Introduction to Algebraic Topology (Part II)
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Lecture - 53
Triangulation of Manifolds

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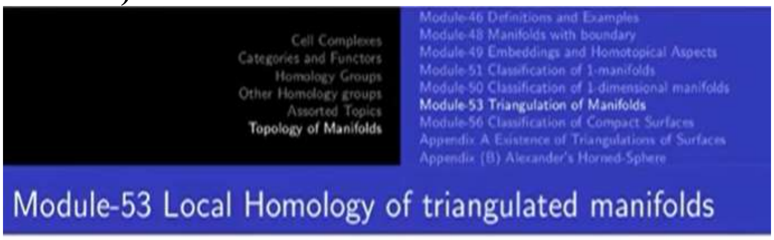

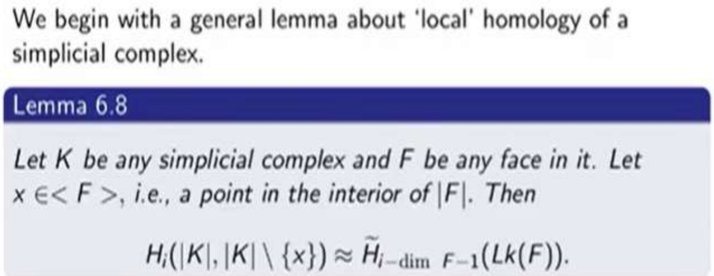

Today we are starting a new topic, triangulation of manifolds. Recall triangulation of any topological space just means that we have a simplicial complex K such that the geometric realisation of the simplicial complex K, denoted by $|K|$, is homeomorphic to the topological space given, okay? Then the homeomorphism will be called a triangulation of X. You want to say, if such a thing exists, then the space is called triangulable okay. So, we are going to study triangulated manifolds. So, in particular it will have the properties of a simplicial complex as well. So, let us begin with the study of local homology of triangulated manifolds. So, first the local homology of any simplicial complex or in general as far as possible, then we will go back to the triangulation of a manifold and then strengthen those results.

So, in general, what is happening to the local homology of any simplicial complex? Okay. So, it is the first lemma here. Let K be any simplicial complex and F be any face in it, let x belong to the interior of F, i.e., x is a point in the interior of $|F|$, this is geometric realization of F, interior of $|F|$ just means the set of all $\sum t_i v_i$, where v_i's are vertices of F, okay, all the t_i's are positive real numbers, with $\sum t_i = 1$, that is element in the interior. The lemma states that the following: H_i of the pair $(|K|, |K| \setminus x)$, obtained by throwing away the point, will be the same thing as $\widetilde{H_{i-\ell-k}}$, the reduced homology of the link of F okay? But the dimension is reduced, the dimension of the link is also equal to dimension of F - 1. So, this is the statement.

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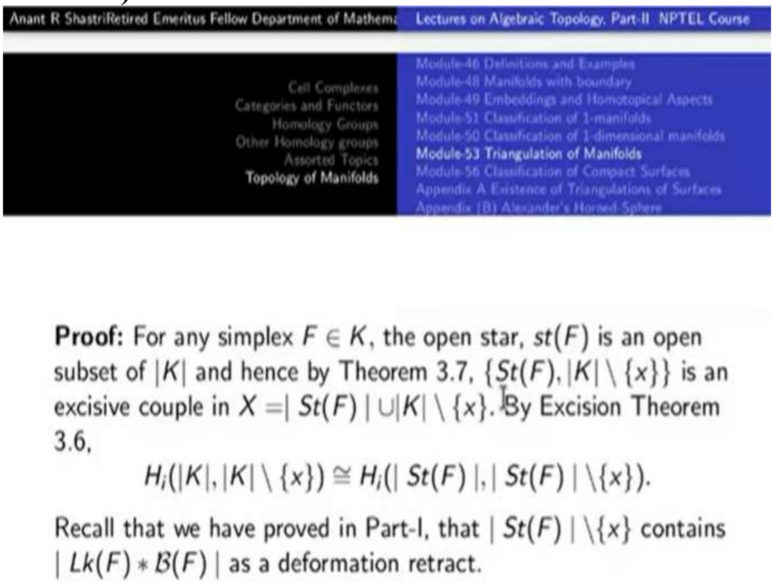

So, recall that for any simplex F in K, the open little star of F, stF, is an open subset of $|K|$. How was it defined? It is the union of interior of simplexes $F \cup G$ in K. First of all, $F \cup G$ must be a simplex in K, then you take F the interior of $F \cup G$ and take the union of all such sets, that is called the open star of F denoted by $st(F)$. The definition is similar to the case when F is vertex, a single point v, $st(v)$ we had defined and this is a generalization okay? Automatically it will be an open subset of $|K|$ itself okay? Therefore, we can apply this theorem 3., the excision theorem Okay? Whenever you have two open sets covering the entire thing, the two open sets will form an excisive couple, viz., the pair $\{St(F), |K| \setminus x\}$ is an excisive couple okay?

Note that here not both of are open. The closed $St(F)$ contains $st(F)$ which is open in $|K|$ and $|K| \setminus \{x\}$ is also open and they cover $|K|$. That is why this is a excisive couple okay, in $X = |K|$. By excision theorem 3.6. $H_i(|K|, |K| \setminus \{x\})$ is the same as $H_i(St(F), St(F) \setminus \{x\})$. So, because, $H_i(X, A)$ is $H_i(B, A \cap B)$, if $\{A, B\}$ is an excisive couple for singular homology. This is the first step. The LHS above is called the homology of X localized at the point x, justified by the above isomorphism. From the whole space X you are able to come to considering a neighbourhood of the point x, $St(F)$ is neighbourhood of every point in interior of F. So, that is the first step. Now let us see further, why it is just the homology of the link Okay?

Recall that we have also proved that star of $F \setminus \{x\}$, where x is single point in the interior of F, will contain the mod of (the link of F join with boundary of F) as a deformation retract. From the point x, you can radially push the space out to the link of F star boundary of F. The star denotes the join of two subcomplexes as a subcomplex. So, it is similar to the definition of star of two topological spaces, wherein you have a disk and their point interior is removed. Then the boundary sphere is a deformation retract. This we have proved in part 1 okay? this is not a very difficult thing here I will assume that one. okay?

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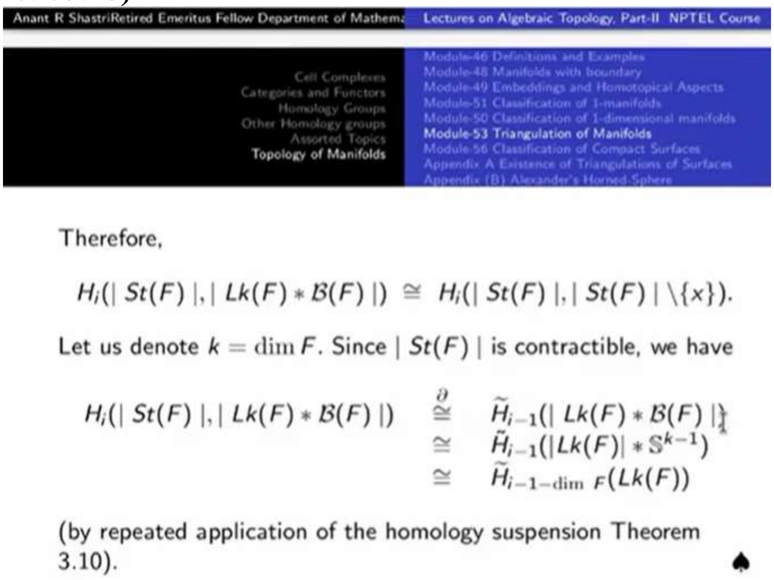

Therefore, in $H_i(|St(F)|, |Lk(F) * \partial F|)$, this is the deformation retract of $St(F) \setminus \{x\}$), therefore, I can replace this one by this one. The situation is again similar to the case of $(\mathbb{D}^n, \mathbb{S}^{n-1})$, whose homology is the same as the homology of the pair $(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\})$, but you cannot use that result but the fact that namely $St(F) \setminus \{x\}$ deforms to this subset, a more general result.

Let us denote by k the dimension of F okay? That means, there are $k+1$ vertices in F. okay? Since $St(F)$ is contractible okay, it is actually a convex subset inside $|K|$. So, this is contractible. What we get from the homology exact sequence of this pair? $\tilde{H}_i(StF)$ will be 0, for all i, and so the connecting homomorphism from H_i of the pair to $\tilde{H}_{i-1}(Lk(F) * \partial F)$ is an isomorphism. So, every third term is zero means the other two terms are isomorphic. That is what happens in the long exact sequence.

I think we used the notation delta for the connecting homomorphism. So you can use the same notation or any other, it does not matter. But note that boundary of a simplex is homeomorphic to a sphere, F is a k-simplex, so, its boundary is homeomorphic to a $(k-1)$-sphere okay? Since mod of the stars is the same as the star of the mods, the underlying topological space of $Lk(F) * \partial F$ is the same thing as the $|Lk(F)|$ starred with $\mathbb{S}^{k-1}$. Okay? But what is the join with $\mathbb{S}^{k-1}$? It is the same as taking iterated suspension k times. Join with $\mathbb{S}^0$ is the same as taking suspension, join with $\mathbb{S}^1$ is the suspension of the suspension and so on. So, you have to suspend k-times, okay?

By suspension isomorphism, you keep coming down k times, $\tilde{H}_{i-1}$ to $\tilde{H}_{i-1-\dim F}$, of the original space, $Lk(F)$. The suspension isomorphism states that in homology of the original space to the homology the suspension but the dimension increased by one. Here we have apply it k-times in the reverse direction. So, this establishes this isomorphism. This lemma is an elementary result which we will be using later on.

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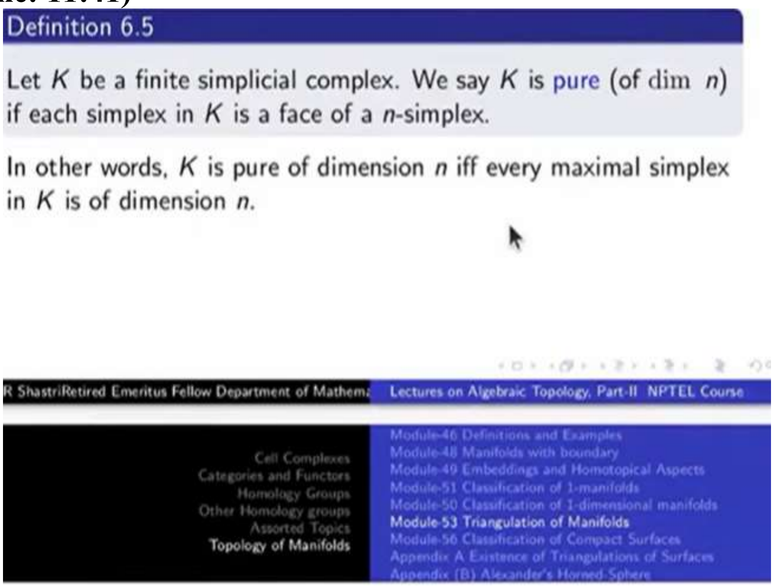

Now, let us make a definition. Take any simplicial complex K. We say it is pure of dimension n (we have done this in the case of CW complexes, this is similar to that), pure of dimension n if each simplex in K is a face of an n-simplex. In other words, all maximal simplexes are of the same dimension n, Okay? So, that is the meaning of pure. This is just some terminology. In other words, K is pure of dimension n and if and only if every maximal simplex in K is of dimension n. So, this could be taken as a definition also.

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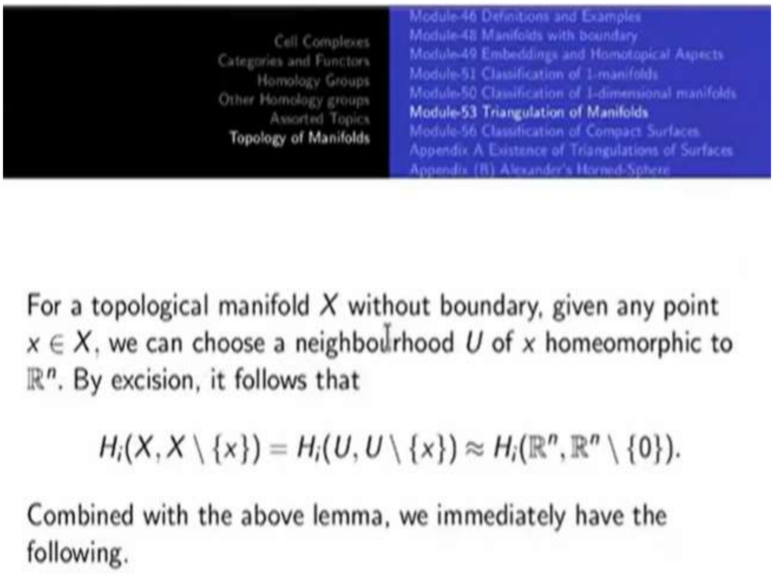

Let us now start with a manifold manifold with or without boundary. So, let us first take the case of without boundary, then given any point $x \in X$ you can choose a neighbourhood U of x homeomorphic to $\mathbb{R}^n$. If x is in the boundary then you would have taken homeomorphic to $\mathbb{H}^n$ the half space, right? So, if x an interior point anyway, then we can take the neighbourhood to be homeomorphic to $\mathbb{R}^n$ itself always, whether X is manifold with or without boundary.

By excision, it follows that $H_i(X, X \setminus \{x\})$ is isomorphic to $H_i(U, U \setminus \{x\})$. But $(U, U \setminus \{x\})$ is homeomorphic to the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ under a homeomorphism from U to $\mathbb{R}^n$ which takes x to 0, a chart at x. Now, combine this result with the previous lemma, we immediately get the following not so obvious result:

Let X be connected compact topological n-manifold without boundary. (I want to emphasise that ∂X is empty.) Let K be a simplicial complex such that $|K|$ is X that is K is a triangulation of X. Then the following statements hold:

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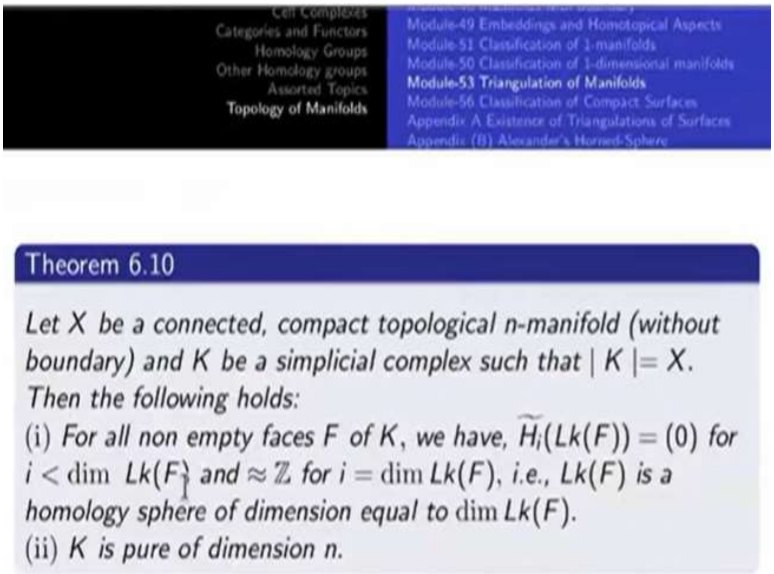

(i) For all non empty faces F of K, we have $\tilde{H}_i(Lk(F))$ is 0 for i less than dimension of the $Lk(F)$ and when i is equal to dimension of the $Lk(F)$, okay? Then it is isomorphic to $\mathbb{Z}$. That is $Lk(F)$ is a homology sphere. Homology sphere means what? The sphere has some homology namely, all the reduced homology, except the n-th homology is 0, right? If the reduced homology of a space is like this, then X is called a homological sphere of dimension n. This is true for the link of any simplex is the statement. We know that the boundary of a simplex F is actually a sphere. However, the link of F in a triangulated manifold $|K|$, may or may not be a sphere but a homology sphere. Also the $Lk(F)$ is of dimension $n-k-1$, where $k = dimF$.

Statement (ii) is that K is pure of dimension n, every simplex must be contained in a n-simplex. This is like invariance of domain and easily follows from invariance of domain. But we shall give a somewhat simpler proof.

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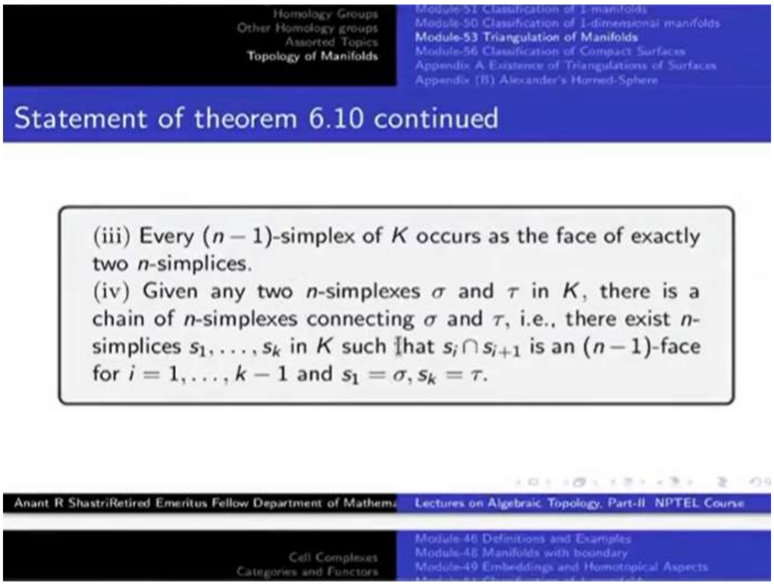

Third statement is somewhat non trivial. Every $(n-1)$-simplex of K occurs as a face of exactly two n-simplexes.

The fourth statement is that given any two n-simplexes σ and τ inside K, there is a chain of n -simplexes connecting σ and τ. What is the meaning of this? This is like a path. Remember in a connected simplicial complex, from one vertex to another vertex, there is a path of edges. Through edges only you can go from any vertex to any other vertex. Similarly, here through n-simplex. So, what is the meaning of this? There is a sequence of n-simplexes $s_1, s_2, \dots, s_k$, so that $s_1 = \sigma, s_k = \tau$ and $s_i \cap s_{i+1}$ is exactly an $(n-1)-$face of both s_i and s_{i+1} for all i. So, that is a similar and somewhat dual to the idea of an edge sequence.

So, statements (ii), (iii) and (iv) are seemingly nothing to do with the discussion of homology. We will go through them, one by one.

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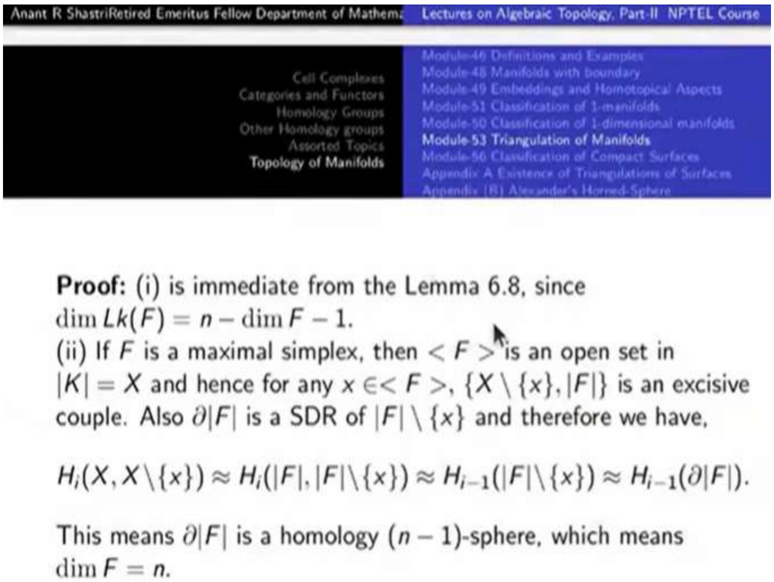

The first one is immediate from the lemma 6.8 by taking any interior point x of F. Since, I can replace $|K|$ by X, we have $H_i(X, X \setminus \{x\|)$ is the same $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ which is the same as $\tilde{H}_{i-1}(\mathbb{S}^{n-1})$. So, only for $i = n-1$, it is infinite cyclic, otherwise it is 0 right? So, that is what I am using here now, in the first part here. Everywhere else it is 0 when i equal to dimension of the link, it is $\mathbb{Z}$. On the other hand this is also isomorphic to $H_i(|K|, |K| \setminus \{x\})$ which, we have is is equal to $\tilde{H}_{i-1-\dim F}(Lk(F))$. The statement (i) follows.

Second statement here is that K is pure of dimension okay. So, how do we prove that? If F is any maximal simplex, then interior of F is an open set in $|K|$. okay? Hence for any x inside interior of F, $\{X \setminus \{x\}, |F|\}$ is an excisive couple. Also the boundary of any simplex is a strong deformation retract of $|F|$ minus the single point. This fact, we have been using several times, because $|F|$ is homeomorphic to a closed disk okay? Therefore, we have $H_i(X, X \setminus \{x\})$ is isomorphic to $H_i(|F|, |F| \setminus \{x\})$, and you can rewrite it in a different way viz., $\tilde{H}_{i-1}(|F| \setminus \{x\})$, using the exact homology sequence, because $|F|$ is contractible okay? Since $|F| \setminus \{x\}$ is homotopy equivalent to the boundary of F, I can replace this one by $\tilde{H}_{i-1}(\partial F)$. This means the boundary of F is a homology $(n-1)$-sphere. Therefore, F must be an n-simplex okay? I started with a maximal simplex then I have concluded that it is an n-simplex Therefore, K is pure dimension n. okay?

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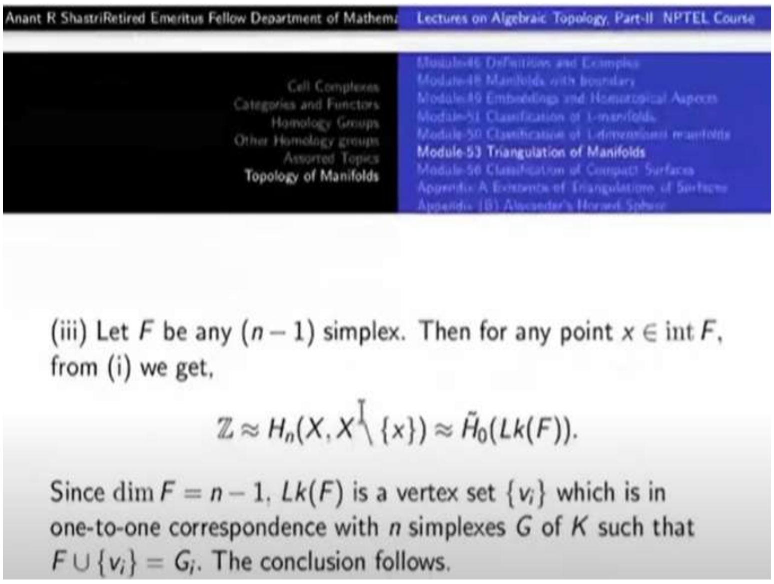

The third statement is every $(n-1)$-simplex in K occurs as the boundary of exactly two of the n-simplexes in K. Like in a triangulation of a circle, every vertex is incident exactly with two edges right? So, the same thing occurs in any triangulation of a manifold of dimension n, that is what we have to prove okay. So, take any $(n-1)$- simplex F in K. Then for any point x in the interior of F, what happens? From (i), we get $\mathbb{Z}$ is isomorphic to $H_n(X, X\setminus\{x\})$ which is isomorphic to $\tilde{H}_{n-1-\dim F}(LkF) = \tilde{H}_0(Lk(F))$. $\tilde{H}_0$ is $\mathbb{Z}$ means that the space exactly two connected components. What is the link of F where F is a $(n-1)$-simplex? Since dimension of K is n, it is of dimension 0. Therefore, $Lk(F)$ is a discrete space consisting of same vertices of K. Since it has two connected components, $Lk(F) = \{u, v\}$ for two distinct vertices u, v in K. This just means that $F \cup \{u\}$ and $F \cup \{v\}$ are the two n-simplexes containing F.

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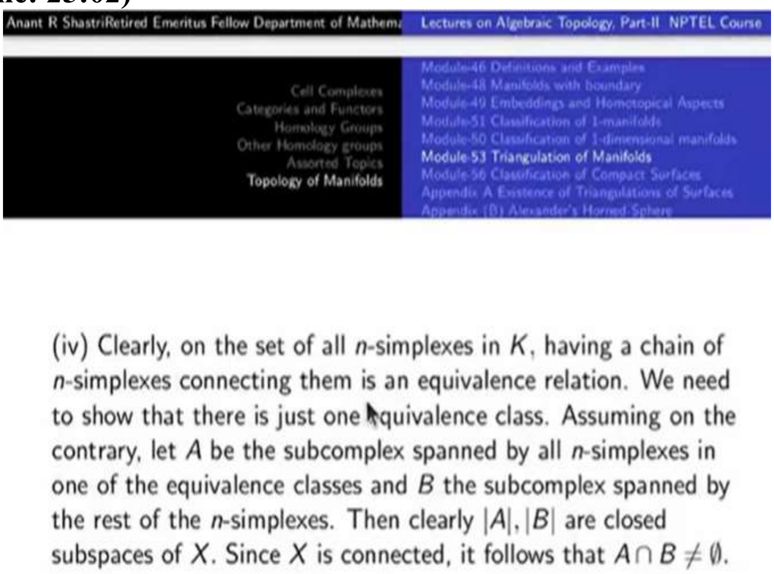

Let us prove the fourth statement now, okay? On the set of all n-simplexes having a chain from one to the other as above defines an equivalence relation, just like any two vertices can be connected by an edge path there if there is a path from here to here, there is a path from there to here, here to there, there to here A to B, B to C and there will be a path from A to C right. So, this is an equivalence relation. We need to show that there is just one equivalence class. Assuming on the contrary, let A be the subcomplex spanned by all n-simplexes in one of the equivalence classes. Start with one n-simplex then look at all the n-simplexes which can be joined to this the simplex through a sequence of simplexes okay? You are assuming that it is not the whole space then what happens is a question right? We need to show that there is just one equivalence class. Assuming on the contrary, let A be the complex spanned by all n-simplexes in one of the equivalence classes. Let B be the sub complex spanned by the rest of the n-simplexes. So, you have two sub-complexes which cover the entire K, okay? Then, $|A|$ and $|B|$ are closed subspaces of $X = |K|$, because they are sub complexes. X is connected and therefore, $|A| \cap |B|$ and hence $A \cap B$ must be non empty, alright? A and B are subcomplexes, so $A \cap B$ itself is a sub complex which is non empty okay? So far we are fine.

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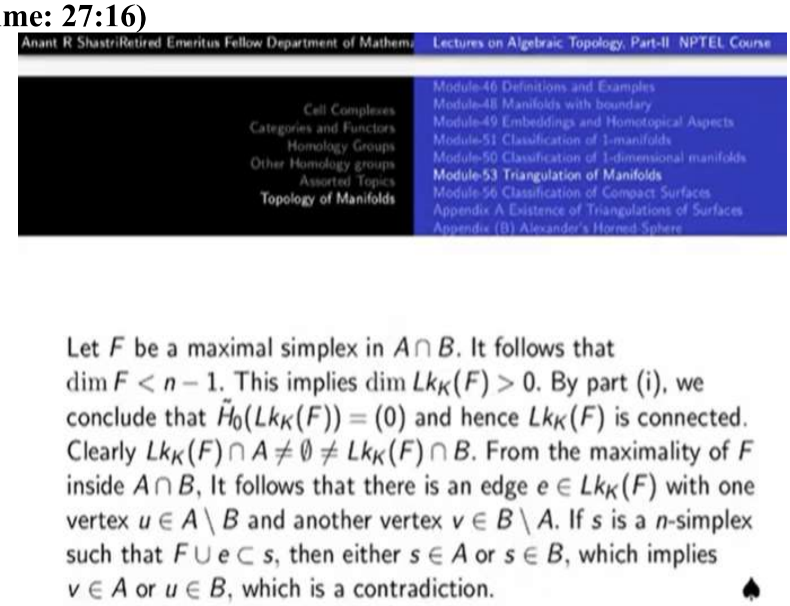

Now, let F be a maximal simplex of $A \cap B$. This may be a vertex or an edge and so on, you do not know yet okay? Take a maximum simplex in the intersection. It follows that dimension of F cannot be n. It must be smaller than n. Why. Because, if it is an n-simplex that will belong to both A and B, right? Because A and B are disjoint union of equivalence classes of n-simplexes.

If F is an $(n-1)$- simplex that means, again since it belongs to both A and B, there must be a n-simplex on this side and another n-simplex on that side of F. That means, the class of A goes beyond A, which is a contradiction. Therefore we are forced to conclude that dimension of F is smaller that $n-1$, okay? Alright. This implies that the link of F in K must be higher dimension than 0. If it is $n-1$ dimension, the link will be 0 dimension and if it is n dimension link will be empty. If it is lower than $(n-1)$, then only link will be of positive dimension because the formula is dimension of F + dimension of link of F +1 $= n$. So, the dimension of the link of F is positive which means that it is a simplicial complex of dimension greater than or equal to 1. But the part (i) says that $\tilde{H}_0(Lk(F))$ must be 0, hence link of F is connected. Clearly link of F cannot be completely inside A nor inside B. Okay? Link of F intersection A non empty, link of F intersection B is also non empty. From the maximality of F, which is very important now, inside $A \cap B$ okay, (I am not taking maximal inside inside the whole of K), in $A \cap B$, it follows that there is an edge e inside F with one vertex u inside A and not in B and another vertex v inside B and not in A. Probably, here you should make a picture of a 2-dimensional simplicial complex (that is the best for any picture) by using an edge path inside $Lk(F)$ running from one vertex in $Lk(F) \cap A$ to another one vertex in $Lk(F) \cap B$.

So, now let s be an n-simplex such that $F \cup e$ is contained inside s, okay? Why does such s exists? Because K is a pure of dimension n, since e is in the $Lk(F)$ and $F \cup e$ is a simplex of K. So, it is contained inside some n-simplex. Then either this s is in A or in B, because all the n-simplexes of K have been divided into these two those set an equivalence class A and other is the union of all other equivalence classes. So, s is either in A or in B okay. So, that is a contradiction now, because it will imply that both u, v are inside A or inside B. but I have chosen u and v such that one is not in B and other one is not in A. That is important here okay?

So, this is a very neat proof I myself enjoyed this one. So, it is simple minded thing like this that you should be able to do soon on your own. Property (iv) of a triangulated manifold itself is made into an axiom. Simplicial complexes which satisfy such an axiom are called pseudo manifolds. I think we will stop here today. So, next time we will study what is a pseudo manifold based on this result Okay? thank you.