**Introduction to Algebraic Topology (Part – II)**
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**Lecture - 51**
**Classification of 1 - Manifolds**

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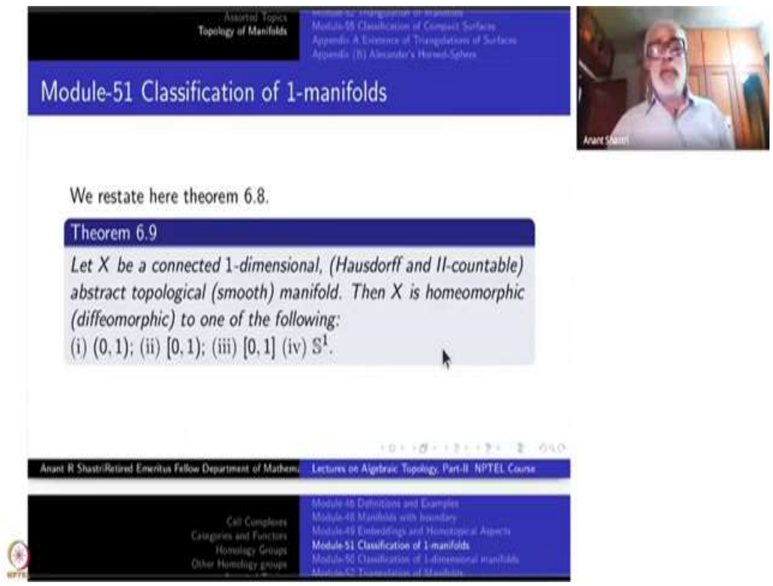


So, continuing with the classification now 1-dimensional manifolds, let me restate our earlier theorem that we wanted to prove okay. So, let $X$ be a connected 1-dimensional, (Hausdorff and II-countable that is a standard assumption Okay? $X$ is an abstract topological manifold. Then $X$ is homeomorphic to one of the following: (i) the open interval $(0,1)$, (ii) half closed interval $[0,1)$, (iii) closed interval $[0,1]$ or (iv) the circle $\mathbb{S}^1$. Recall that we also reduced proving this theorem to the case of proving when boundary of $X$ is empty Okay? Then we have to prove that it is either an open interval $(0,1)$ or the circle $\mathbb{S}^1$. Okay?

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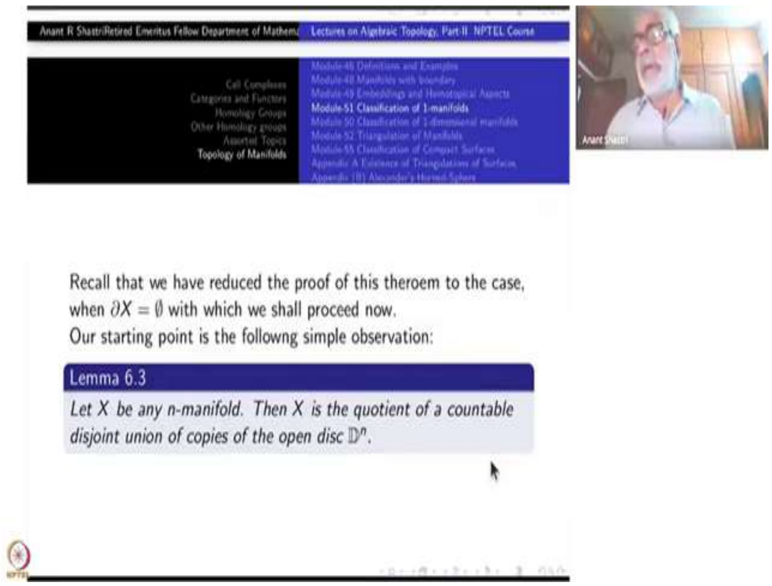


So, where do we start off? We start off with the definition and of course, later on we will say look at this is connected and it is Hausdorff and II-countable etc, keep on using these properties. okay? So, the definition gives you immediately the following lemma, namely: Indeed, more generally, take any $n$-manifold (not necessarily 1-dimensional manifold), then $X$ is the quotient of a countable disjoint union of copies of the open disc $\mathbb{D}^n$. Okay? All manifolds are like that.

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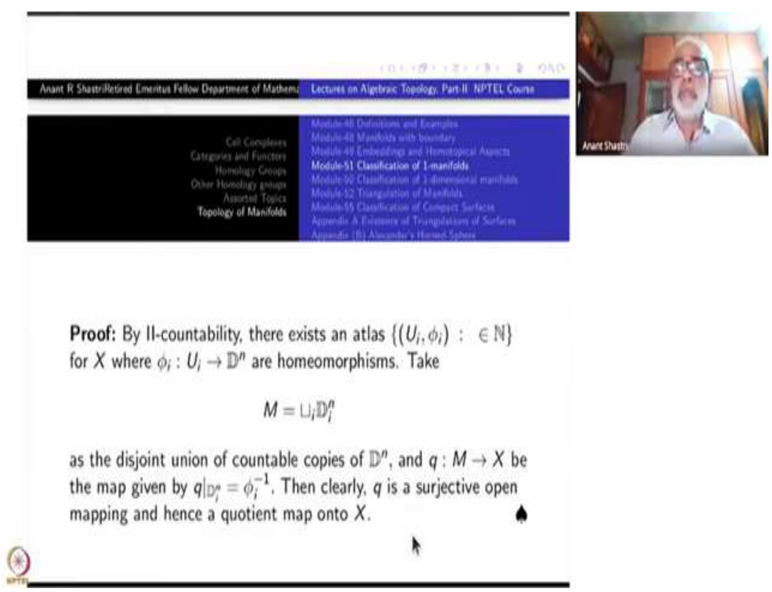


By II-countability, there is a countable atlas. Let us call this $\{(U_i, \phi_i)\}$, $i$ belonging to the set of natural numbers, or a finite subset of it. For each $i$, $\phi_i$ is from $U_i$ to $\mathbb{D}^n$ is a homeomorphism. Now you take $M$ to be the disjoint union of $\mathbb{D}^n_i$'s of countably many copies, of standard open unit discs indexed by natural numbers. Okay, from $M$, I can define a quotient map now onto $X$, namely, $q$ from $M$ to $X$ be the map given as follows: on each copy $\mathbb{D}_i^n$, take it as the inverse image of $\phi_i$. From there, I can define a quotient map now. Namely $q$ from $M$ to $X$ be the map given by on each copy $\mathbb{D}_i^n$ take it as the inverse image of $\phi_i$.

Remember what our $\phi_i$? $\phi_i$ is from $U_i$ to $\mathbb{D}^n$. So, $\phi_i^{-1}$ will be from $\mathbb{D}^n$ to $U_i$ subset of $X$. So $q$ restricted to $\mathbb{D}_i^n$ is $\phi_i^{-1}$. Since $\mathbb{D}_i^n$ are mutually disjoint, this is well defined and $q$ is continuous. Because this is an atlas, means union of $U_i$'s is the whole of $X$, it follows that $q$ is surjective. But $q$ is also an open mapping being a homeomorphism onto open subsets, restricted to each copy. Any surjective open mapping is a quotient map. So the proof of the lemma is over.

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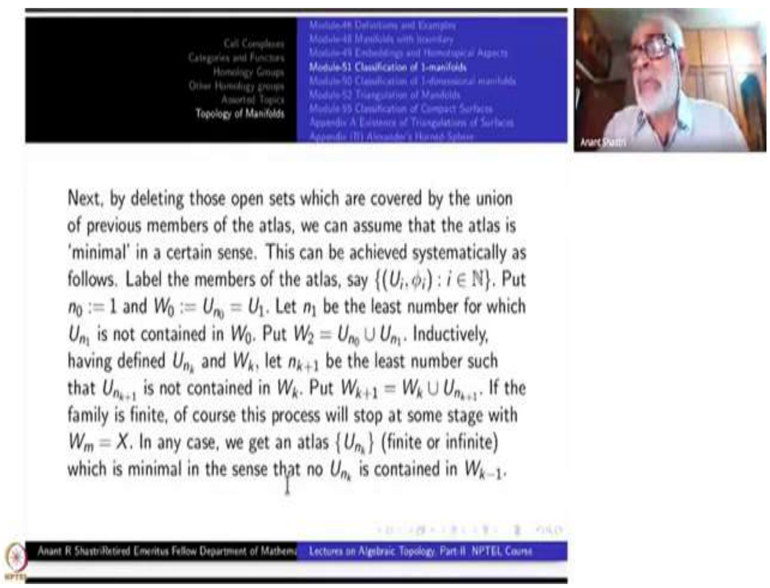


So, next we would like to make some bargaining here. We do not want all those $U_i$'s you know, some of them may be useless and so on. What is the meaning of being useless? For covering the space $X$. So, this is what we want to do. By deleting those open sets which are already covered by the union of previous members in the list $U_1, U_2, \ldots$ If $U_2$ is already covered by $U_1$, what is the point of taking $U_2$? Just delete it. $U_2$ and go to $U_3$, and relabel it as $U_2$ and so on. Suppose, $U_1, U_2, U_3, U_4$ cover $U_5$, then what is the use of keeping $U_5$? Delete it from the list so on.

So, delete those member, which are already covered by all the previous members. this is what I want to do in a systematic way. So, start with any listing $\{U_i\}$ of charts. Put $n_0 = 1$ and $W_0 = U_1$

. Let $n_1$ be the least number $i$ for which $U_{n_1}$ is not contained in $W_0$. Put $W_1 = W_0 \cup U_{n_1}$. Keep doing this. Suppose you have defined $U_{n_k}$ and $W_k$, let then $n_{k+1}$ be the least number such that $U_{n_{k+1}}$ is not contained in $W_k$. Put $W_{k+1} = W_k \cup U_{n_{k+1}}$.

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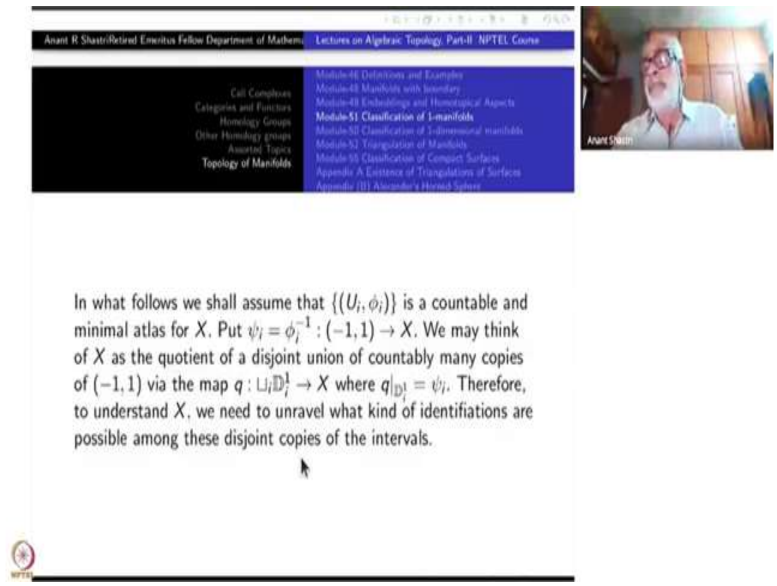


Coming back to the case when dimension of $X$ is 1, we may assume that the atlas $\{(U_i, \phi_i)\}$ is minimal in the previous sense, okay? Put $\psi_i = \phi_i^{-1}$ from the open interval $(-1,1)$ to $X$. We are in the special case when $n = 1$ and therefore, $\mathbb{D}^1 = (-1,1)$. By the previous lemma, you may think of $X$ as a quotient of the disjoint union of countably many copies of the interval $(-1,1)$. Okay? Let $q$ be the quotient map where $q$ restricted to the $i^{th}$-interval is $\psi_i$, the inverse of the coordinate chart. (These are actually called local parametrizations for $X$.) Therefore, to understand our manifold $X$, what we need to do is to unravel what kind of identifications are taking place under this quotient map. Any quotient map is same thing as introducing certain equivalence relation on the domain. So, what are these equivalence classes? Which pair of points are identified? That is what we have to understand. Alright?

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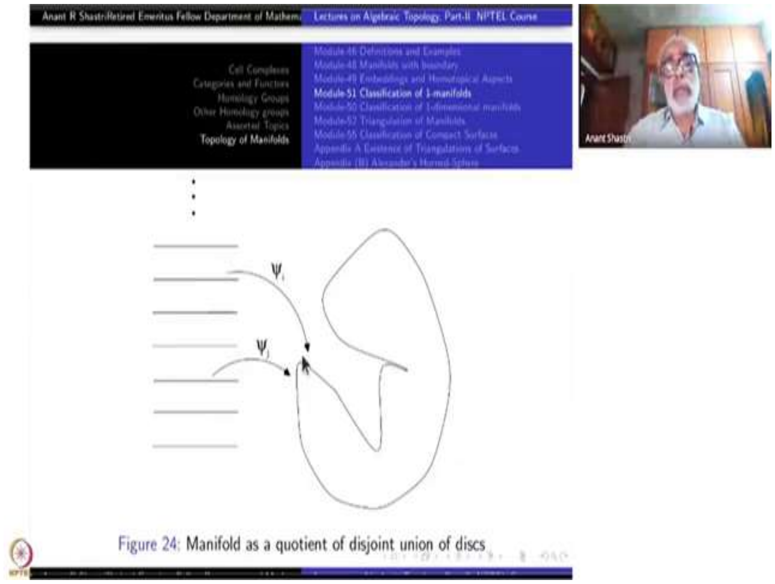


Therefore, we start with this is picture. This is some 1-dimensional manifold. Possibly these are all copies of the interval $(-1,1)$. And there are these parameterizations this part will cover something this part will cover something Okay. So $X$ is quotient of disjoint union of copies of $(-1,1)$. $U_i$ is an open subset equal to $q$ of the $i^{th}$ copy of the interval and so $U_i = \psi(-1,1)$.

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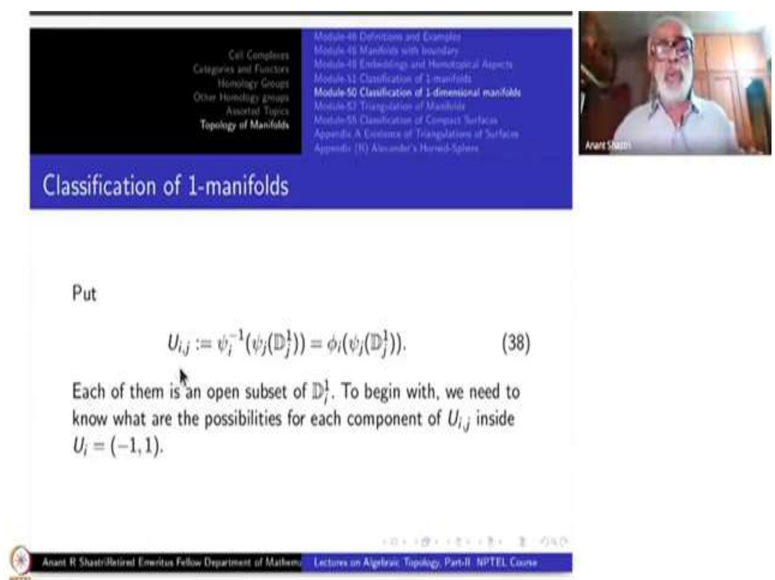


For each pair of indexes $i, j$, put $U_{i,j} = \psi_i^{-1}(U_j) = \psi_i^{-1}(\psi_j(\mathbb{D}^1))$. The each $U_{i,j}$ is an open subset of $(-1,1)$. It may be empty. No problem. It will never be equal to the whole of $(-1,1)$, because of the minimality of the list.

To begin with we need to know what are the possibilities for each component of $U_{i,j}$. It is better to think of $U_{i,j}$ being subset of the $i$-th copy of $\mathbb{D}^1$. We have to understand how the two homeomorphisms $\phi_i$ and $\phi_j$ are related. Surprisingly, it is possible to do this because we are dealing with 1- dimensional case, homeomorphisms of intervals to intervals are better understood.

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So, let me illustrate this with an example. Okay? This example tell you most of the story that may acutally happen.

(A) Suppose the atlas consists of just two charts only, $i = 1, 2$. Concentrate just on $U_{1,2}$ and $U_{2,1}$ to begin with, okay? So, I am assume that $M$ is the quotient of $\mathbb{D}^1_1$ disjoint union $\mathbb{D}^1_2$ and $q$ from $M$ to $X$ is the quotient map. Then let us consider some simpler subcases here. Namely, the following cases.

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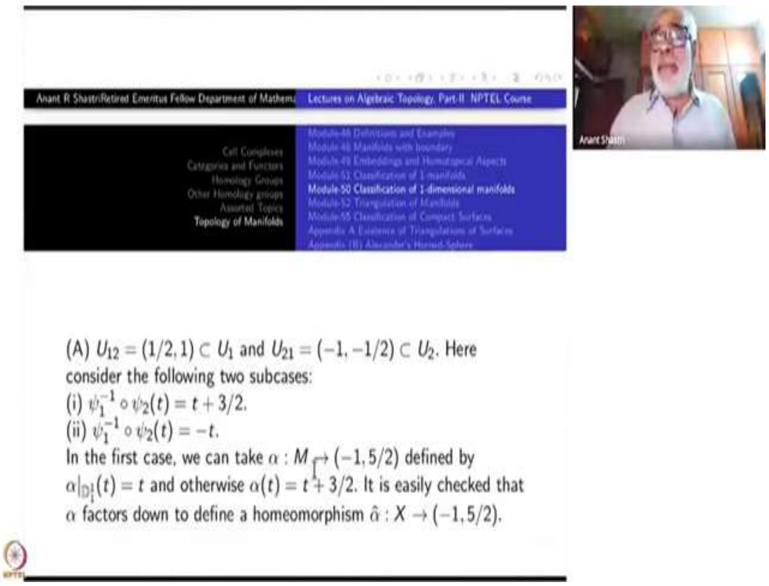


The first case (i) is $U_{1,2}$ is the open interval $(1/2, 1)$ contained in $D_2$. and $U_{2,1}$ is the open interval $(-1, -1/2)$ contained in $D_1$. That is a possibility. If I go from here to here by $\psi_2$ and then follow it by $\psi_1^{-1}$, I would get from here a homeomorphism of an interval to an interval. What could this homeomorphism be? Let us say that it is something like $t$ going to $t + 3/2$. We have to be careful here. Under this map, $-1$ will go to $1/2$ and $-1/2$ will go to $1$. Okay? All this I am just assuming I am not proving anything here about this homeomorphism, but taking the easiest cases.

(ii) Or it may be the other way round, namely, $t$ mapsto $-t$. A simpler one than the first one Okay. So, what happens under these two cases? Let us understand okay?

In the first case, we can take $\alpha$ from $M$ to $\mathbb{R}$ as follows: $M$ is what? It is the disjoint union of these two intervals $D_1$ and $D_2$ okay? Take $\alpha$ to be $t$ mapsto $t$ on first copy $D_1$ and on the second copy, let it be $t$ mapsto $t + 3/2$, translating by $3/2$. Then on $U_{1,2}$ and $U_{2,1}$, $\alpha$ is compatible with the identifications. Check that this map, factors down to define a homeomrphism of $X$ onto the interval $(-1, 5/2)$. So here is the picture for that.

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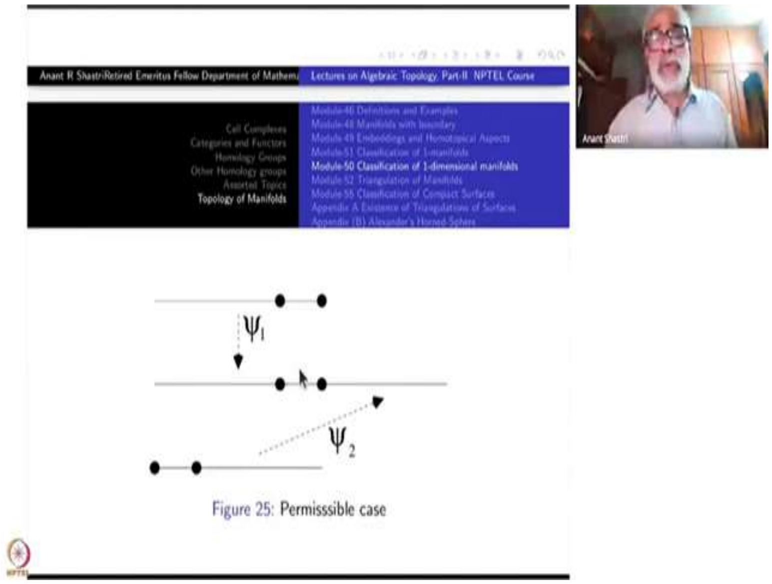


So, this is $1/2$ to $1$. This is your $U_{1,2}$ here. This is $U_{2,1}$ and this is your $X$ now. To begin with, I do not know what $X$ is, but I actually got it this way okay. The quotient space has to be this just identify with this $1$ by shifting the $t$ going to $3/2$. This goes to $t$ here, but this goes to $3/2$ times this $1$. So, they agree a point here is identify the point here, but both agree on that part. So, therefore, these two parts patch up together to give you a map from $X$ to this whole interval from $-1$ to $5/2$. So we have proved that the manifold $X$ is again open interval. Its length may be larger. Do not worry, we are not interested in the length okay?

Let us see what happens in the second case, which looks like simpler. $t$ going to $-t$, you are just reversing the direction. So, in the picture I can just write this way.

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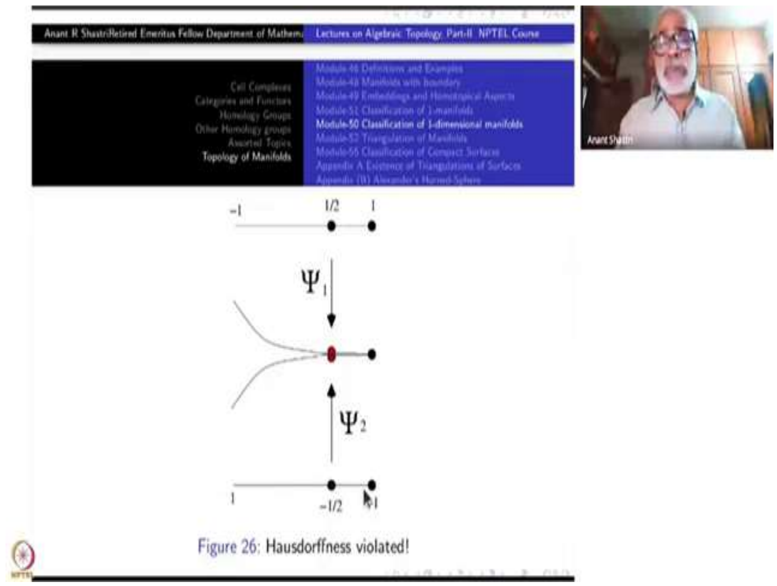


So, this is $-1$ to $1$, the other way around, okay? Now, a point $1/2$ here goes to $-1/2$ here. That means $\psi_1(1/2,1)$ is equal to $\psi_2(-1,-1/2)$ in the reverse direction. The rest of $\psi_1(D_1)$ and $\psi_2(D_2)$ remain disjoint in $X$. So, you have to draw the picture just like this.

What happens to the points $1/2$ in $D_1$ and $-1/2$ in $D_2$? They are not identified. Only open interval are identified. The point shown with a black bullet is not to be included in $X$. But here I put a red bullet which represents two distinct points, these points are not identified. The open intervals on one side of these points are identified what happens with those two points. Both of them are in the closure of this common region which is homemorphic to an open interval. Therefore Hausdorffness of $X$ is violated here.

Conclusion is that we do not want to accept that Hausdorffness is not violated, but we want to say that our own assumption is wrong, namely, we cannot have the second case at all: $\psi_1^{-1}$ of $\psi_2$ will not be the homeomorphism which sends $t$ to $-t$.

The first case is acceptable and gives you a longer interval. So, the intersection of two coordinate patches are like this then the identification like the first one are allowed, whereas the ones like the second case are not allowed. That is the lesson from this example.

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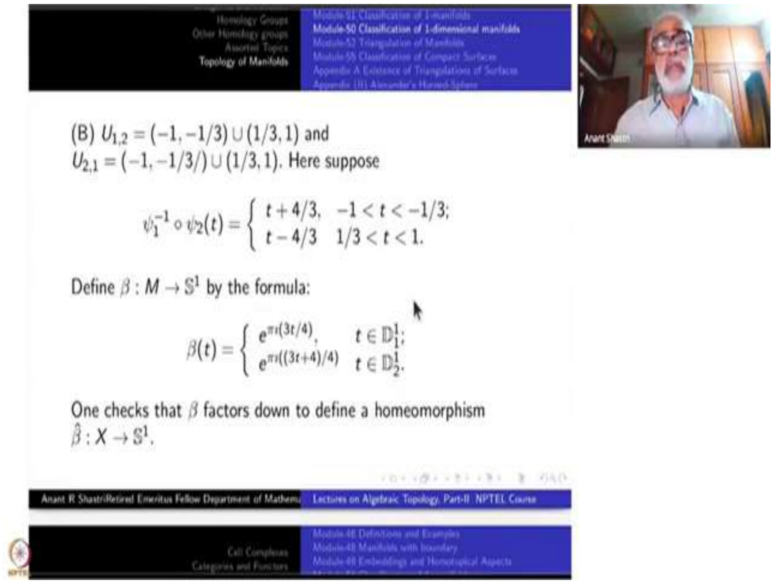


I will tell you one more example, which is much more interesting one okay? So, this is done in (B). Here I am assuming $U_{1,2} = (-1,-1/3) \cup (1/3,1)$. So, $U_{1,2}$ consists of two connected components. In (A), it was connected. Here it is not connected, but has two components. okay? Of course then $U_{2,1}$ should be also have two components. We assume that $U_{2,1} = (-1,-1/3) \cup (1/3,1)$ as well. This is just an assumption to make the case simpler okay. Now suppose $\psi_1^{-1} \circ \psi_2$ is given by (I have made two cases) $t \mapsto t + 4/3$ on the first component, and $t \mapsto t - 4/3$ on the second.

I did not make a picture of this one because this is much much easier actually. So, I will let you make a picture for yourself. So, if using your picture itself you may understand what is happening. The homeomorphism is interchanging the two components and within each of them it is just a shift, a translation homeomorphism. Okay? That is the case I am looking at okay?

So, we define $\beta$ from $M$ which is the disjoint union of $D_1$ and $D_2$ into $\mathbb{S}^1$, by the formula, namely, on $D_1$, take $t$ going to $e^{\pi i 3t/4}$. See on the interval $(-1,1)$, if you take $e^{piit}$, it will cover the entire circle exactly once right? But here I am covering only $(3/4)$-th of it. That explains the factor $3t/4$ in the formula. Okay? The first one covers that much of $\mathbb{S}^1$, rest of the part should be covered by the second part, namely, take beta on $D_2$ to be $t$ mapsto $e^{\pi i((3t+4)/4)}$, which means I am rotating the first map through an angle $\pi$. Clearly then $\alpha$ on $M$ covers the whole of $\mathbb{S}^1$, $\beta$ from

$M$ to $\mathbb{S}^1$ is a surjective mapping alright. What are the pair of points that go to the same point under $\alpha$? They are precisely given by the map $\psi_1^{-1} \circ \psi_2$. Therefore, what happens is that $\beta$ factors down from $M$ through the quotient map $q$ to a map $\hat{\beta}$ from $X$ onto $\mathbb{S}^1$ which is injective also. Therefore, $\hat{\beta}$ will be a homeomorphism okay.

So, what we have done is taking two arcs like this and then forming this kind of identification, a portion here overlapping with a portion and similarly here. That will give you a circle. So, that is what is happening here okay. So, I think these two examples explain the whole thing whatever I wanted to tell you. The key here is that, though these are looking like very special examples, they will take care of most of the cases that can occur. So, I have to make a general statement here now, okay. That is, a general fact indicated by these examples. We formalize this in the following two lemmas, the proof will be completely obvious. However, I implore you to write down the proofs on your own. So, take it as an assignment, okay. So, these are the statements.

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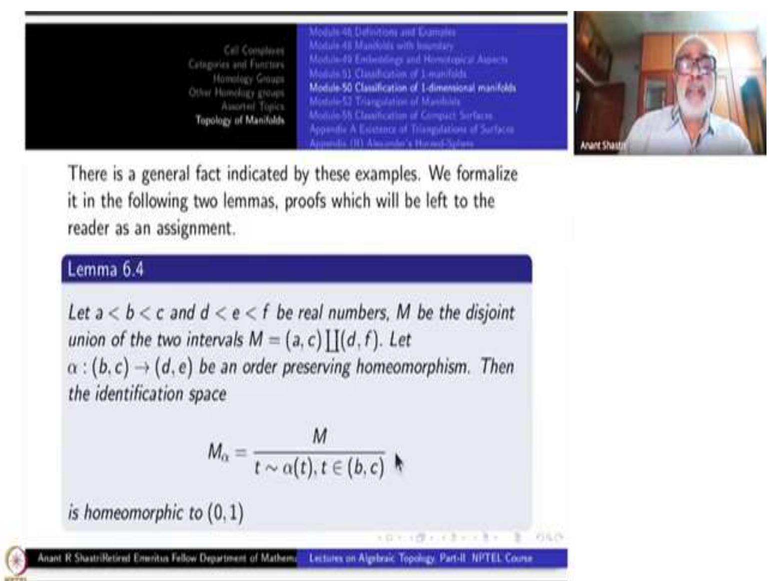


Let $a < b < c$, and $d < e < f$ be real numbers. (Now, I am not even using that they some specific numbers like $-1, 1$ etc., okay? So, this is the whole idea because for any open interval is homeomorphic to any other open interval. So, that is why I am writing arbitrary numbers $a < b < c, d < e < f$.

Let $M$ be the disjoint union of the two intervals $(a, c)$ and $(d, f)$ okay? Let $\alpha$ from $(b, c)$ to $(d, e)$ be an order preserving homeomorphism. $(b, c)$ is one end of the first interval and $(d, e)$ is also one end of the second interval. The difference is that in the first one it is the terminal end and in the second one it is the initial end. This is similar to the first subcase of case (A). In the second case, we had an order reversing homeomorphism which was not good. So, all that you need to assume is that $\alpha$ is order preserving homeomorphism.

Then the identification space $M_\alpha$ obtained from $M$, by identifying $t$ with $\alpha(t)$ where $t$ is inside the open interval $(b, c)$, (this you denote it by $M_\alpha$ to indicate the role played by $\alpha$), will be homeomorphic to the open interval $(0, 1)$. Okay?

Here I do not even start with the union of two disjoint copies of $\mathbb{D}^1$. No specific intervals are mentioned anywhere. Okay? I do not care either $0$ to $1$. Because any two open interval are homeomorphic to each other, I can always say it is homeomorphic to $(0, 1)$. Similarly, you can have the second statement which will give you a circle. Okay? Exactly similarly. So, this is the first case which in which (A1) and (A2) not occurring. So, case (B) is discovered by this one. Okay

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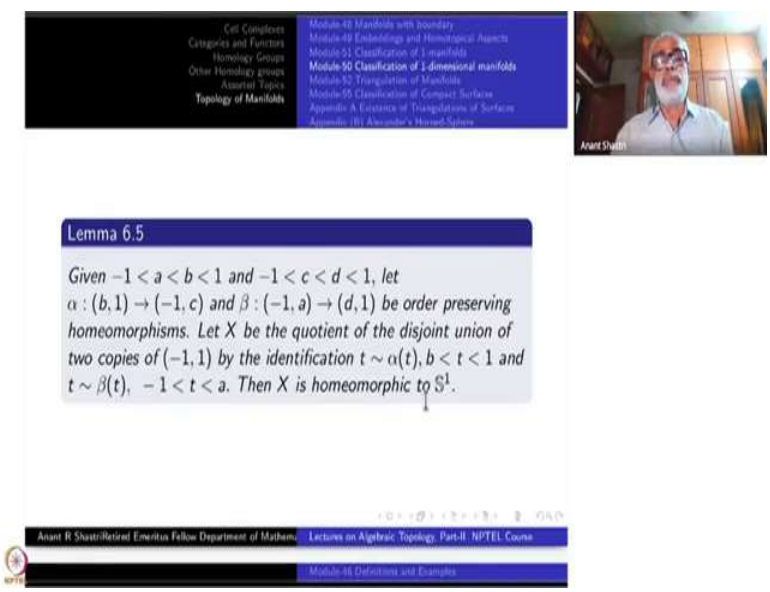


However as in the example, I have made the second statement a little more specific, by taking $M$ to be disjoint union of two copies of $\mathbb{D}^1$. Given real numbers $-1 < a < b < 1$, and

$-1 < c < d < 1$, let $\alpha$ from this end $(b, 1)$ to the initial end $(-1, c)$ and $\beta$ from $(-1, a)$ to $(d, 1)$ be order preserving homeomorphisms. So, both of them are order preserving homomorphisms but interchange the ends. Let $X$ be the quotient of the disjoint union of the two copies of $(-1, 1)$ by the identification, $t$ with $\tilde{\alpha}(t)$ for $t$ inside $(b, 1)$ and $\tilde{\beta}(t)$ for $t$ inside $(-1, c)$. Then $X$ is homeomorphic $\mathbb{S}^1$, okay?

For a proof, you have to correctly copy what you have done in the example above, without using the specific values such as $4/3$ etc. Of course. That is the challenge okay? So, we are going to use these two lemmas. So, further things will be taken next time. Let these two examples sink properly in your mind. Next time we will continue with the proof of the classification. Thank you.