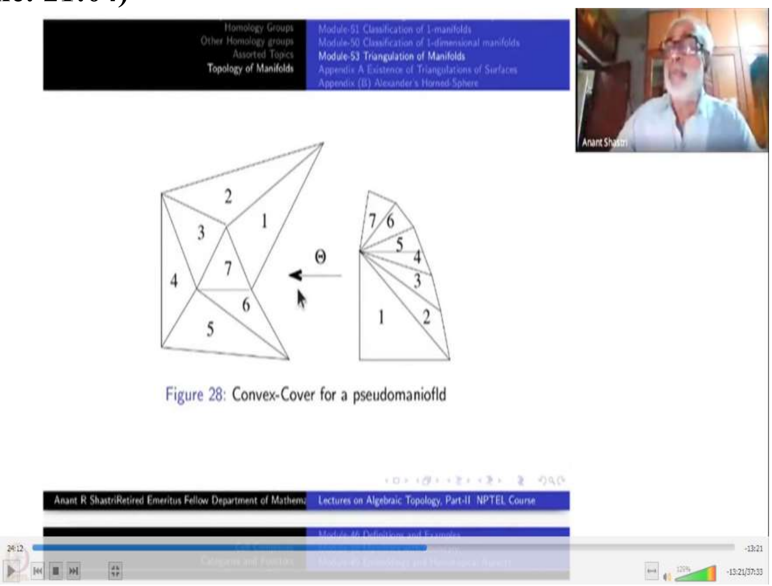

So, here is the picture of a worst scenario. Even in this case, condition (c) will be true. To begin with you have your K, us say, consisting of 7 of the triangles here in the picture. So, whatever labelled could be, it is quite arbitrary except that it satisfied a certain condition. So, for example, I have started from here and gone like this labeling them 1,2,,3,4,5,6, and 7.

So, accordingly I construct the $\mathcal{P}_i$'s here. At the 7 the stage, look at the triangle σ_7. It is sharing one facet here with σ_6 and then I have constructed correspondingly a convex polygon, this will be my point w and that will be mapped to this point and extended linearly. The construction is over now, why condition (c) is true? What does the condition (c) say? You take the entire interior of of this polygon, add any of the boundary facets here any one of them, on the union Θ_7 must be injective. (Once you prove injectivity, it will be automatically a homeomorphism.) That is what you have to show. So, suppose I have taken this edge which is already in the in $\mathcal{P}_6$. Since condition (c) is true for Θ_6, so injectivity follows.

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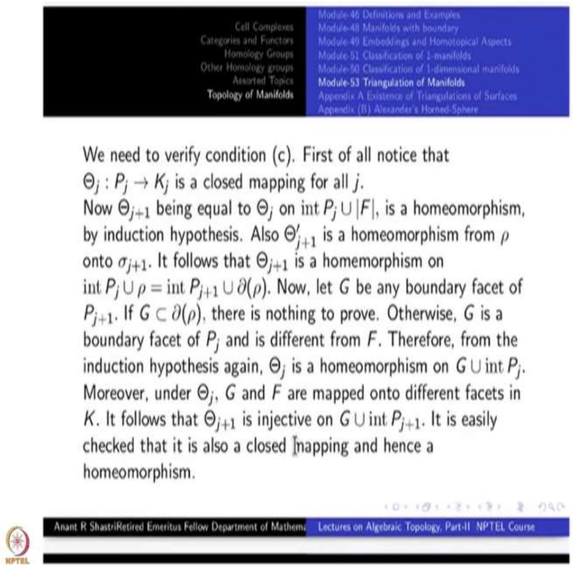

So, more generally, if the added boundary facet is already in $\mathcal{P}_j$, there is nothing to prove. But suppose the extra facet is is a facet of ρ. Then we use the fact that Θ_j is a bijection from the interior of $\mathcal{P}_j \cup F$ and Θ' is is a bijection onto σ_{j+1} and their image is precisely equal to F', to conclude that Θ_{j+1} is injective. Here we are appealing to a general topological fact, viz., if f is a continuous map on $A \cup B$ and f from A to C and f from B to D are homeomorphism and $f(A \cap B) = C \cap D$ then f from $A \cup B$ to $C \cup D$ is a homeomorphism.

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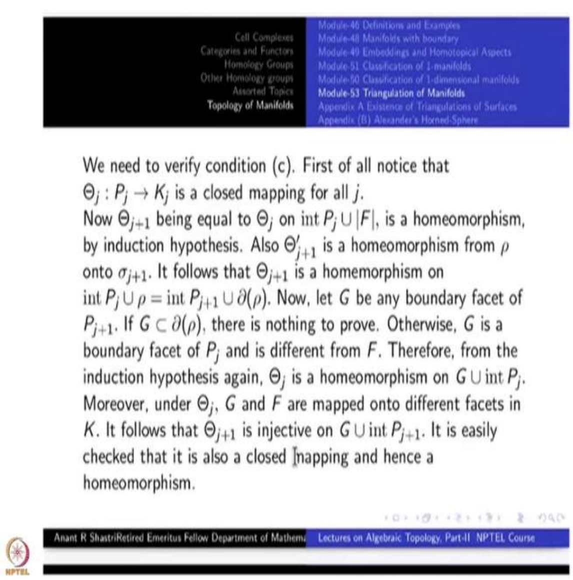

Finally, what do you do? Take $\mathcal{P}$ to be $\mathcal{P}_k$, where k is the number of n-simplexes in K, and define Θ from $\mathcal{P}$ to K by taking it to be Θ_j on $\mathcal{P}_j$ for each j. It is well defined because of condition (b). Clearly, Θ is a surjective closed mapping and hence is a quotient map. It follows that Θ is a homeomorphism in the interior of $\mathcal{P}$. If you take any one facets of $\mathcal{P}$, there also it is injective. But now I am going to say something better.

The image may not be a boundary facet of K. Why? because this facet may be covered by another facet also. It may happen that two distinct facets of $\mathcal{P}$ are mapped onto the same facet in K. Label them in pairs F_i and F_i'. (There will not be a third one mapped onto the same facet. Why? Because each facet in K is the subset of at most two n-simplexes of K.)

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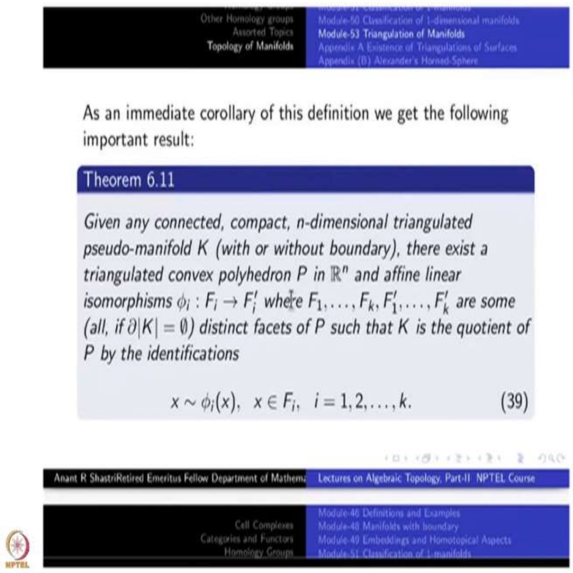

So, label them F_i and F_i'. Put $\psi_i = \Theta$ restricted to F_i and $\psi_i' = \Theta$ restricted to F_i' and $\phi_i = (\psi_i')^{-1} \circ \psi_i$ from F_i to F_i'. It follows that each ϕ_i is a linear isomorphism and we have $\Theta(x) = \Theta \circ \phi_i(x)$ for all $x \in F_i$. This completes the proof of the theorem.

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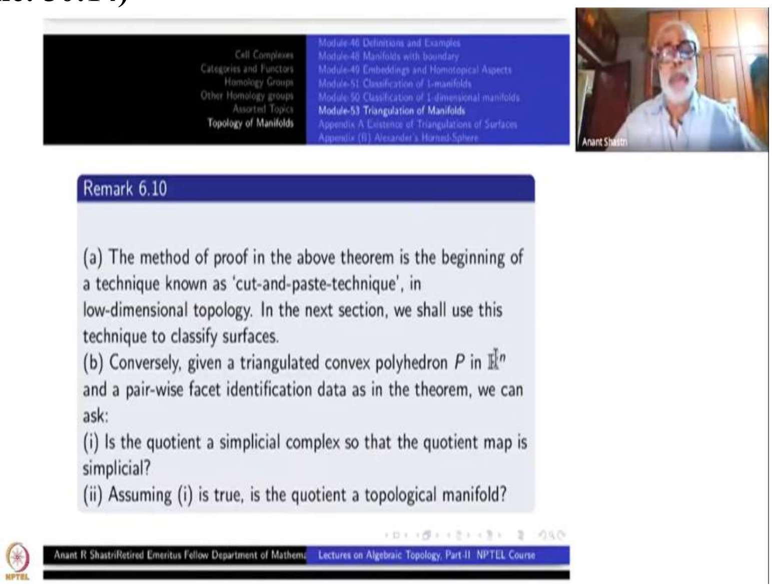

So, let me make a few remarks before closing up. If K' is the quotient of $\mathcal{P}$ obtained by only face identifications as in the theorem, then clearly K' is a pseudo-manifold and Θ factors to define a quotient map f from K' to K. In general, this f may not be injective only because there may be further relations inside the $(n-2)$-skeleton of $\mathcal{P}$ which are not consequences of the face relations.

The method of proof in the above theorem is the beginning of a technique known as cut and paste technique. You started with a triangulated pseudo-manifold, you perform a few cuts along its facets in any order you like, you then paste them back exactly wherever you have cut but in a different order. You get back the same space. You have the freedom to change the size of the pieces through affine isomorphism, because we are topologies you are not doing any geometry here. So, you can change the size of the simplex but wherever you have cut it from you have to place it in the corresponding thing there. This freed can be used in different ways in different contexts.

So, this is something which is somewhat strange in the sense that in topology you are not supposed to cut things. That will appear as if you are doing something discontinuous. Continuity has to be retained. Wherever you have cut that is only temporary you are pasting along the same spaces upto homeomorphism we are pasting the same parts that is why it is allowed. It is called cut and paste technique. Almost half a century of mathematicians have used this especially in dimension 3 very fruitfiully.

Dimension 3 he says become a big industry a lot of results are proven. Yet the final aim namely, of proving Poincare conjecture in dimension 3 was not achieved, from this technique. So that is why I am calling it a low dimensional topology, which usually means the study of 3 and 4 dimensions. In the next lecture, we shall use this technique to classify surfaces.

Now, we are going to discuss the converse. Given triangulated convex polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ and a pairwise facet identification data as in the above theorem, we ask a few questions:
(i) Is the quotient a simplicial complex so that the quotient map is simplical. This is the first question which is answered quite easily anyway.
(ii) Let us take the second question. Is the quotient a topological manifold?

Starting with any pseudo manifold K which may not be a manifold, the above theorem gives you the convex polyhedron and the quotient map as above. But we have not proved that the quotient is obtained by precisely the facet relations. That is why this is a non trivial question. That is why it is a good question. Because now you have to think of putting some extra conditions of your choice to obtain an affirmative answer.

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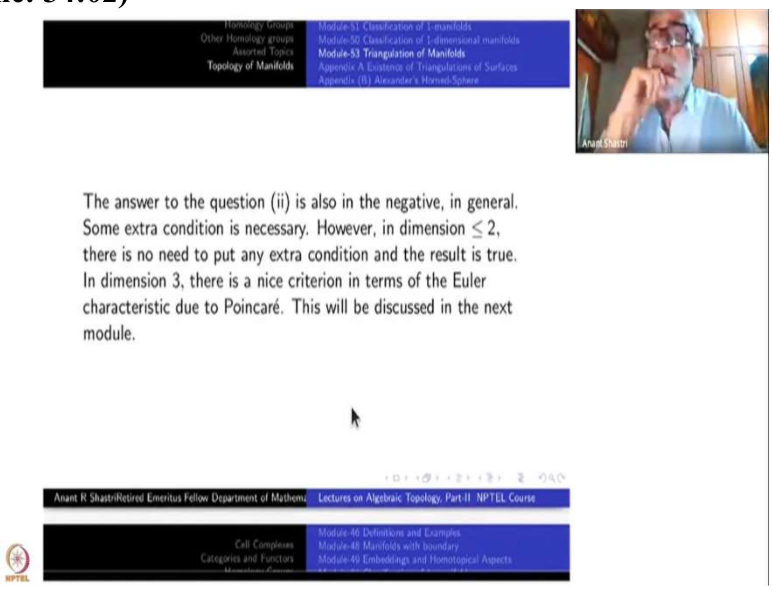

So, let me discuss it for a few minutes only. The answer to (i) is in the negative in general. As a counter example, you may take $n = 1$, then the quotient is either itself or a CW complex with one vertex and one 1-cell.

However, if we take the second barycentric subdivision of $\mathcal{P}$ which is already assumed to be triangulated, then the quotient map induces a simplicial structure on K such that the quotient map becomes simplicial. I will leave this one to you because I am not going to use it in this in this course anyway.

As the answer to second question is also negative in general. But in dimension 1, it is obviously true. In dimension 2 also, it is tried but requires some proof. So, in dimension 3 onwards, this is some some extra condition which is necessary and sufficient, in terms of the Euler characteristic.

And this condition is due to Poincare. It is a very interesting one. I guess perhaps, how Poincare tried to classify all compact three dimensional manifolds. So, next time we will discuss this result due to Poincare and another result which is not related to this question but just to fit the module, and this result is due to Munkres. Thank you.