Introduction to Algebraic Topology (Part-II)

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Lecture - 50
Homotopical Aspects continued

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Last time, we proved one of the hardest results about topological manifolds, namely, if any closed subspace of $\mathbb{R}^N$ is locally contractible then it is a neighbourhood retract.

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By the very definition, a manifold is locally Euclidean and locally Euclidean means that it is locally contractable. By the embedding theorem which we proved partly and which we are assuming that every manifold is a subspace of $\mathbb{R}^N$ for some N. Combining these two results and we can derive a few homotopical properties of arbitrary topological manifolds. So, let us take care of some of them. So, this is the first thing that we derive:

Any topological manifold is a retract of a locally finite countable CW complex W, of some finite pure dimension N. Moreover, if X is compact, then W can be chosen to be finite. So, this is the basic theorem that we have we can derive it from whatever we have done previously.

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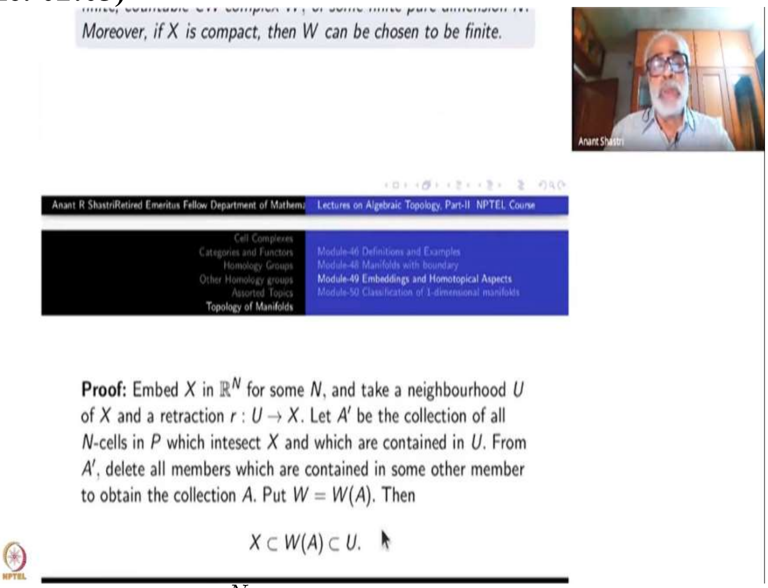

Start with an embedding of X in $\mathbb{R}^N$. Take a neighbourhood U of X and a retraction r from U to X. So, this is what the earlier theorem allows us. In fact, that theorem was more elaborate only weaker version of that we are using here. There is a neighbourhood and that neighbourhood and a retraction is there is all I am using here. Now, go back to result giving CW complex structures out of lattice structures. That will be used again.

Let A' be the collection of all N-cells in P which intersect X and which are contained in U. If an N-cell is too big, we do not want them in A', they must be contained inside the open subset U and they must intersect X. From this A', delete all members which are contained in some other member of A', to get the sub collection A.

Once a big N-cell Q is taken inside A, no smaller N-cells will be taken inside A. Faces of Q will be taken of course. Put $W = W(A)$ to be the union of all members of A. (This W is different from the W of the previous theorem by the way. Do not confused that one which was $U \setminus X$. Here it is a which is a neighbourhood of X itself, though not an open set. Then X

is contained inside this $W(A)$ which is contained in, U because all members of A are contained in U.

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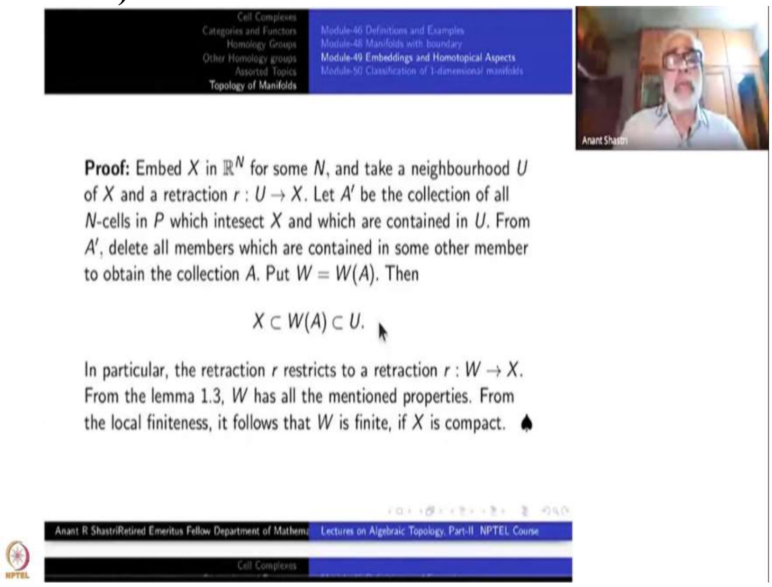

There is a retraction r from U to X, so, you can restrict r to $W(A)$. And our previous result tells you $W(A)$ has all the prescribed properties, namely, it is a countable, locally CW complex of pure dimension N. Pure of dimension N means what? All the cells in $W(A)$ are contained inside some N-cell in $W(A)$. That is obvious because we have started with the union of all the N-cells in A and then take faces of them in giving the CW structure.

From the local finiteness of W, it also follows that W is finite if X is compact. For each point $x \in X$, there is a neighbourhood V_x which intersects only finitely many N-cells will intersect. As x varies over X, all members of A are accounted for. Since X is compact, we can take a finite subcover of $\{V_x\}$ and these finitely many neighbourhoods will tell you that there are only finitely many members in A. That gives W is a finite CW complex. So, this big theorem comes without much hard work now.

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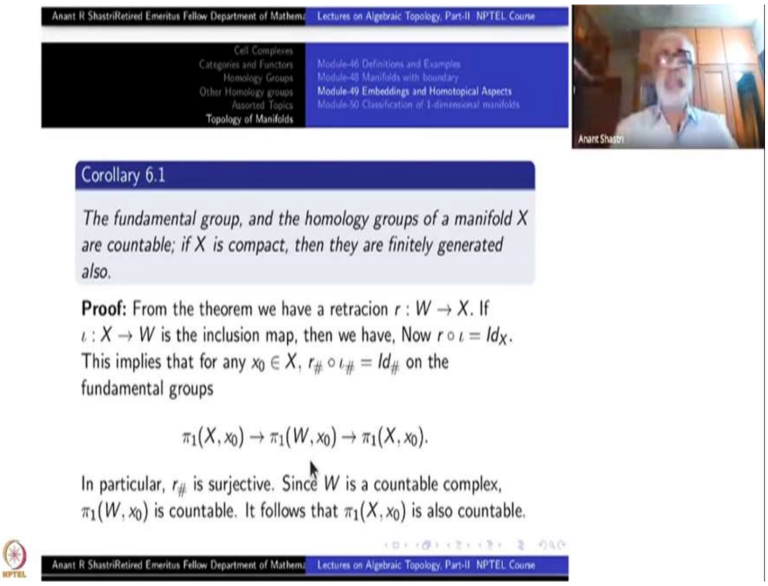

Now, an interesting corollary is that the fundamental group, and the homology groups of any manifold X are all countable and if X is compact, they will be finitely generated also. You may notice that every manifold has at most countably many path components. So, without loss of generality, we can and do assume that X is path connected. It immediately follows that $W = W(A)$ is also path connected.

So, what is the proof? It is very straightforward from the theorem. We have a retraction r from W to X. If i from X to W is the inclusion map, r is a retraction means what? $r \circ i$ is the identity of X. Passing to the fundamental group level, we get $r_\# \circ i_\#$ is $identity_\#$. On the fundamental group level what is that? $\pi_1(X, x_0)$ goes to $\pi_1(W, x_0)$ under $i_\#$ and then $r_\#$ comes back to $\pi_1(X, x_0)$. In particular, this implies that $r_\#$ is surjective. The composite is identity map implies this must be surjective.

But W is a countable CW complex. So there are countably many generators of π_1. Indeed you have look at only the 1-skeleton W, which has only finitely many 1-cells. Alright? Any group which is generated by a countable set is itself countable. Therefore, $\pi_1(W)$ is countable. Since $r_\#$ is surjective homomorphism, it follow that $\pi_1(X)$ is also countable.

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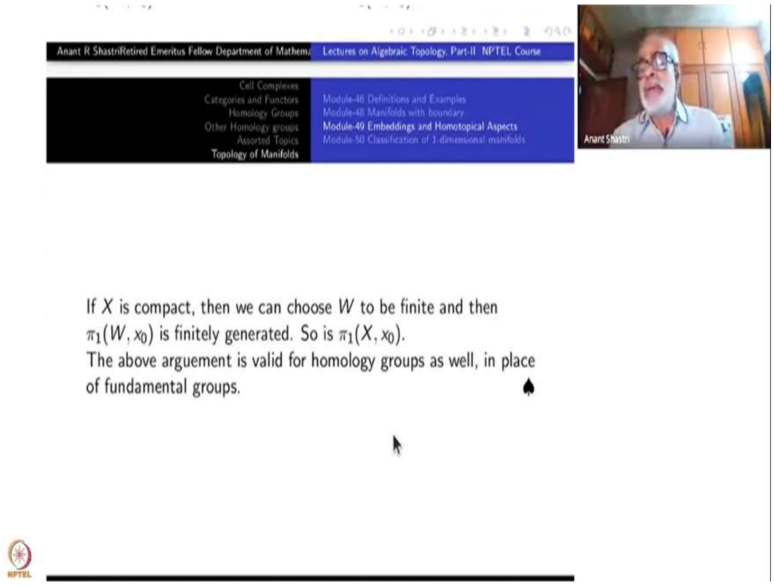

For the next part, suppose X is compact then we have seen that W can be chosen to be a finite CW complex. Then the 1-skeleton is finite therefore, there are only finitely many generators of $\pi_1(W)$. Since $r_\#$ is surjective homomorphism, the same holds for $\pi_1(X)$ also.

So, this is about the fundamental group. The argument is ditto for homology group as well. Instead of π_1 if you put H_*. The CW chain groups of W itself are all countably generated and vanish beyond dimension $N + 1$. And in the compact they are finitely generated also. So, the same conclusion holds for homology of W as well as X.

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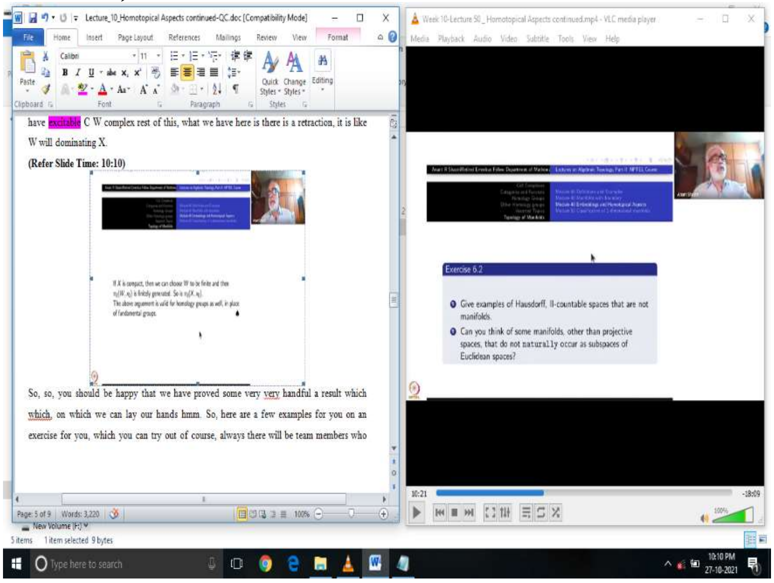

So, you should be happy now that we have proved a very useful result. So, here are few examples and exercise for you, which you can try out of course. Always there will be team of tutors who will help you out.

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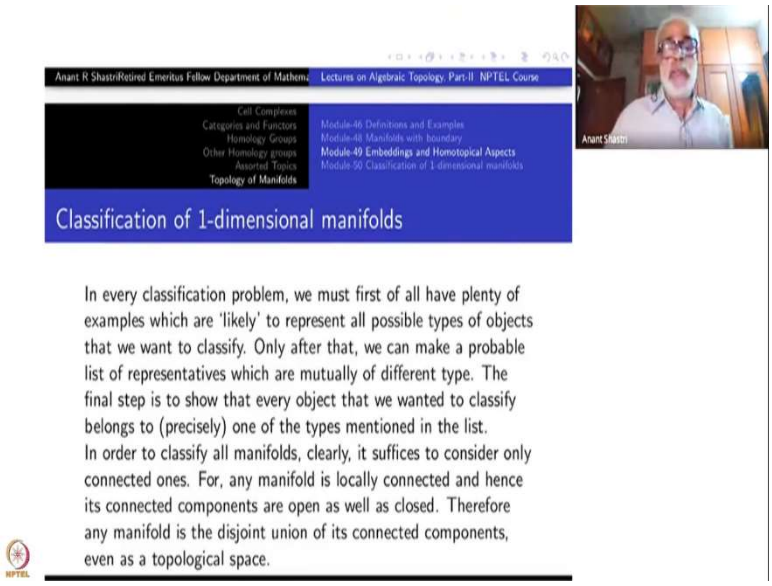

Now, let us go to the next topic. Let us begin that topic of classification of 1-dimensional manifolds. It will be again somewhat lengthy topic. So, this is again another very useful topic on which we can put our hands. And you will feel that you have learned something.

So, more generally, let us look at any classification problem in science. First of all what do we do? We have to collect samples, plenty of them. And when we are somewhat sure of having collected a large variety of these samples and we feel that it must be covering all possible all variety of objects, then only we go for a systematic classification of members. Namely, we must decide upon the meaning of classes by specifying certain properties such as connectedness or compactness and so, on in our case.

For instance, in biological classifications, we specified what is a protozoa what is vertebrata and so on animals were classified into various phylums. So, the final step will be that you should determine all possible `mutually different types' of objects, the classes must be mutually disjoint.

In all possibility, in the run for that, you may find that you have not collected yet one or two types of probable objects. Your study is pointing at probable existence of more types than in your collection. So, you search for such objects. What do you mean by search? Any scientific study always leads to such searches. Just like how the classification of fundamental elements in Chemistry. So, when you systematically carry out the search you may feel that there must be something missing here. So, you search for it and then actually find those missing objects! That is the way, by trial and error, any scientific classification is carried out.

Now coming back to the study of manifolds. Any manifold is locally connected and hence, its connected components are all open as well as closed. Therefore, every manifold is as a topological space, the disjoint union of its connected components. Therefore, studying connected manifolds is enough. why?

Because then you can take disjoint union of the members you have listed to get any other non connected manifold. So, we always consider only connected manifolds. This is not the case with compactness. There is no way of actually reducing the general classification to compact case, though understanding the compact case fully is a must and will help in the general case of non compact manifolds. However, considering the compact cases first simplifies our task for technical reasons. So right now, as far as 1-dimension manifolds are concerned, we are only putting connectivity condition, that's all.

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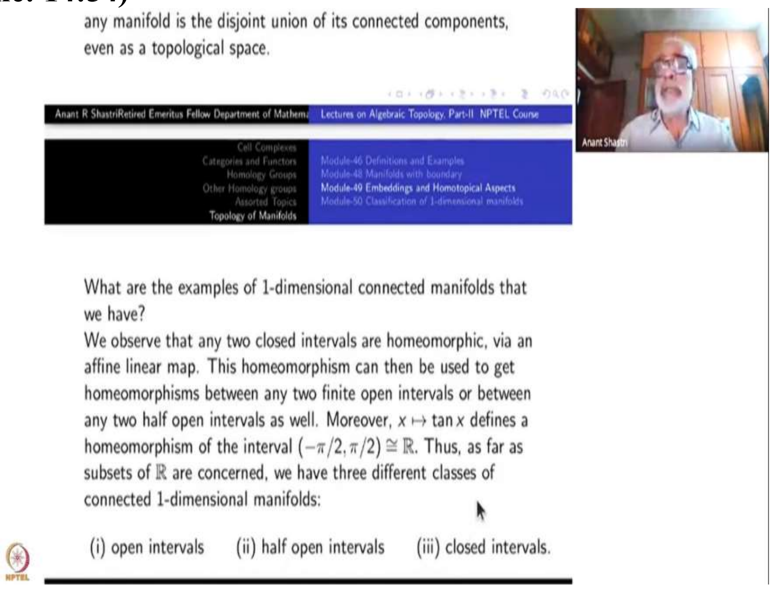

So what are examples of 1-dimensional connected manifolds we have. You have to collect the samples first. So, you can take any open subset of $\mathbb{R}$. They are manifolds. $\mathbb{R}$ itself is one. Any open subset of $\mathbb{R}$ which is connected. So, they are only open intervals. But now there are different kinds of intervals, open intervals, half closed intervals and closed intervals. And that is the end of all types of connected one manifolds inside $\mathbb{R}$.

So, by the why did we not specify the end points of these intervals. Within this collection of samples, we note that any two open intervals are homeomorphic to each other, whether finite or infinite. Similarly any half closed interval is homeomorphic to any other half closed interval. Any closed interval, is homeomorphic to any other closed interval. This much we know already. Actually there are linear homeomorphisms between any two of them in each of the above three cases. From a finite interval to an infinite one, we have maps such as x to $tan(x)$, which gives gives you a homeomorphism from $(-\pi/2, \pi/2)$ to the whole of $\mathbb{R}$. So, equipped ourselves with these informations, we are sure that there are only three types of manifolds dimension 1, which are connected and which are inside $\mathbb{R}$, viz., open intervals half closed intervals and closed intervals.

What happens when you go out of $\mathbb{R}$? Just go to $\mathbb{R}^2$. There are many other kinds of 1-dimensional manifolds there. All conic sections will come, circles, ellipses, parabolas and so on. A pair of lines, of course, if they are parallel then they are 1-manifold but not connected, otherwise they intersect and hence they are not manifolds. Similarly hyperbolas have two connected components each of them is a 1-manifold.

You immediately perceive that parabola, hyperbola and pair of non intersecting straight lines are all covered by the real line itself each being either homeomorphic to $\mathbb{R}$ or two copies of $\mathbb{R}$.

Circles and ellipses form a different type. Of course you also know that all of them are homeomorphic to each other and hence form a single class, even though geometrically a circle is a different type of object from an ellipse.

Next we can consider triangles. No problem. Again they are all homeomorphic to a circle. Same is true a square, a pentagon, boundary of any convex polygon.

So this way we can keep on collecting more and more examples and while doing so, check whether they form a new type. However, we seem to get no more types, even if we go to $\mathbb{R}^3, \mathbb{R}^4$ etc. You may have very twisted curves having no specific shape or geometry. But we do not seem to get any different types of connected 1-manifolds. So at this stage, we would like to ask whether we really have reached the dead end.

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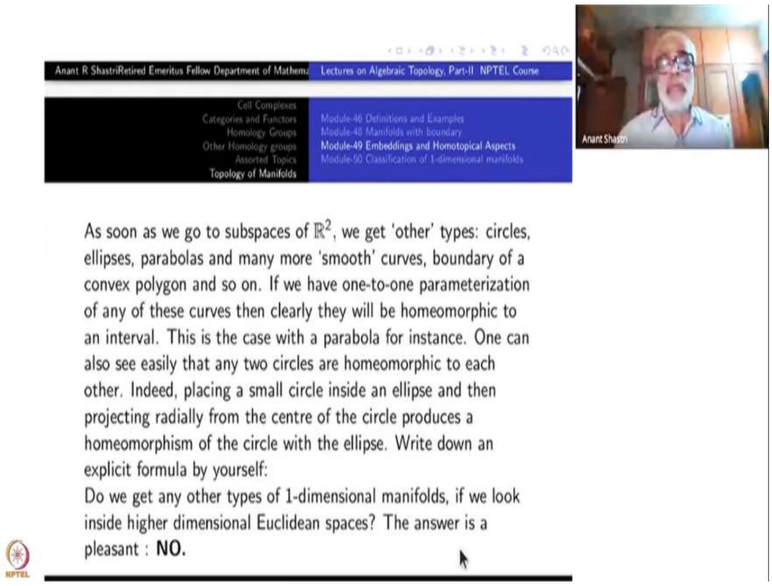

So, do we get any other types of 1-dimensional manifolds if we look for them inside, $\mathbb{R}^3, \mathbb{R}^4, \ldots, \mathbb{R}^N$ and so on? We do not have to look outside Euclidean spaces, because any topological manifold is a subspace of $\mathbb{R}^N$. In fact, everything 1-manifold will be inside some $\mathbb{R}^3$ itself, though we have not proved the embedability of any 1-dimensional manifolds inside $\mathbb{R}^3$. A pleasantly surprising answer is that we have already exhausted the list-there are no more of them.

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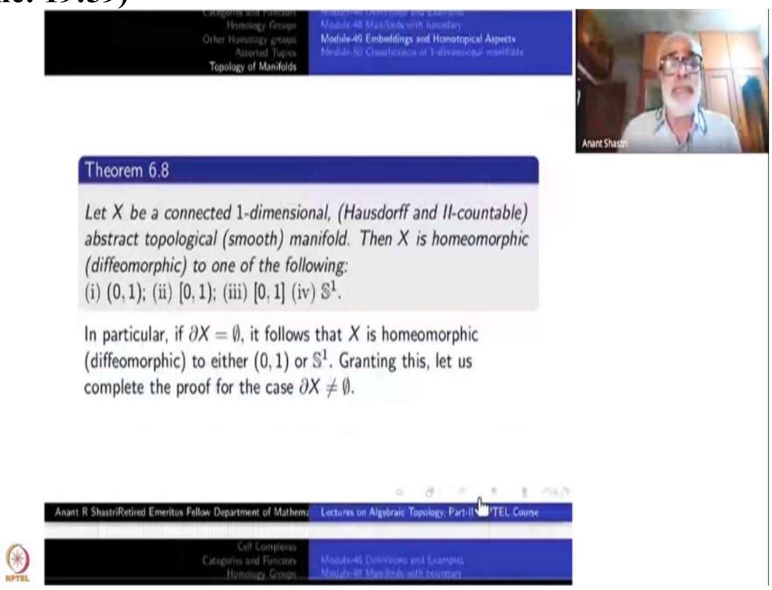

So here is a statement of the theorem which we would like to prove. While proving we may get some other insight and then maybe we find a new type that is possible of course, but this is a conjecture. So, this is in a conjecture format. This is what we want to prove what is that that?

Let X be a connected 1-dimensional manifold (remember that they Hausdorff, and II-countable, that is in the back of our mind). So, here I am stating a result even for the smooth case also, but you can ignore the smoothness part right now. Then, X is homeomorphic to one of the following four types: open interval $(0,1)$, half-closed interval $[0,1)$, closed interval $[0,1]$ or the circle $\mathbb{S}^1$. One can easily see that each of these type is a different one. They are themselves not homeomorphic to each other.

If they were, then I would mention them separately in the list. I will cut down my list. The big problem is to prove that this list is complete. Why there is not a fifth element here? That is the hard thing to prove.

So, in particular what does this mean? Suppose boundary of X is empty. That happens only for the first and the last members. For the second and third, the boundary is non empty. So, we must first of all prove that any 1-dimensional, connected manifold without boundary is homeomorphic to the open interval $(0,1)$ or the circle $\mathbb{S}^1$. Suppose, we have proved this part. Then we can complete the proof of the theorem by considering the case when the boundary is non empty. This can be done in different ways. Here is one method.

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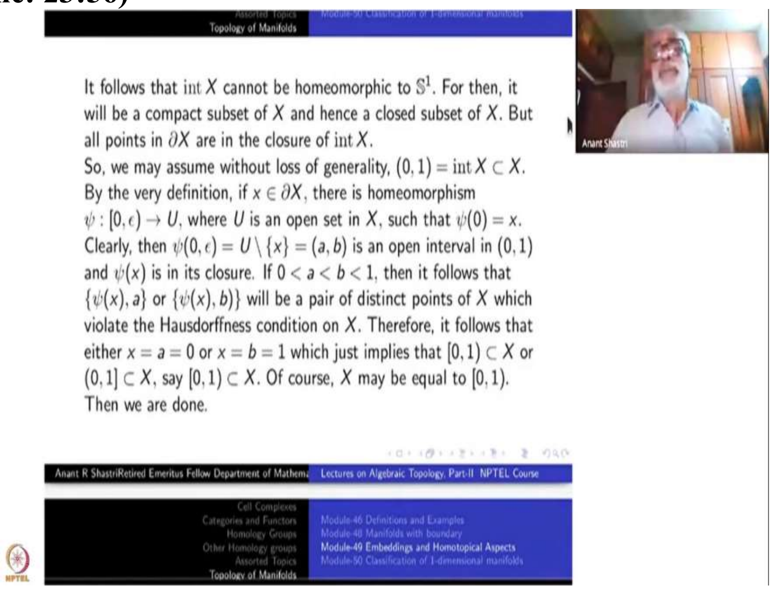

Let X be a 1-dimensional connected manifold with non empty boundary. Its interior is again a 1-dimensional connected manifold without boundary. Therefore, $int(X)$ must be (homeomorphic to) $(0,1)$ or the circle $\mathbb{S}^1$. Do you agree? Now can the circle $\mathbb{S}^1$ be the interior of a connected manifold with boundary?

In that case $int(X)$ being homeomorphic to $\mathbb{S}^1$ is compact subset of X and hence a closed subset of X. The boundary points of any manifold are in the closure of $int(X)$ and so the whole of X is equal to $int(X)$. That is a contradiction.

So we are are left with the case when $int(X)$ is $(0,1)$. Yes, this is possible of course. You just look at this list. The list itself tells you two such possibilities. The open interval $(0,1)$ is the interior of both $[0,1)$ as well as $[0,1]$. Are there any other possibilities for X? X could be $(0,1]$. But $(0,1]$ is homeomorphic to $[0,1)$ by the reflection in the point $1/2$. So, that is not a new one.

So, it can be very easily seen that you cannot have a third point as an interior point as a boundary point of this. A detailed proof is given in the slide. (However, there is a typo, replace $\psi(x)$ by $\psi(0)$). The argument used here will be repeated again in the sequel. So, for today let us take this for granted.

Next time we will continue with the proof of the first part. Thank you.