Introduction to Algebraic Topology (Part-II)
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Lecture - 49
Embeddings and Homotopical Aspects continued

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So, far, in the study of manifolds, we introduced the notion of a topological manifolds, gave some examples and then introduced the notion of manifolds with boundary also. We have also shown that every manifold is paracompact including the manifolds with boundary. We also showed that the boundaries of a bounded manifold have what are called Collar neighbourhoods. So, today we shall study some homotopic aspects.

A key to that is the result that every topological manifold is a subspace of some large Euclidean space. That is the meaning of embedding. So, every manifold can be embedded inside a Euclidean space. So, as such, it seems that in the definition of a manifold, we do not have to go out of $\mathbb{R}^n$, and could have taken only topological subspace for $\mathbb{R}^n$ which are locally Euclidean, such as a circle or a spheres union of lines and so on.

However, in practice what happens is manifolds many not arise may not occur naturally as subspace of $\mathbb{R}^n$. They arise in different forms, especially when as quotients of some familiar objects. Then it is a burden to see them first of all as subspaces of $\mathbb{R}^n$ even before identifying them as manifolds. So, the abstract definition has this advantage.

So, let us anyway do this embedding theorem, which will itself help in the study of other aspects of manifolds. As we shall see this single result has several implications on topological and homological properties of a manifold. Being a subspace of some Euclidean space is itself something very special.

Every n-manifold is homeomorphic to a closed subset of $\mathbb{R}^{2n+1}$. Start with a manifold of dimension n. Irrespective of how complicated it may be, you do not have to go for a very large m to get an embedding into $\mathbb{R}^m, m = 2n + 1$ will do. So, this is quite tight. There are examples wherein you may not be able to do it in $\mathbb{R}^{2n}$ or even lower than that. So, we should stick to that.

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For every topological manifold X, if you looked at all the embedding inside $\mathbb{I}^{2n+1}$ is dense in the space function space $(\mathbb{I}^{2n+1})^X$. What is this function space, all continuous functions from X to $\mathbb{I}^{2n+1}$. Out of which you take only the embeddings, that subset would be dense in the entire space. The topology on this function space is the compact-open-topology, or you can call it the topology of uniform convergence.

If you want to closed embeddings, then you have to take X to be a compact subset because $\mathbb{I}^{2n+1}$ is compact and closed subset of a compact set is compact. So this is the result. I have chosen not to prove this result. The proof is quite lengthy and complicated and not very illuminating either. Therefore, we shall only state a very mild form of this theorem, only for compact spaces and we shall be liberal with this the dimension as well namely $\mathbb{I}^N$ for some large N and not bother to find one which depends only on n. So that is what we are going to do now.

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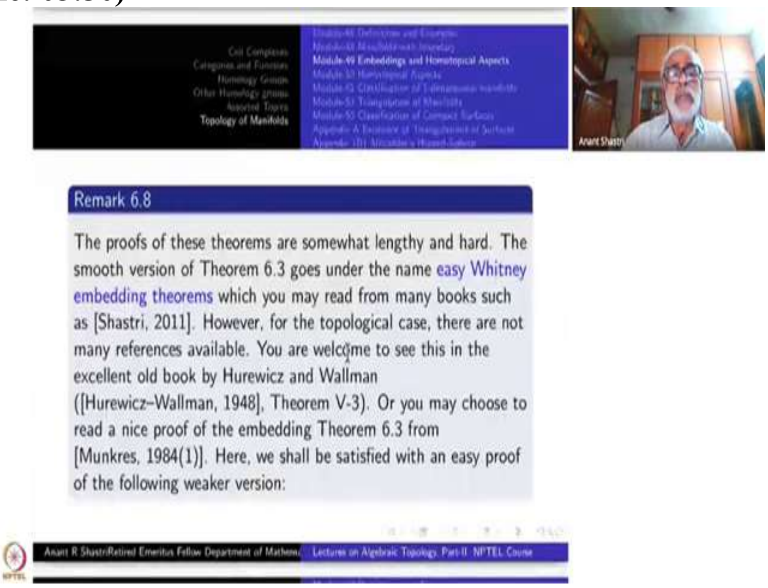

However here is reference. The proof of these theorems are somewhat lengthy. You know there are smooth versions which are slightly easy. They are under the name easy Whitney embedding theorems which have easier proofs also. You may read them from many books such as my own book on differential topology.

However, for the topological case, there are not many references available you are welcome to see this in an excellent old book by Hurewicz-Wallman. I have given the reference here. It is a wonderful book. Or you may choose to read a nice proof of embedding theorem 6.3 from Munkres' book. We shall be satisfied with an easy proof of the following weaker version.

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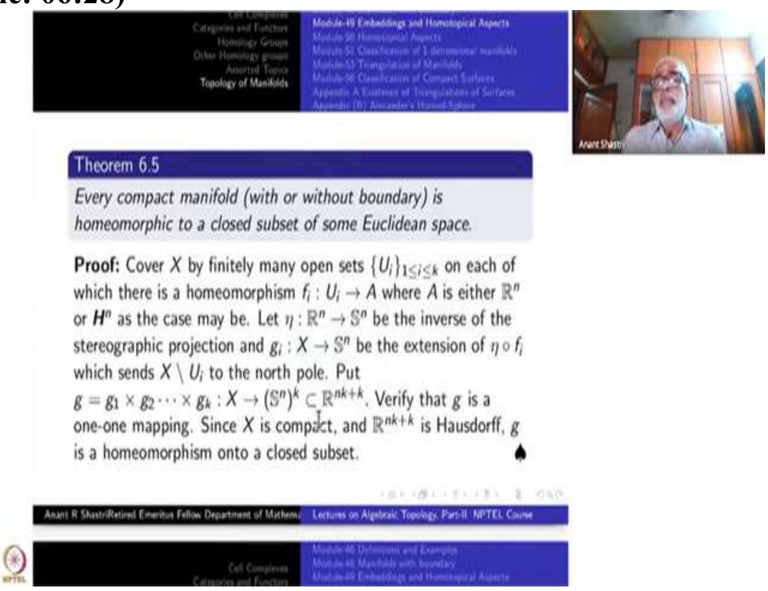

Namely, every compact manifold (with or without boundary) is homeomorphic to a closed subset of some Euclidean space $\mathbb{R}^m$. So, we are not bothered about how large m has to be taken and that is why the proof is very easy. Let us see how. For each $x \in X$, choose an open neighbourhood of U_x of $x \in X$ and a homeomorphism f_x from U_x to A, where A is the whole of $\mathbb{R}^n$ or the whole of $\mathbf{H}^n$, according as x in in the interior or boundary of X. Since X is compact, there will be a finite atlas and we label them $\{(U_i, f_i)i = 1, 2, \ldots k\}$.

Let η from $\mathbb{R}^n$ to $\mathbb{S}^n$ setminus the north pole be the inverse of the stereographic projection ϕ from $\mathbb{S}^n$ setminus the north pole to $\mathbb{R}^n$, which we know is a homeomorphism. Now look at $\eta_i \circ f_i$ from U_i to $\mathbb{S}^n$ and let g_i from X to $\mathbb{S}^n$ be the extension of η composed with f_i which sends the entire of $X \setminus U_i$ to the north pole. Arguments involving 1-pt compactification or imply using properness of homeomorphisms, you can easily check that g_i are continuous.

Put g equal to the product function $g_1 \times \cdots \times g_k$ from X to $\mathbb{S}^n \times \cdots \times \mathbb{S}^n$ (k copies), I have got some continuous functions g_i. What is the property of these continuous functions restricted to each U_i? They are one-one mappings, they embeddings, but outside of U_i, they are constant functions. But look at g. We will see that this map is finally what we want. It is an embedding of X into $(\mathbb{S}^n)^k$.

It is enough to verify that g is one-one. Then since X is compact, automatically g will be a homeomorphism on its image and hence an embedding, since the codomain is Hausdorff. The proof will be over.

Verifying that g is a one-one mapping, I have left to you as an exercise in the slide. But now I will do that in a minute. So why is g a one-one map tell me. You ask why it is not. For that there must two distinct points x, y in X such that $g(x) = g(y)$. If both x and y happened to be inside the same U_i, then $g_i(x)$ will be different from $g_i(y)$, g_i is injective on U_i. But then $g(x)$ not equal to $g(y)$. So x is in U_i and y is not in U_i. But then $g_i(x)$ is not equal to $g_i(y)$ which is equal to the north pole. Therefore once again $g(x)$ not equal to $g(y)$.

So, now we will use this property that every manifold can be realized as a subspace of some Euclidean space, even though we proved it only for compact manifolds.

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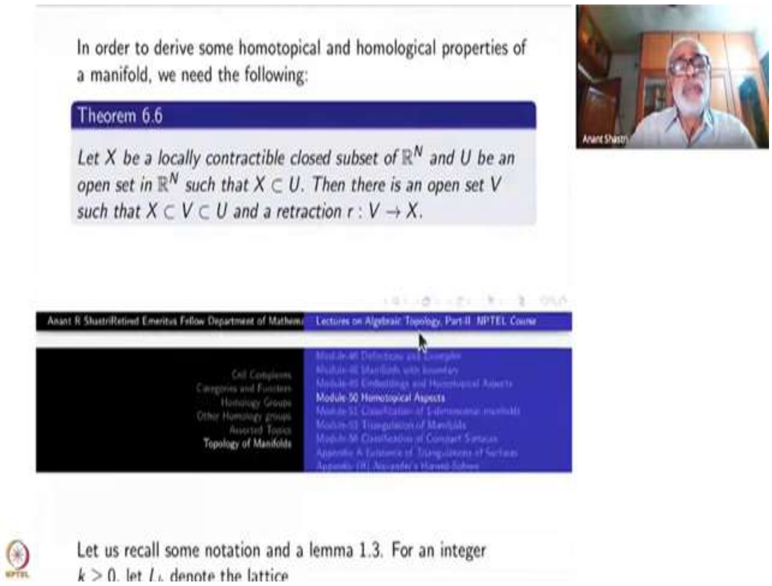

So, in order to derive some homotopical and homological properties of manifold, the first thing we prove is the following theorem and perhaps this is the only result that we can prove today. So, have some patience because this is slightly longer.

Let X be a locally contractible, closed subset of $\mathbb{R}^N$ and U be an open subset of $\mathbb{R}^N$ such that X is inside U. That means this U is a neighbourhood of our subset X. What is the assumption on X? It is a closed subset and it is locally contractible. Then there is an open subset V of $\mathbb{R}^N$ such that X is inside V contained in U and a retraction r from V to X. In other words, every closed subset of $\mathbb{R}^n$ which is locally contractible is a neighbourhood retract.

In fact, you can say that it is a deformation retract and so on. That is what leads to our concept of co-fibration etc. I am not trying to prove such a strong result here. Only retraction every closed subset which locally contractible is a retract of a neighbourhood and that neighbourhood can be chosen as small as you please this is the meaning of this theorem.

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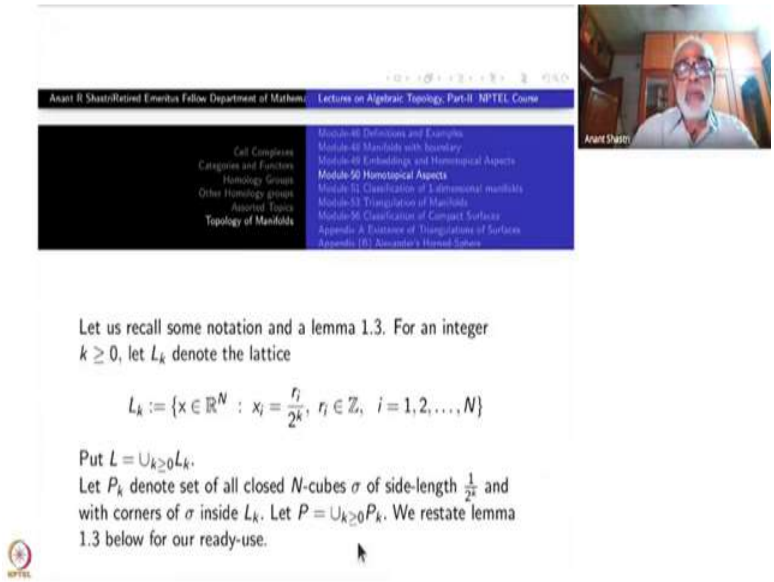

Let us see how we are going to do that. So, recall our notation and lemma for the so called lattice structure in $\mathbb{R}^N$. Remember that, it was used to obtain examples of CW-complexes and so on and while proving CW-approximation theorem we have used that one. So, let us recall them from Module 4B.

So, this k is fixed integer, $k \geq 0$. L_k denotes the set of points $x \in \mathbb{R}^N$, all of whose coordinates x_i are rational numbers of the form an integer divided by 2^k. So, L_0 is the the set of so called lattice points, with all coordinates as integers, L_2 has points with cooridanates half integers and so on. The lines planes you know drawn at length at the interval 1 at all the integer points N_2 will be at half integer points.

Let P_k denote set of all closed N-cubes sigma of side-length $1/2^k$ with their corners inside L_k and P equal to union of all P_k's. I am just recalling all these things which we have done earlier in Module 4B and we restate a theorem for our ready use here.

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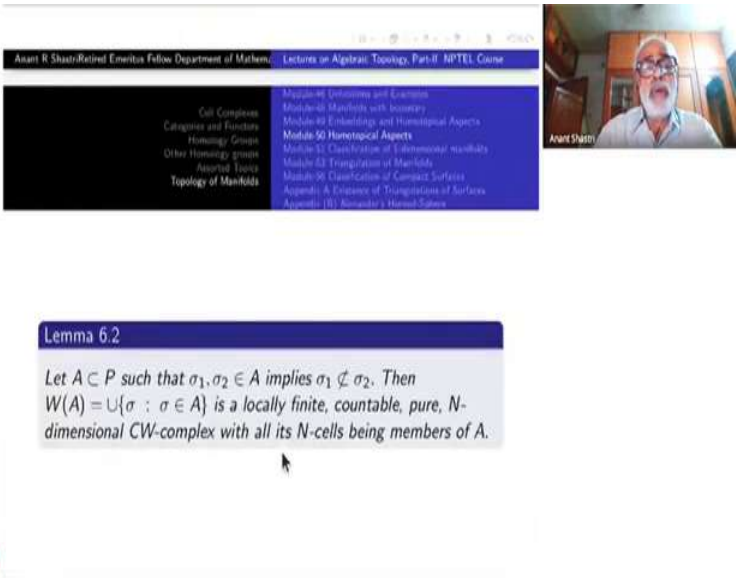

(Editors note: The speech diverts quite a bit from the slides here, almost reproving the theorem in a slightly different way but for the proof of the theorem either of them will do. So we keep both versions for the benefit of the students.)

So a lemma: Take a subset A of P such that σ_1 and σ_2 belong to A implies σ_1 is not contained in σ_2. (Remember that these σ_i's are N-dimensional cubes of differerent sizes in $\mathbb{R}^N$. So, for instance if we have $(1/4)$-th size cube a $(1/8)$-th size cube it is likely that the second one may be a subset of the first one. That should not happen. They may intersect each other. That is allowed.) Let $W(A)$ be the union of all the σ's belonging to A. Then the subspace $W(A)$ of $\mathbb{R}^N$ has a locally finite, countable, pure, N-dimensional CW-complex structure with all its N-cells being precisely the members of A.

We have seen that this CW-structure actually can be further `cut-down' into a simplicial complex by cutting each N-cube into a simplicial complex. So, that is the lemma that is relevant for now. We directly go to the proof of the theorem. I have to produce this open set U' as well a retraction r.

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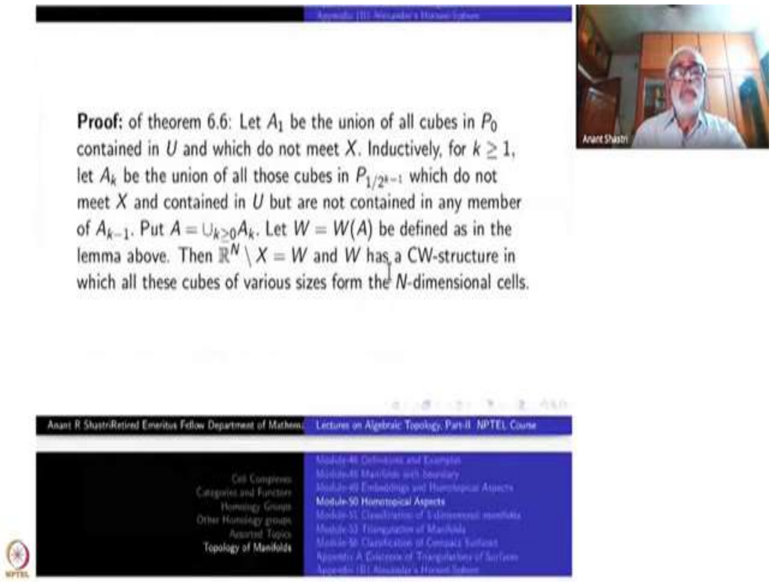

So, let A_0 be collection of all cubes in P_0 (P_0 is what N-cubes of side length 1, with corners as integer coordinates) contained in U and which do not meet X. This set may be empty, I do not care. Let A_1 be the collection of all cubes in P_1 contained in U and which do not meet X and not contained in any member of A_0. Inductively, for $k \geq 1$, let A_k be the collection of all those cubes in P_k contained in U, not meeting X and which are not contained in any member of A_i, for $i < k$. Put A equal to union of all A_i's.

Let $W = W(A)$ be defined as the lemma. By the very definition A will satisfy the condition of this lemma, namely, σ_1 and σ_2 are in A, will imply that σ_1 is not contained in σ_2. Once a cube is admitted inside A, no smaller cubes inside it will be taken. So, that is why this condition is automatically satisfied.

Therefore, we can apply the lemma. $W = W(A)$ will be defined as the union of all $\sigma \in A$. Clearly, W is a subset of $U \setminus X$. W has a CW structure in which all these N-cubes of various sizes in A. Given any point $x \in U \setminus X$, which is open in $\mathbb{R}^N$, we can choose k sufficiently large so that there is a member sigma of P_k which contains x and σ contained in $U \setminus X$. Therefore, σ belongs to A_k and x belongs to W. Therefore, $W = V \setminus X$.

Put W_k equal to the union of all cubes A_i for $i < k+1$. Then each W_k will be a closed subset of $\mathbb{R}^N$, being a union of a locally finitely family of closed sets. Also W= union of all W_k. So the funny thing here is that to begin with we are not doing anything to X but the complement of X in U has been given a nice CW structure, these cubes of various sizes all of them N-dimension cells, smaller and smaller maybe smaller which keep coming nearer and nearer X

but will not intersect it.

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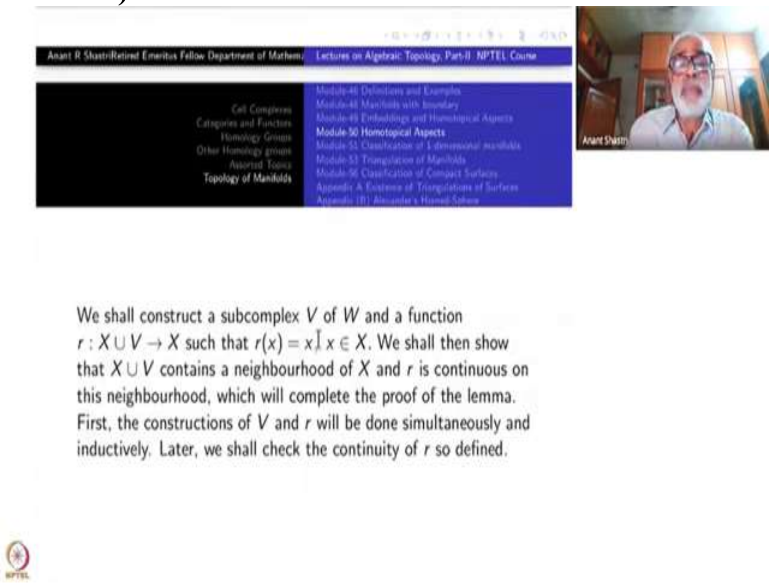

So, that is what it is now. Let us see how we shall construct an open neighbourhood U' of X in U and a retraction r on U' which will complete the proof of the theorem 6.6. First we are actually going to construct a subcomplex K of W, and a retraction r from $X \cup K$ to X. (Here we take the liberty to write the same symbol K for the complex K as well as the underlying subspace $|K|$).

We shall then show that $X \cup K$ contains an open set U' which contains X and r is continuous on U'. That may be a wishful thinking and it takes sometime to prove each of them. Let us go ahead with this. First the constructions of K and r will be done simultaneously and inductively, keeping in mind that r should be continuous on a suitable subset.

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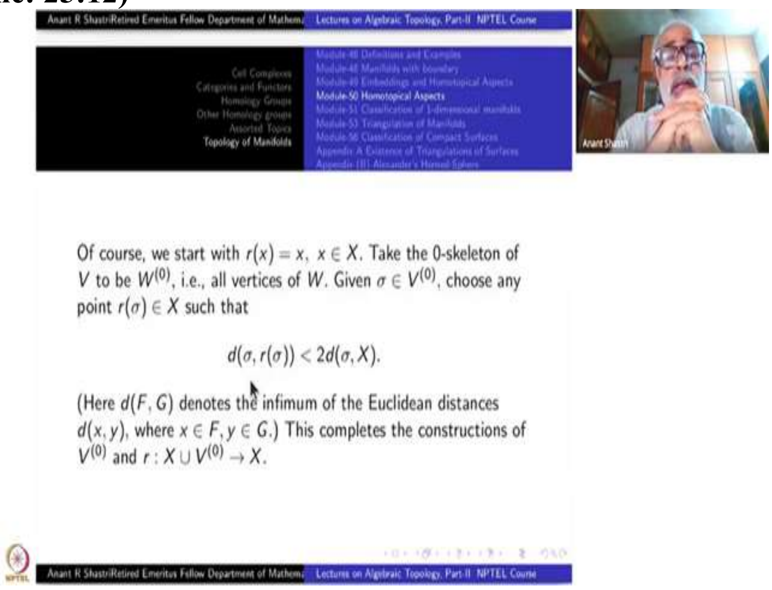

Of course, we start with $r(x) = x$, because there is no other choice, since r has to be retraction onto X. So, $r(x) = x$. Take the 0-skeleton of K to be $W^{(0)}$ (Do not confuse it for W_0.) We are going to define a subcomplex of K of W. So, take all the 0-cells of W. These are some lattice points, away from X.

Given a 0-cell σ, choose $r(\sigma)$ to be any point in X such that the distance between σ and $r(\sigma)$ is less than twice the distance between X and σ. Remember σ in W does not intersect X which is a closed set so the distance is positive. Here the distance between two subsets A, B of $\mathbb{R}^N$ is nothing but the infimum of all $d(a, b)$ where a ranges over A and b ranges over B. Therefore such a point $r(\sigma)$ exists.

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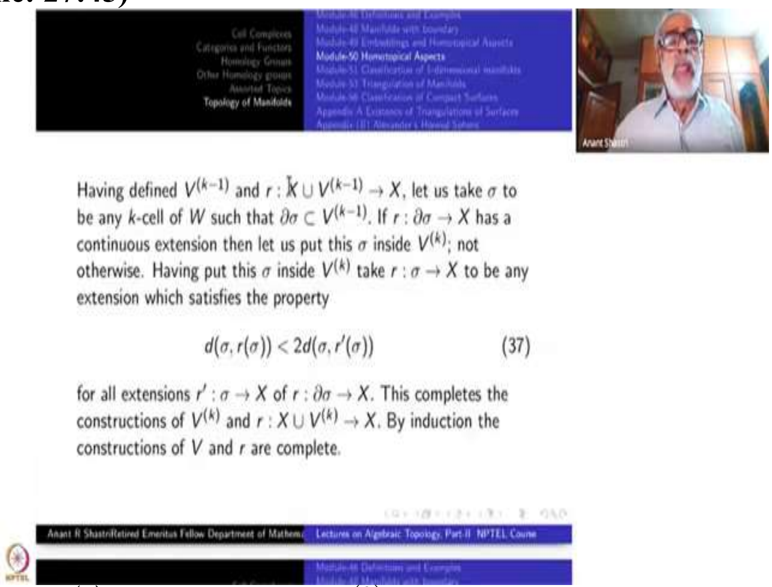

The construction of $K^{(0)}$ and r from $X \cup K^{(0)}$ to X is over. Actually, here you can easily see that this r has to be continuous function because on X it is identity and $K^{(0)}$ is discrete. But let us not discuss continuity right now. Let us go ahead. Having defined $K^{(k-1)}$ and r from $X \cup K^{(k-1)}$ to X, first of all, let us take σ to be any k-cell in W such that its entire boundary is contained in $K^{(k-1)}$. You know σ is a k-dimensional box, its boundary consists of $(k-1)$-dimensional boxes. (For instance, for $k = 1$, you could select any 1-cell in W, because you have admitted all the 0-cells of W inside $K^{(0)}$. But wait.)

Now look at r restricted to $\partial(\sigma)$, which is already defined by the induction hypothesis. If this extends continuously to a function from σ to X, then and then only admit σ inside $K^{(k)}$. In general, we do not know if there is such a continuous extension. So this is a non-vacuous condition. If there is no such extension, do not take this cell σ in $K^{(k)}$.

Now, choose r from σ to X to be any continuous extension which satisfies the property that amongst all such continuous extensions r', you must take one such that distance between σ and $r(\sigma)$ is less than twice the distance between σ and $r'(\sigma)$ for all such extensions r'.

You see if you take the set of all distances between σ and $r'(\sigma)$ where r' ranges over the non empty set of continuous extensions of r restricted to $\partial(\sigma)$, and look at its infimum, it will be positive being bigger that distance between σ and X. However, we do not know whether this infimum can be attained. So, I have doubled these distances so that one of them among all extensions r' from σ to X, will definitely satisfy this condition, because a positive number cannot be smaller than twice itself. So, there must be one r which satisfies this property. We can now extend r over the whole of $K^{(k)}$ continuously. By induction, the construction of K which is equal to union of all its skeletons $K^{(k)}$ and r is over.

It remains to prove a number of things, that $X \cup K$ contains a neighbourhood U' of X and r is continuous on U'. Once you have done that, the proof is over. So, let us do that.

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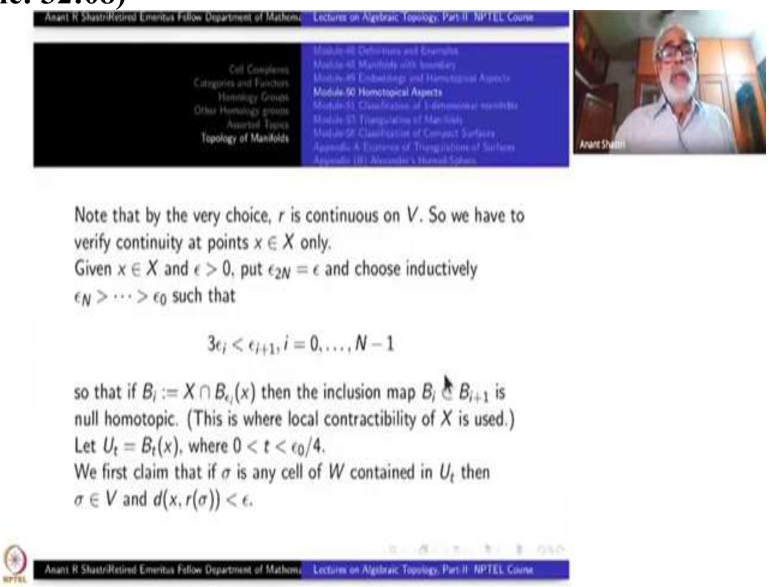

Note that by the very choice r is continuous on K because for any CW complex a function is continuous iff restricted to each cells it is continuous. So, we have to verify continuity at points of X. You see r restricted to X is identity. That it is continuous on X. However, that does not mean that as a function from $X \cup K$ to X, it is continuous at x. So many people make this mistake, since r restricted to X being identity continuous. (The trouble is because

X is not open in $X \cup K$.)

But we do not need r to be continuous on the whole of $X \cup K$. So, we must achieve continuity at points of X but we may cut down the space $X \cup K$. Remember that. So, given x belonging to X and ϵ positive, let us put ϵ_{2N} equal to ϵ. So go on choosing $\epsilon_i > 0$ in the reverse order, for $i = 2n-1, 2n-2, \ldots$ till you hit $i = 0$, such that $3\epsilon_i$ is less than ϵ_{i+1}.

Put B_i equal to $X \cap B_{\epsilon_i}(x)$, where $B_\epsilon(x)$ is the the standard open ball in $\mathbb{R}^N$. Further, we need ϵ_i to satisfy the condition that inclusion map B_i to B_{i+1} is null homotopic. So, how do you ensure this? This is where local contractibility of X has to be used. What is the meaning of local contractibility: given any neighbourhood of any point in X, we must have a smaller neighbourhood such that the inclusion map of the smaller in the larger one is null homotopic. So, I do not say that this ϵ_i is exactly one third of ϵ_{i+1}. No, it is possible to choose ϵ_i smaller than that so as to satisfy this homotopy condition also.

Let U_t be $B_t(x)$, where $0 < t < \epsilon_0/4$. After choosing the last ϵ_0, I am going to further divided it 4 and take a positive t smaller than that. As a step toward continuity, we claim that if σ is any cell of W, contained in this U_t, then σ will be automatically in V, and the distance between x and $r(\sigma)$ (remember that $r(\sigma)$ was some point of X) will be less than ϵ. The original ϵ looks very big now, because we have come down to $\epsilon_0/4$. We shall prove this by induction on the dimension of the σ, by proving slightly stronger claims in the inductive steps.

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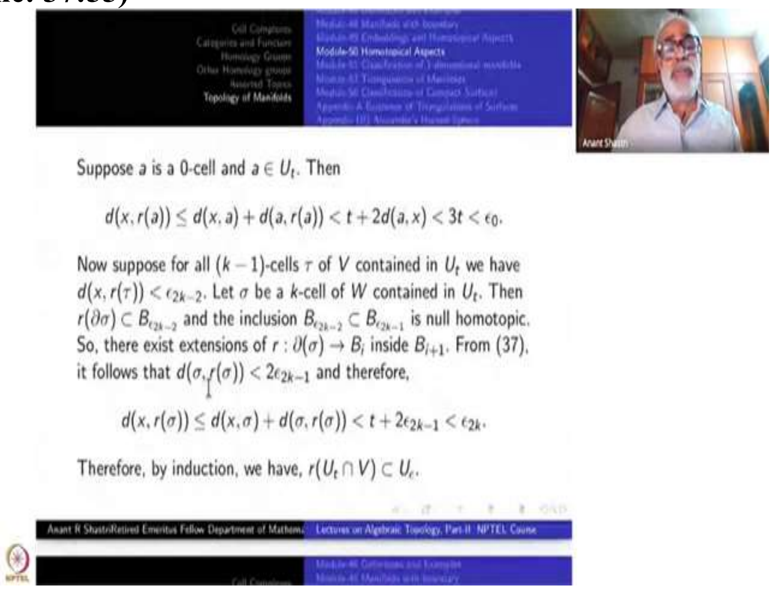

Suppose a is a 0-cell in W and U_t. Already by definition $K^{(0)} = W^{(0)}$ and so a belongs to V. Now the distance between x and $r(a)$ is less than the distance between x and a plus the distance between a and $r(a)$. Since x and a are inside U_t, the first term is less that t and second one is, by the definition of r, less than twice distance between a and x. So the sum is less than $3t < \epsilon_0$, because $t < \epsilon/4$. Of course ϵ_0 is less than ϵ. So, the inductive step for 0 is proved.

Now, suppose for all $(k-1)$-cells τ of W contained in U_t, we have proved that τ is in K and that distance between x and $r(\tau)$ is less than ϵ_{2k-2}. We want to prove this to be less than ϵ but we need to have this stronger inductive hypothesis to go to the next step. Any way we have proved it for 0-cells. Let now σ be a k-cell of W contained in U_t. Then boundary of σ will be contained inside $B_{\epsilon_{2k-2}}$ from the induction hypothesis since the boundary of a k-cell in W is the finite union of $(k-1)$-cells W. Also, we have the inclusion map B_i to B_{i+1} is null homotopic. (That is important now.) Therefore, you can always extend r which is defined on $\partial(\sigma)$ to a map on σ into $B_{\epsilon_{2k-1}}$ continuously. So, first of all σ belongs to K. Moreover, from (43), it follows that the distance between σ and $r(\sigma)$ will be less than $2\epsilon_{2k-1}$. Therefore, once again, the same thing happens here, viz., $d(x, r(\sigma))$ is less than $d(x,\sigma) + d(\sigma, r(\sigma))$. This is always less than t and the second term is less than $2\epsilon_{2k-1}$ so the sum is less than ϵ_{2k}. Therefore, by induction, we have proved the claim, because ϵ_{2k} is always less that ϵ for all $k = 0, \ldots, N$.

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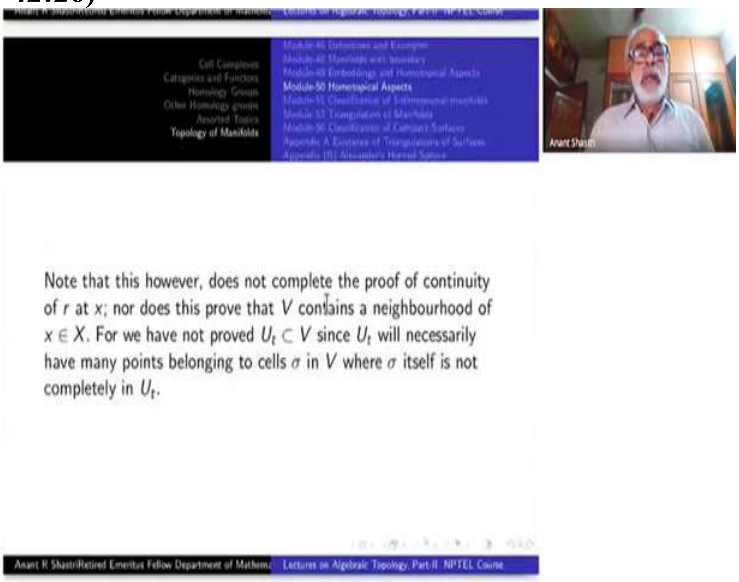

Note that this however does not complete the proof of continuity of r at x. Nor does this prove that $X \cup K$ contains a neighbourhood of X in $\mathbb{R}^N$. But this is a good step toward continuity. We have to prove one more step for we have not yet proved that U_t itself is contained inside K. Since U_t may have many points belongings to the cells σ in K where σ itself is not completely inside U_t.

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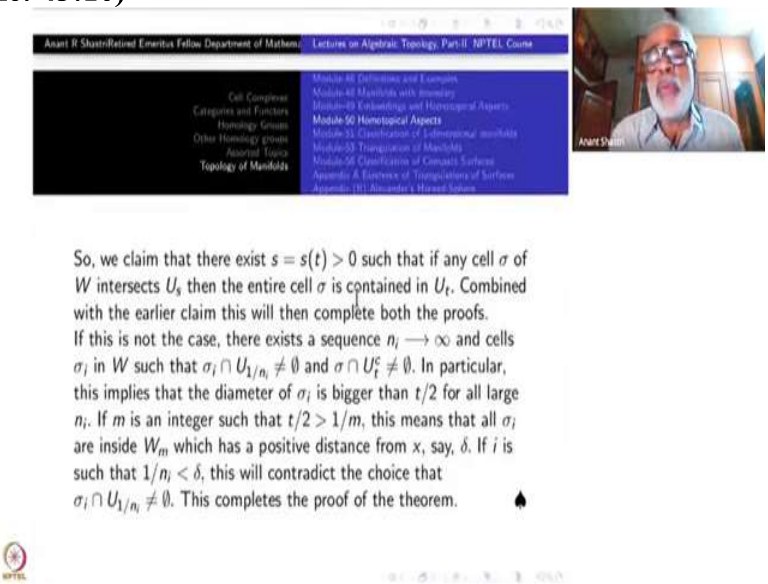

So, we claim that there exists $0 < s = s(x,t) < t$ (depending upon x as well as t) such that if any cell σ of W intersects U_s, then the entire cell σ is contained in U_t. This would clearly imply that $r(U_s)$ is contained in U_ϵ, since we have proved earlier that $r(U_t \cap K)$ is contained in U_ϵ. Therefore the continuity at the point x for the function r from $X \cup K$ to X follows.

By choosing ϵ to be any fixed number, say $\epsilon = 1$ for all $x \in X$, and taking $U' =$ union of all $U_{s(x,\epsilon_0/5)}$, we get an open subset of $X \cup K$ which is a subset of U and contains X. This will then complete the proof. So, it remains to prove the existence of $s(x,t)$ as claimed.

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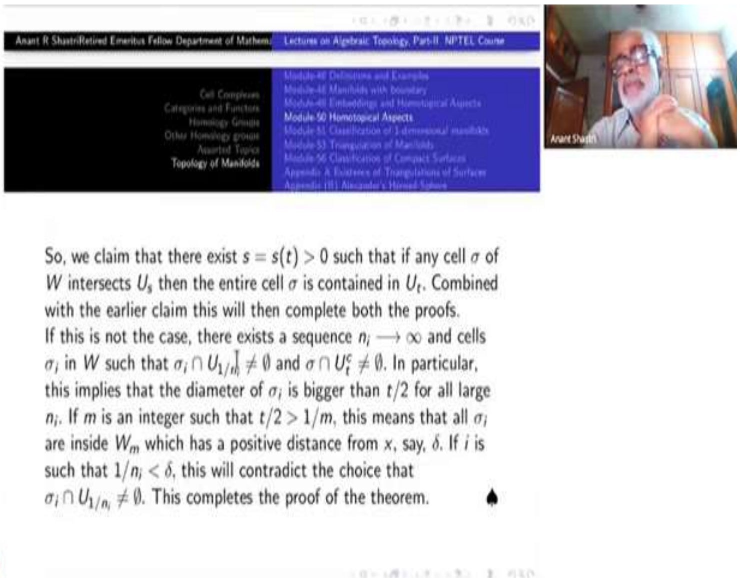

So, if this is not the case what happens? That means that there exists a sequence n_i converging into infinity such that none of the numbers $s_i = 1/n_i$ satisfy this property, meaning, there are cells σ_i in W such that σ intersects both U_{s_i} as well as U_t^c. In particular, this implies that the diameter of σ_i is bigger than $t/2$ for large i. viz., for all i such that $1/n_i < t/2$. Note that the diameter of any N-cells in P_k is equal to $\sqrt{N}/2^k$. So choose an integer k such that diameter of any N-cell in P_k is smaller than $t/2$. Then none of these σ_i are in P_m for any $m < k$ and hence they are contained in W_k. But W_k has a positive distance say, $\delta(x)$ from x. Therefore if $1/n_i < \delta(x)$, it is a contradiction to the fact that σ_i intersects U_{1/n_i}. This proves the theorem 6.6.

Using this theorem, next time we shall prove one of the very important things answering a question that we had raised earlier, namely, whether' the fundamental group of any topological manifold is countable.

That is the homotopical aspect of this one that is what retraction is already some kind of homotopy though we have not proved that it is a deformation retraction, that is not needed. So, let us stop here. Thank you.