Introduction to Algebraic Topology (Part – II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture - 52
Classification of 1 – Manifolds continued

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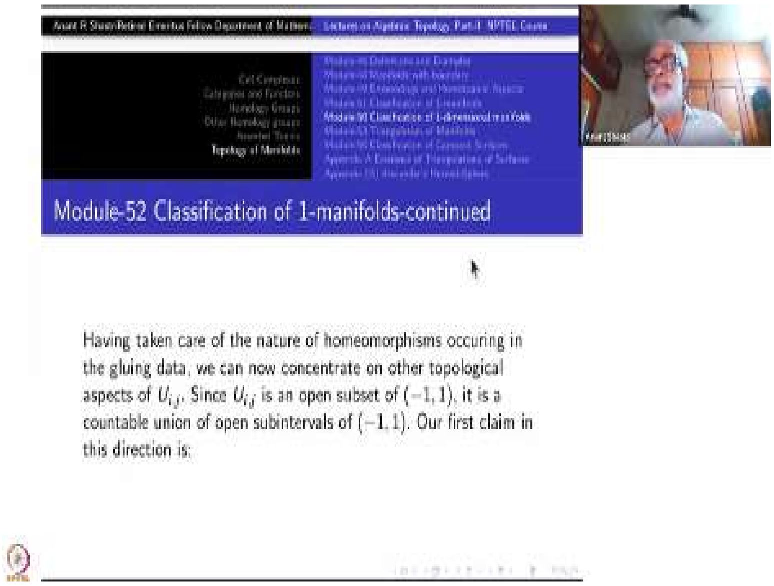

Continuing with the classification of 1-dimensional manifolds, recall that we have represented every manifold as the quotient of a disjoint union of open disks, all of them of 1 dimension, and in order to study the topology of the manifolds, we need to understand the equivalence relation defining these quotient maps which are expressed by what are called the transition functions. And we have done a beginning of this study, with two motivating examples and two lemmas, which actually, take care of a big chunk of what is happening there okay?, namely, the nature of homeomorphisms occurring in the gluing data okay?

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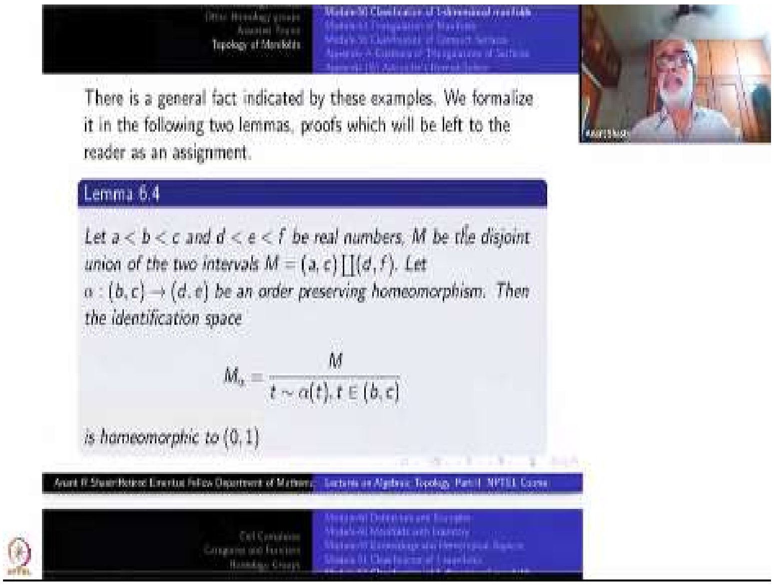

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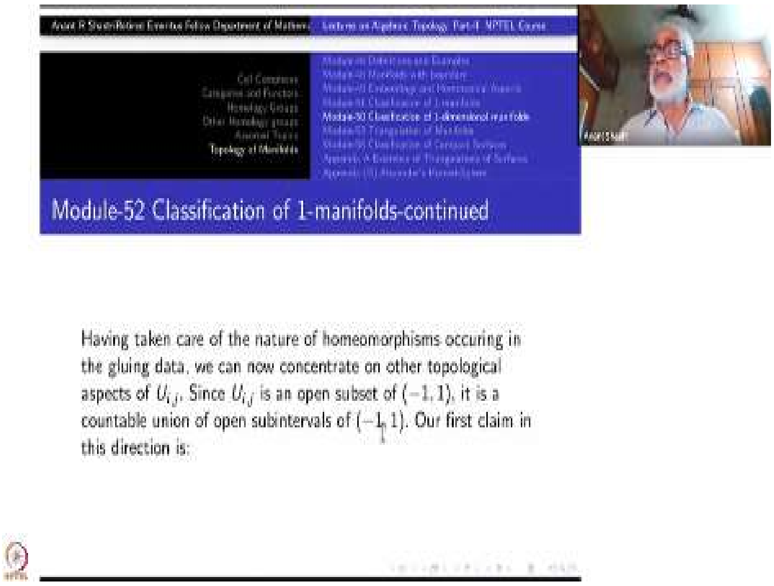

We can now concentrate on other aspects of these transition functions, $\psi_i^{-1} \circ \psi_j$ from $U_{j,i}$ to $U_{i,j}$ where these $U_{i,j}$ are open subsets of different copies of $\mathbb{D}^1$. The first thing you would like to know is how many components $U_{i,j}$ has. Since $U_{i,j}$ is an open subset of $(-1, 1)$, it is a countable union of open sub intervals of $(-1, 1)$ and each component is homeomorphic to $(-1, 1)$. (This is one of the biggest advantage of being in 1-dimension. Such a neat description is not possible in higher dimensions.) Our next claim in this direction is the following:

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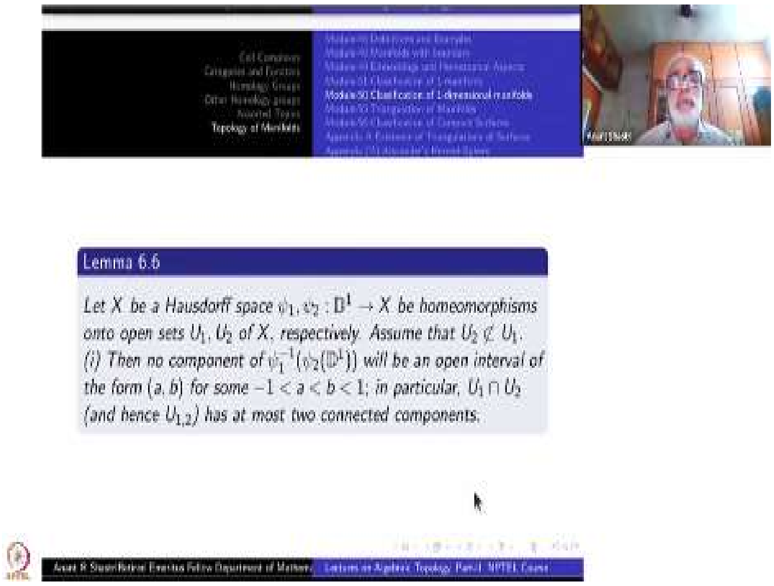

Let X be a Hausdorff space and ψ_1, ψ_2 from $\mathbb{D}^1$ to X be any homeomorphisms onto open subsets U_1 and U_2 of X respectively. Assume that U_2 is not a subset of U_1. The first claim is that:

(i) There is no component of $\psi_1^{-1}(\psi_2(\mathbb{D}^1)) =: U_{1,2}$ which is an open interval of the form (a, b) for some $-1 < a < b < 1$, i.e., strictly inside the interval $(-1, 1)$. In particular, this will imply that $U_1 \cap U_2$ okay, can have at most 2 components. This is the statement of course, we will see the proof soon.

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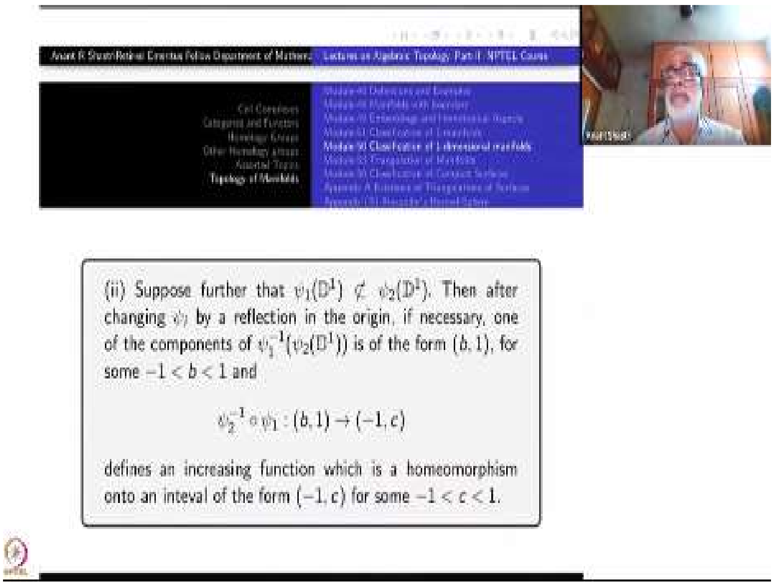

The second part of the statement of this lemma is that: Assume further that U_1 is not contained in U_2. One more condition now.

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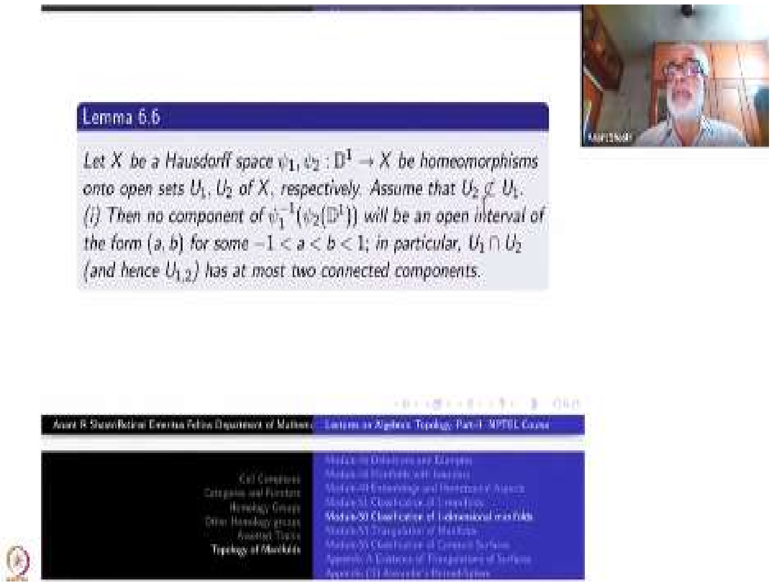

Here we are assumed that U_2 is not contained in U_1, but now we want the other way inclusion also not valid, so assume that also.

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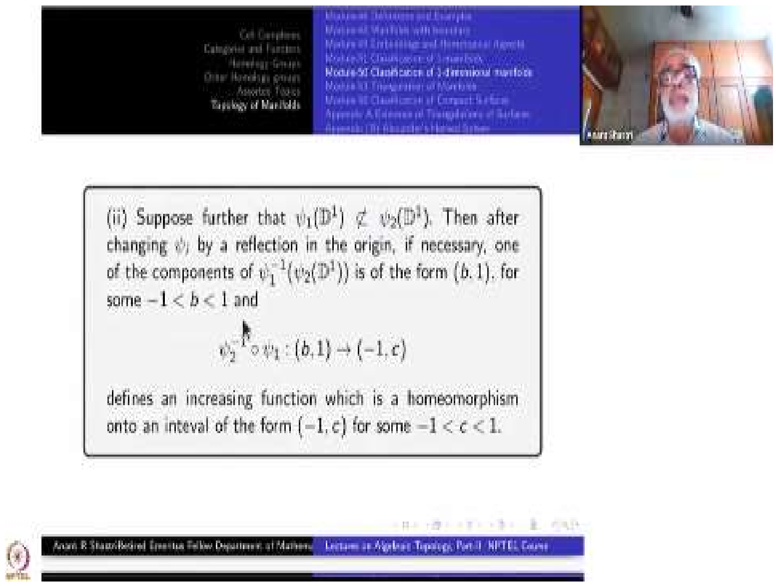

Then after changing ψ_i by a reflection about 0 in the domain of ψ_1, if necessary, one of the components of this $U_{1,2}$ is of the form $(b, 1)$ for some b strictly between -1 and 1 and $\psi_2^{-1} \circ \psi_1$ from $(b, 1)$ to $(-1, c)$ is order preserving homeomorphism onto the interval $(-1, c)$, where c is in the open interval $(-1, 1)$, Okay? So, this is the second part of the lemma.

So, this lemma is a little more elaborate than our earlier lemma. Together they more or less complete the local analysis of what is happening inside each $\mathbb{D}^1$. So, let us go through the proof of this.

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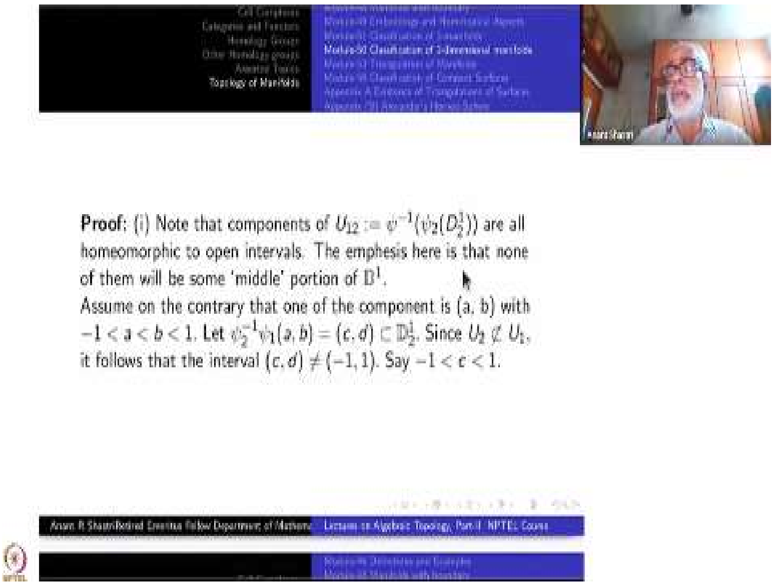

The first thing is to note that components of $U_{1,2}$ are all homeomorphic to open intervals. I am repeating this. That is all. The emphasis here is that none of them will be some middle portion of $\mathbb{D}^1$. It will not be equal to (a, b) where $-1 < a < b < 1$. So, this is the claim here. Now, suppose that is not true, namely, there is one component of $U_{1,2}$ of the form (a, b) as above.

Now consider the restriction of the homeomorphism, $\psi_2^{-1} \circ \psi_1$ to (a, b). Suppose its image is (c, d), which is a subset $\mathbb{D}^1$, Okay? Since U_2, which is the image of ψ_2, is not contained in U_1, okay? It follows that (c, d) is not the whole of $(-1, 1)$. Therefore, we must have $c \neq -1$ or $d \neq 1$. Let us say that $-1 < c < 1$. Okay? The other case will be similar. So, we are claiming that this leads to a contradiction.

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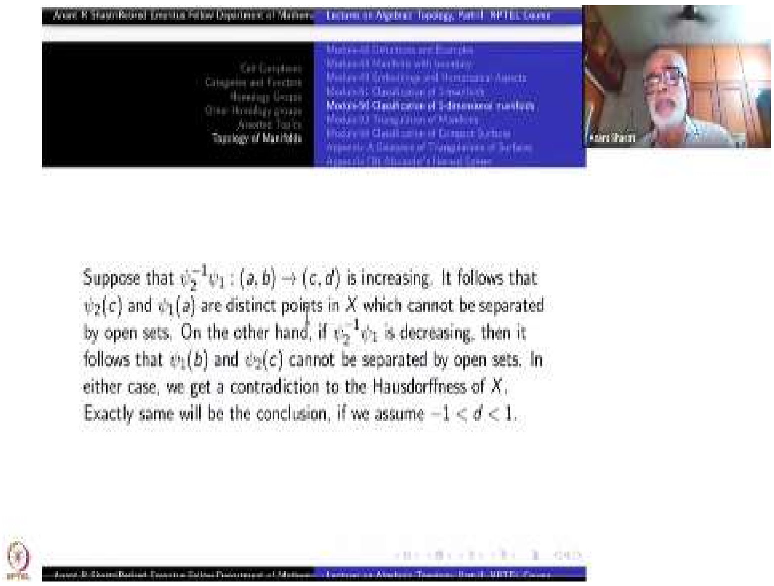

Suppose $\psi_2^{-1} \circ \psi_1$ from (a, b) to (c, d) is increasing. It follows that $\psi_2(c)$ and $\psi_1(a)$ (remember ψ_1 and ψ_2 are defined on the whole of $\mathbb{D}^1$, and so $\psi_2(c)$ and $\psi_1(a)$ make sense) are distinct points of X. They are not identified with each other. Also, they cannot be separated by open sets, because every neighbourhood of $\psi_2(c)$ will contain $\psi_2(c, c+\epsilon_2)$ which will intersect every neighbourhood of $\psi_1(a)$ which will contain $\psi_1(a, a+\epsilon_1)$ for some positive ϵ_i that will contradict the Hausdorffness of X. You have used this argument in the earlier lemma. So, this this is what happens here again.

On the other hand, suppose now, this homeomorphism is decreasing. Means what? Points near a are mapped onto points near d and points near b are mapped to points near c. Then it follows that $\psi_1(b)$ and $\psi_2(c)$ will be very very nearer, but the two are distinct points and they cannot be separated by open sets in X.

In either case, we get a contradiction to the Hausdorffness of X. Okay? So exactly similar things would happen with $\psi_2(d)$, if we assume $d < 1$. So, we have full contradiction, contradiction to what, contradiction to the fact that one of the component is of the type (a, b) a middle portion.

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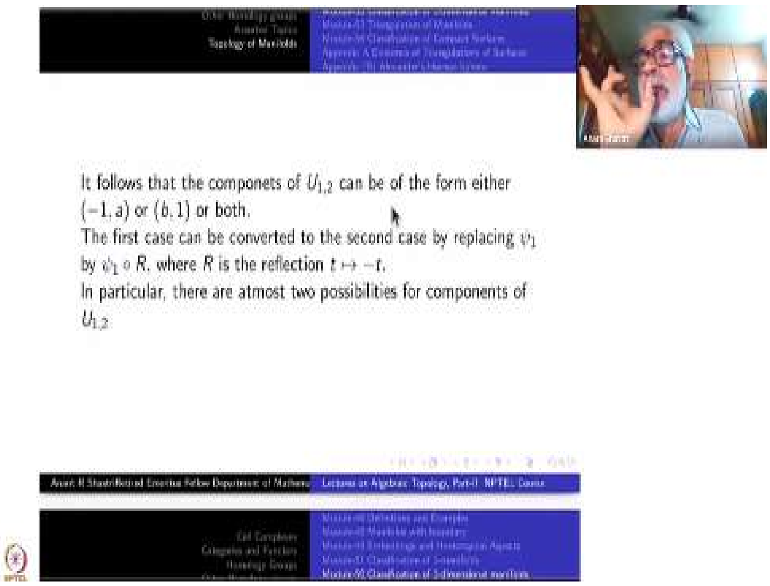

What is the meaning of this? That means, components of $U_{1,2}$ must be of the form $(-1, a)$ or $(b, -1)$ or where a, b are in $(-1, 1)$. So, there are only two possibilities. In particular, $U_{1,2}$ may have just one component which could be either of the two type, or $U_{1,2}$ may have two components in which case on one of them is of the form $(-1, a)$ and the other $(b, 1)$. In particular, $U_{1,2}$ has at most two components. This completes the proof of (i). That gives you a better picture of what map happen.

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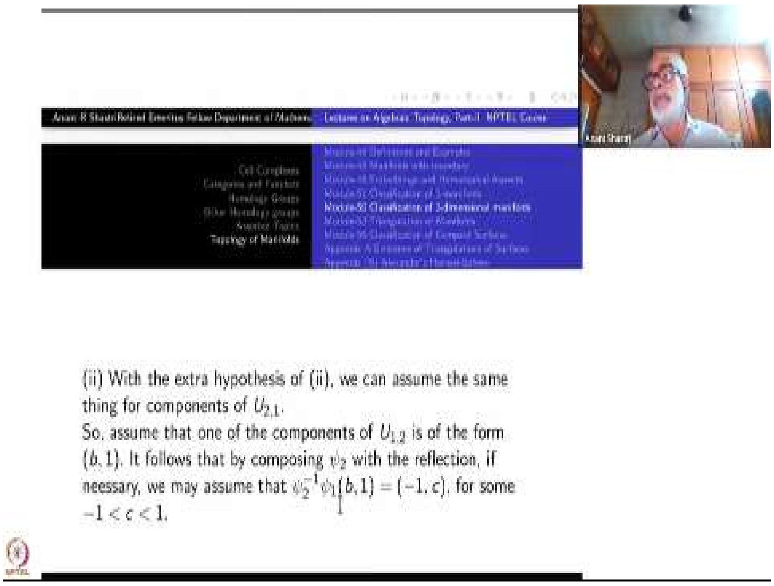

(ii) Now, under the additional hypothesis namely U_1 is not contained inside U_2 okay, it follows that there are at most two components of $U_{2,1}$ and they are of the form $(-1, c)$ or $(d, 1)$ or both. Before going further, note that, $U_{1,2}$ and $U_{2,1}$ are both homeomorphic to $U_1 \cap U_2$. Hence the two have same number of components.

Now, for the sake of definiteness, assume that one of the components of $U_{1,2}$ (it may the only component) is of the form $(b, 1)$, by replacing, if necessary, ψ_1 with $\psi_1 \circ R$, where R is the reflection in 0. Now $\psi_2^{-1}(psi_1(b, 1))$ is some component of $U_{2,1}$, we do not know what is its form, there are two possibilities. By replaces ψ_2, if necessary, with $\psi_2 circR$, we shall assume that it is of the form $(-1, c)$.

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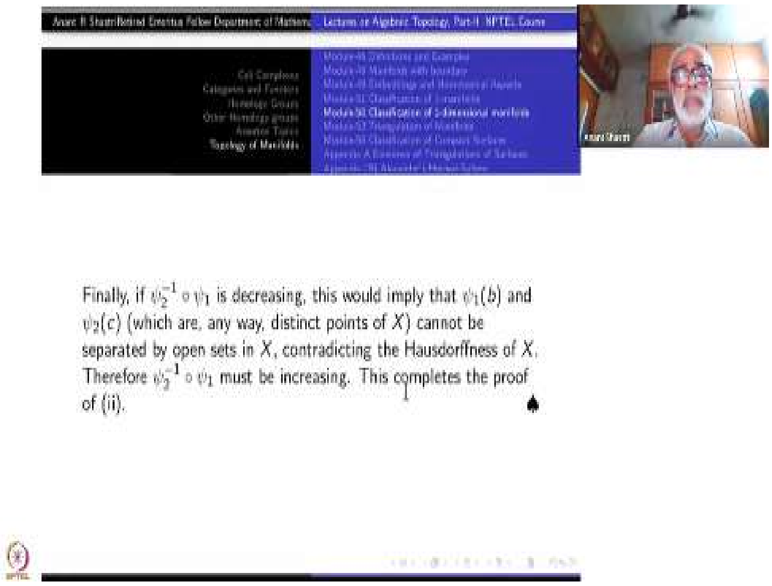

Now comes the question whether $\psi_2^{-1} \circ \psi_1$ from $(b, 1)$ to $(-1, c)$ is increasing or decreasing. Suppose this is decreasing. Decreasing means what? Points near b go near c and points near 1 go near to -1. Points 1 and -1 cause no problem. But look at $\psi_1(b)$ and $\psi_2(c)$, there can be a problem right? So, these two will be distinct inside X, but cannot be separated by open sets. Argument is similar to what we have before. Therefore, $\psi_2^{-1} \circ \psi_1$ must be increasing okay? from $(b, 1)$ to $(-1, c)$. This is the what we wanted to happen, so, that completes the proof of this lemma okay?

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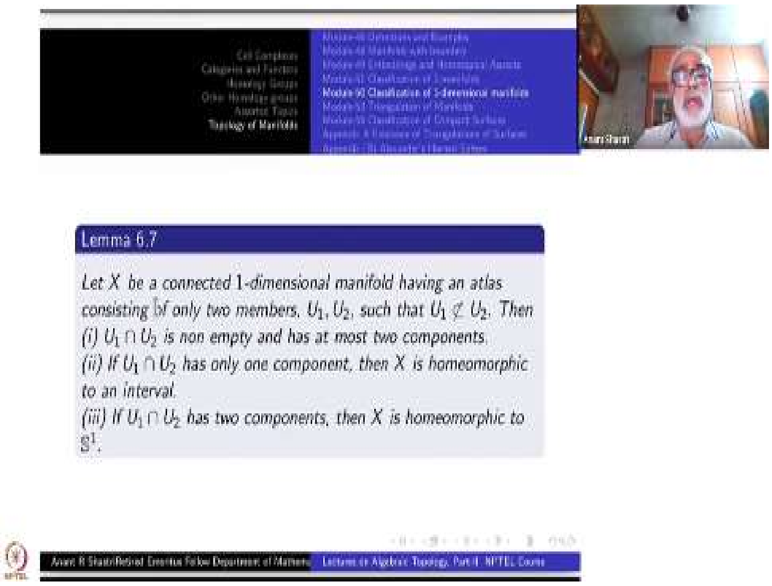

So, now we can consolidate the whatever observation we have done so far, in the form of another lemma. Let X be a connected 1-manifold. (So far connectivity of X was not necessary, X is automatically Hausdorff okay? Now, I want to make a clear picture.) Now, let us take a connected manifold X, having an atlas consisting of only two members. Then I want to write down a complete description of X upto a homeomorphism. So, only two members $(U_i, \psi_i), i = 1, 2$. Assume that U_1, U_2 are both proper subset of X (strictly speaking this condition is not necessary.)

(i) Then their intersection $U_1 \cap U_2$ is non empty and has at most 2 components. (This much is what we have seen already okay?)

(ii) If you $U_1 \cap U_2$ has only one component, then X is homeomorphic to an interval. Of course an open interval because we are assuming all the time that X is without boundary.) So, this is part of our first lemma which we did yesterday.

(ii) $U_1 \cap U_2$ has two components then X is homeomorphic to $\mathbb{S}^1$.

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This is the second lemma that we had okay? So, this is nothing new here. But let us go through this one again carefully. So, we are claiming that $U_1 \cap U_2$ non empty, first of all. Why? Because if there are only two open sets covering X, if they are disjoint would mean X is disconnected okay? So it must be non empty. That part is slightly new. And then it can have only one component or two components okay? So, this part (ii) is exactly same as lemma 6.4 and 6.5.

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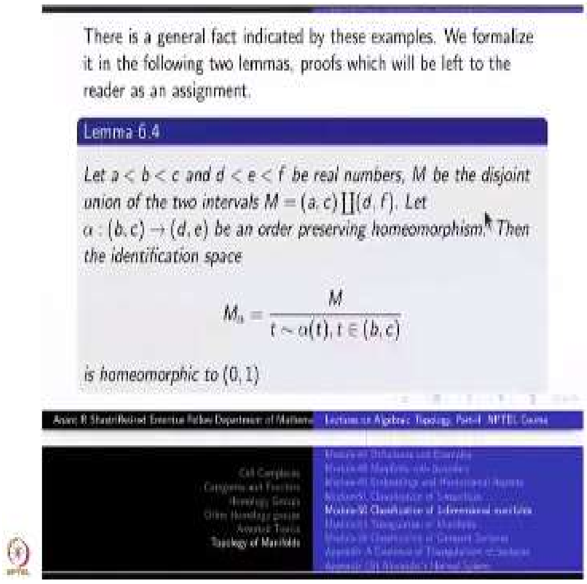

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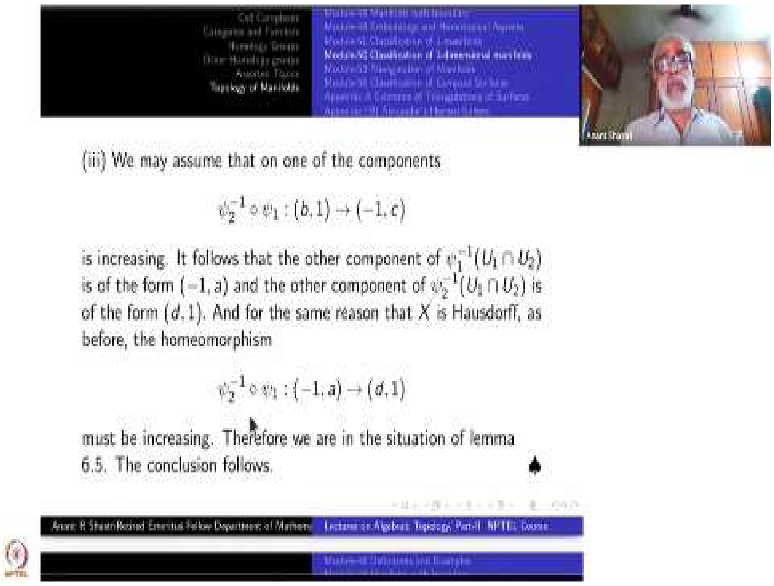

So, the third part. Here you may assume on one of the components, okay? $\psi_2^{-1} \circ \psi_1$ from $(b, 1)$ to $(-1, c)$ is increasing. This much is seen in the previous lemma, actually, I do not have to repeat it. It follows that on the other component $\psi_2^{-1} \circ \psi_1$ is from $(-1, a)$ to $(d, 1)$. Applying the same reason to its inverse, it follows that this is also increasing. Therefore, we are in a situation of lemma 6.5. Okay, the conclusion follows.

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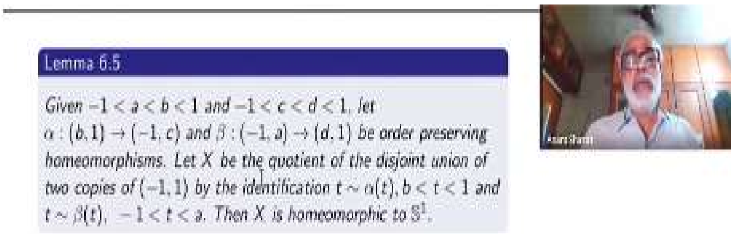

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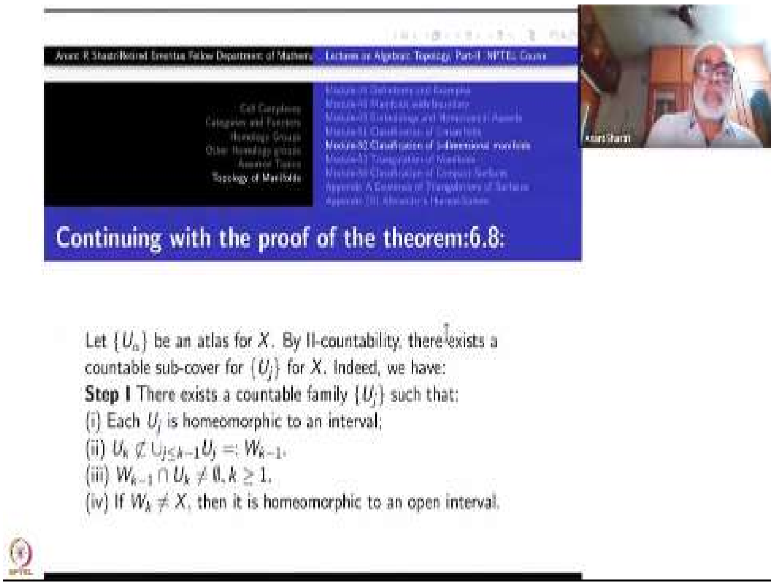

So, now, we can continue with the proof of the theorem. Earlier we had labeled the open sets $U_1, U_2, U_3, \ldots$ such that that U_i is not contained in the union of the previous ones. U_{i+1} and so on. Actually all that was not necessary, because, we are now going to prove something stronger than that which is necessary. That is the I step.

By second countability, we have a countable open cover $\{U_j\}$ of X, each U_j is homeomorphic to an interval. We want to arrange them in the following way. What is this?

(i) There exists countable family $\{U_j\}$ such that each U_j homeomorphic to an interval this part is Okay?

(ii) Second one is that the k^{th} member is not contained in the union earlier $k-1$ members, which I am denoting by W_{k-1}. This is the definition of W_{k-1}.

(iii) W_{k-1} intersects U_k for $k > 1$.
(iv) Finally the fourth condition is: if W_k is not the whole of X, then it is homeomorphic to an interval.

So, this is what I am going to claim okay. So, this is my first step okay. Let us see how we prove this one.

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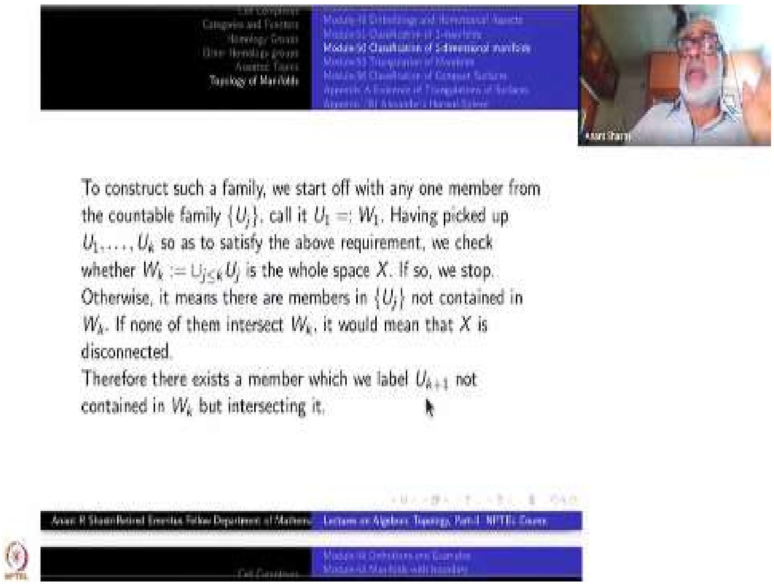

To construct this, we start off with any one member from the family $\{U_j\}$, call it $U_1 = W_1$. Having picked up $U_1, U_2, \dots, U_k$ so as to satisfy our requirements (i), (ii), (iii) and (iv), inductively, okay, then I want to pick up U_{k+1}, we check first of all whether W_k which is the union of U_j's up to k, is the whole space X or not. If it is the whole space there is nothing more to be done. Even if, there are other members in the family, they are all contained in W_k. So, we stop. Otherwise, it means that there are members U_j not contained in W_k. Okay, if none of them intersect W_k, then what happens? X will be the disjoint union of two open sets viz., W_k and the rest of the U_j's. That would mean that X is disconnected. So that cannot happen. So, at least one member which we have not taken yet must be intersecting W_k and contained in W_k. Okay?

Label that one as U_{k+1} Okay? This U_{k+1} is not contained in W_k, but it intersects W_k. Okay? So, (i), (ii) and (iii) are satisfied upto $k+1$.

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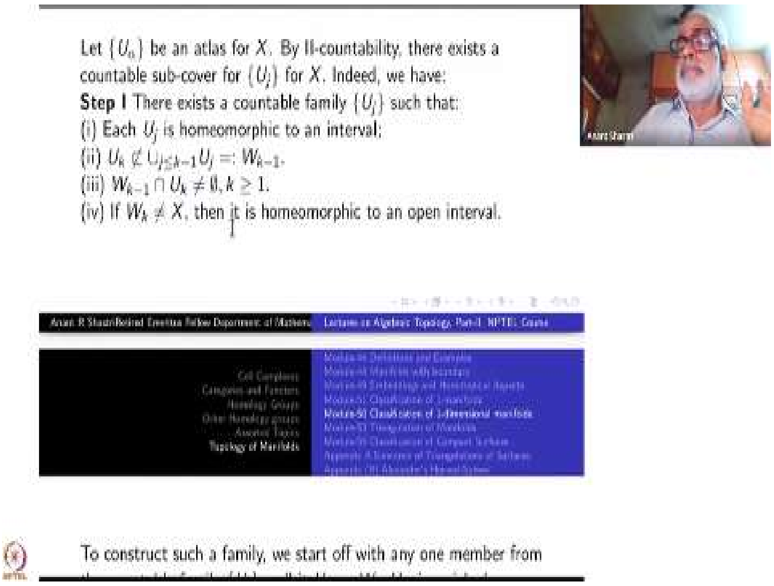

By induction hypothesis, W_k is an interval. I want to show that W_{k+1} is also an interval okay? Provided it is not the whole of X. okay? So, this is what I have yet to prove.

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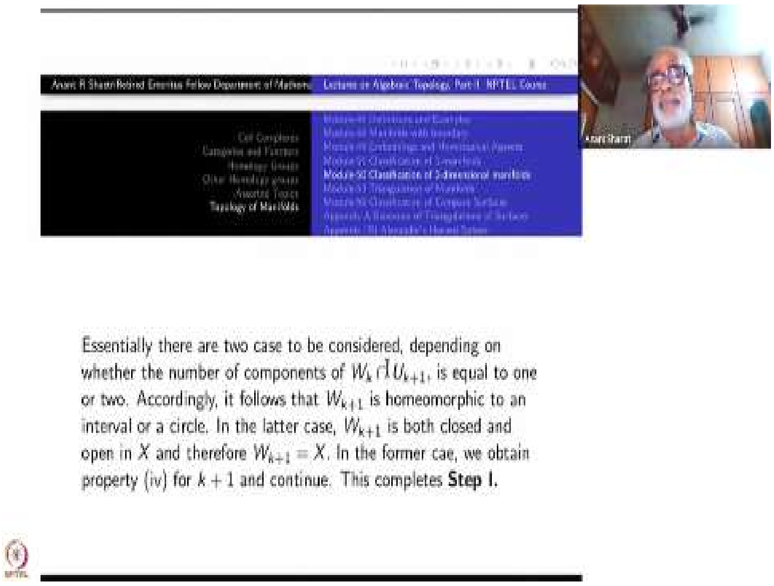

Essentially, there are two cases to be considered depending on whether the number of components of $W_k \cap U_{k+1}$ is one or two. This follows from a previous lemma. These are the only two possibilities. Okay? Again by the previous lemma, it follows that if there is only one component then W_{k+1} is homeomorphic to an interval, being the union of two intervals patched up along a common subinterval. And if there are two members, it will be homeomorphic to a circle, full circle right? If it is a full circle, what happens? W_{k+1} will be both open as well as closed in X. But X is connected. So, W_{k+1} is the whole space. If I assume that W_{k+1} is not equal to the whole space X, then it cannot be a circle and hence$k+1$

must be an interval. So therefore inductively all these things have been satisfied for now. That means the construction of step 1 is over.

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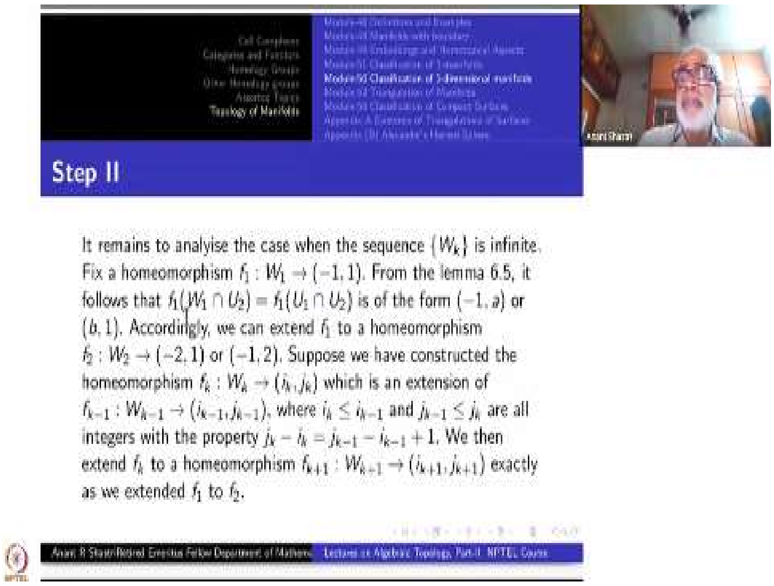

So let us finally complete step 2. Step 2 is to analyse what happens when we keep on running this one infinitely. If it stops at any finite level, k then W_k is equal to X and there are two case, either it is an open interval or a circle so the problem is over. If it keeps on going for infinitely terms, then what happens? That is the case right? So, to analyse this case, we fix a homeomorphism f_1 from W_1 to $(-1,1)$. Okay. Pick some homeomorphism because W_1 is nothing but U_1. From the lemma 6.5, it follows that $f_1(W_1 \cap U_2)$ is nothing but $f_1(U_1 \cap U_2)$ and hence has at most two components each of the form $(-1,a)$ or $(b,1)$ right? Accordingly, we can extend f_1 to a homeomorphism f_2 from W_2 to a larger interval than $(-1,1)$ either on the left side, say onto $(-2,1)$, or on the right side, say $(-1,2)$, which on you do not know but only one of them. The same will happen at each i. So, both ways you have to keep extending f_i to f_{i+1} from W_{i+1} to some larger and intervals.

Suppose you have constructed the homeomorphism f_k from W_k to an open interval, (i_k, j_k) where i_k, j_k are some integers. (Remember each W_k is a proper subspace of X. It may not be a subspace of $\mathbb{R}$, we are constructing homeomorphisms of W_i's into open intervals inside $\mathbb{R}$, Okay?)

U_{k+1} maybe coming and intersecting the left side of the interval or the right side of the interval. Accordingly you have to extend f_k to f_{k+1}. Thus what we get, inductively is that i_k will be smaller than or equal to i_{k-1}. Similarly, j_{k-1} will be smaller that or equal to j_k.

Beginning with $i_1 = -1$, and $j_1 = 1$, all i_k will be negative and all j_k will be positive. So, they are all integers with one additional property that the difference between $j_k - j_{k-1}$ and $i_k - i_{k-1}$ is 1 okay. Therefore, okay once you have extended this way from f_k to f_{k+1} and W_{k+1} to (i_{k+1}, j_{k+1}) we keep extending like this just the way we have extended f_1 to f_2 Okay.

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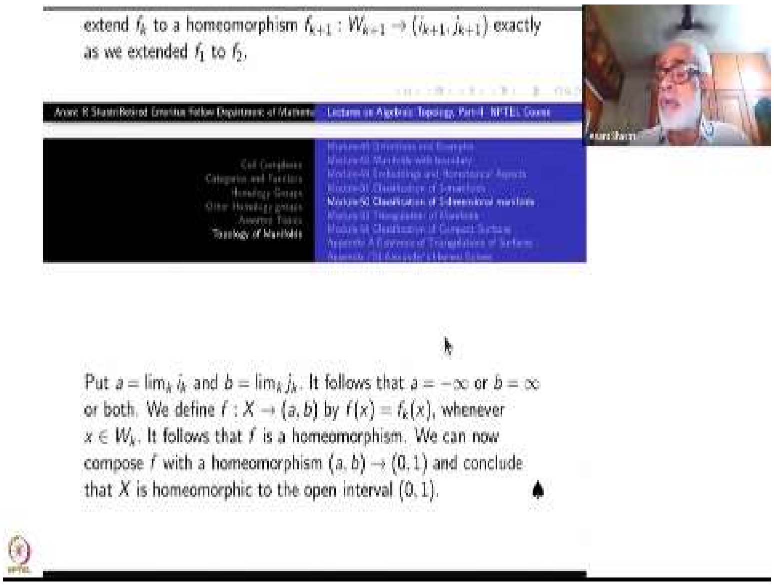

So, we have this sequence of homeomorphims f_k, each f_{k+1} being an extension of f_k, we have to understand what happens in the limiting case. Let a be limit of the sequence $\{i_k\}$ and b be the limit of the sequence $\{j_k\}$. It is possible that one of them becomes a constant after a certain stage but not the both, or both the sequences are unbounded. Accordingly, we have three cases. (i) either a is a negative integer and b is ∞ or (ii) a is $-\infty$ and b is a positive integer, or a is $-\infty$ and b is ∞. In all these cases (a, b) is an open interval in $\mathbb{R}$, of infinite length.

Whatever it is, we define f from X to (a, b) by the rule $f(x) = f_k(x)$, where x itself is in W_k. Once it is in W_k, it is also in W_{k+1}, W_{k+2} etc. But $f_{k+1}(x)$ will agree with $f_k(x)$, etc, so f is well defined okay? So f is well defined on each interval, and is a homeomorphism already. That holds for the entire X also. Finally, you can compose f with a homeomorphism of (a, b) to $(0, 1)$, because any open interval or an open ray is homeomorphic to $(0, 1)$. So, conclude that X is homeomorphic to an open interval.

So finally, starting with a connected 1-manifold, (which is Hausdorff and II-countable) we have shown that it is either homeomorphic to $(0, 1)$ or to the circle okay? Now, finally to sum up, we have already taken care of the case of X having non empty boundary. How did we complete the proof? The boundary may consist of just one point, which could be 0 or 1, it may have two points. The boundary points can be at most 2. Okay? So, in either case, X will be one of the two things, $[0, 1)$ the $[0, 1]$, since $(0, 1]$ is anyway homeomorphic to $[0, 1)$. This completes the entire classification problem for 1-manifolds. Thank you.