

**Introduction to Algebraic Topology (Part-II)**  
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**Lecture - 49**  
**Embeddings and Homotopical Aspects continued**

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Cell Complexes  
Categories and Functors  
Homotopy Groups  
Other Homotopy groups  
Algebraic Topology  
Topology of Manifolds

Manifolds: Definitions and Examples  
Manifolds with Boundary  
**Module-49 Embeddings and Homotopical Aspects**  
Module-50 Homotopical Aspects  
Module-51 Classification of 1-dimensional manifolds  
Module-52 Triangulation of Manifolds  
Module-53 Classification of Compact Surfaces  
Appendix A Existence of Triangulations of Surfaces  
Appendix B Alexander's Horned Sphere

**Module-49 Embeddings and Homotopical Aspects**

The following result tells us that after all, we could have just stuck to the study of subsets of Euclidean spaces for studying manifolds. As we shall see, this single result has several implications on topological, homotopical and homological properties of a manifold.

**Theorem 6.3**

*Every  $n$ -manifold is homeomorphic to a closed subset of  $\mathbb{R}^{2n+1}$ .*

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So, far, in the study of manifolds, we introduced the notion of a topological manifolds, gave some examples and then introduced the notion of manifolds with boundary also. We have also shown that every manifold is paracompact including the manifolds with boundary. We also showed that the boundaries of a bounded manifold have what are called Collar neighbourhoods. So, today we shall study some homotopic aspects.

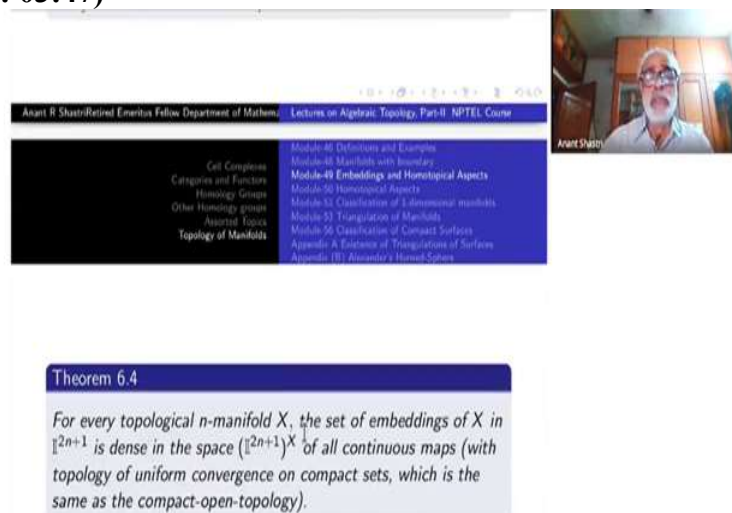
A key to that is the result that every topological manifold is a subspace of some large Euclidean space. That is the meaning of embedding. So, every manifold can be embedded inside a Euclidean space. So, as such, it seems that in the definition of a manifold, we do not have to go out of  $\mathbb{R}^n$ , and could have taken only topological subspace for  $\mathbb{R}^n$  which are locally Euclidean, such as a circle or a spheres union of lines and so on.

However, in practice what happens is manifolds many not arise may not occur naturally as subspace of  $\mathbb{R}^n$ . They arise in different forms, especially when as quotients of some familiar objects. Then it is a burden to see them first of all as subspaces of  $\mathbb{R}^n$  even before identifying them as manifolds. So, the abstract definition has this advantage.

So, let us anyway do this embedding theorem, which will itself help in the study of other aspects of manifolds. As we shall see this single result has several implications on topological and homological properties of a manifold. Being a subspace of some Euclidean space is itself something very special.

Every  $n$ -manifold is homeomorphic to a closed subset of  $\mathbb{R}^{2n+1}$ . Start with a manifold of dimension  $n$ . Irrespective of how complicated it may be, you do not have to go for a very large  $m$  to get an embedding into  $\mathbb{R}^m$ ,  $m = 2n + 1$  will do. So, this is quite tight. There are examples wherein you may not be able to do it in  $\mathbb{R}^{2n}$  or even lower than that. So, we should stick to that.

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The screenshot shows a video lecture interface. At the top, there is a header bar with the text "Anant R Shastri Retired Emeritus Fellow Department of Mathematics" and "Lectures on Algebraic Topology, Part-II, NPTEL Course". Below this is a table of contents with two columns. The left column lists: "Cell Complexes", "Categories and Functors", "Homology Groups", "Other Homology groups", "Algebraic Topology", and "Topology of Manifolds". The right column lists: "Module-46: Definitions and Examples", "Module-48: Manifolds with Boundary", "Module-49: Embeddings and Homotopical Aspects", "Module-50: Homotopical Aspects", "Module-51: Classification of 1-dimensional manifolds", "Module-52: Triangulation of Manifolds", "Module-53: Classification of Compact Surfaces", and "Appendix A: Examples of Triangulations of Surfaces" and "Appendix B: Alexander's Horned Sphere". In the top right corner, there is a small video feed of the lecturer, Anant Shastri. Below the table of contents, there is a slide titled "Theorem 6.4" which states: "For every topological  $n$ -manifold  $X$ , the set of embeddings of  $X$  in  $\mathbb{I}^{2n+1}$  is dense in the space  $(\mathbb{I}^{2n+1})^X$  of all continuous maps (with topology of uniform convergence on compact sets, which is the same as the compact-open-topology)."

For every topological manifold  $X$ , if you looked at all the embedding inside  $\mathbb{I}^{2n+1}$  is dense in the space function space  $(\mathbb{I}^{2n+1})^X$ . What is this function space, all continuous functions from  $X$  to  $\mathbb{I}^{2n+1}$ . Out of which you take only the embeddings, that subset would be dense in the entire space. The topology on this function space is the compact-open-topology, or you can call it the topology of uniform convergence.

If you want to closed embeddings, then you have to take  $X$  to be a compact subset because  $\mathbb{I}^{2n+1}$  is compact and closed subset of a compact set is compact. So this is the result. I have chosen not to prove this result. The proof is quite lengthy and complicated and not very illuminating either. Therefore, we shall only state a very mild form of this theorem, only for compact spaces and we shall be liberal with this the dimension as well namely  $\mathbb{I}^N$  for some

large  $N$  and not bother to find one which depends only on  $n$ . So that is what we are going to do now.

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Cell Complexes Categories and Functors Homology Groups Other Homology groups Algebraic Topology Topology of Manifolds	Module-40: Definitions and Examples Manifolds with Boundary <b>Module-49: Embeddings and Homotopical Aspects</b> Module-50: Homotopical Aspects Module-51: Classification of 1-dimensional manifolds Module-52: Triangulation of Manifolds Module-53: Classification of Compact Surfaces Appendix A: Examples of Triangulations of Surfaces Appendix B: Alexander's Theorem
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Remark 6.8

The proofs of these theorems are somewhat lengthy and hard. The smooth version of Theorem 6.3 goes under the name **easy Whitney embedding theorems** which you may read from many books such as [Shastri, 2011]. However, for the topological case, there are not many references available. You are welcome to see this in the excellent old book by Hurewicz and Wallman ([Hurewicz-Wallman, 1948], Theorem V-3). Or you may choose to read a nice proof of the embedding Theorem 6.3 from [Munkres, 1984(1)]. Here, we shall be satisfied with an easy proof of the following weaker version:


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However here is reference. The proof of these theorems are somewhat lengthy. You know there are smooth versions which are slightly easy. They are under the name easy Whitney embedding theorems which have easier proofs also. You may read them from many books such as my own book on differential topology.

However, for the topological case, there are not many references available you are welcome to see this in an excellent old book by Hurewicz-Wallman. I have given the reference here. It is a wonderful book. Or you may choose to read a nice proof of embedding theorem 6.3 from Munkres' book. We shall be satisfied with an easy proof of the following weaker version.

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Cell Complexes Categories and Functors Homology Groups Other Homology groups Algebraic Topology Topology of Manifolds	Module-40: Definitions and Examples Manifolds with Boundary <b>Module-49: Embeddings and Homotopical Aspects</b> Module-50: Homotopical Aspects Module-51: Classification of 1-dimensional manifolds Module-52: Triangulation of Manifolds Module-53: Classification of Compact Surfaces Appendix A: Examples of Triangulations of Surfaces Appendix B: Alexander's Theorem
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Theorem 6.5

*Every compact manifold (with or without boundary) is homeomorphic to a closed subset of some Euclidean space.*

**Proof:** Cover  $X$  by finitely many open sets  $\{U_i\}_{1 \leq i \leq k}$  on each of which there is a homeomorphism  $f_i : U_i \rightarrow A$  where  $A$  is either  $\mathbb{R}^n$  or  $H^n$  as the case may be. Let  $\eta : \mathbb{R}^n \rightarrow S^n$  be the inverse of the stereographic projection and  $g_i : X \rightarrow S^n$  be the extension of  $\eta \circ f_i$  which sends  $X \setminus U_i$  to the north pole. Put  $g = g_1 \times g_2 \times \dots \times g_k : X \rightarrow (S^n)^k \subset \mathbb{R}^{nk+k}$ . Verify that  $g$  is a one-one mapping. Since  $X$  is compact, and  $\mathbb{R}^{nk+k}$  is Hausdorff,  $g$  is a homeomorphism onto a closed subset.

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Namely, every compact manifold (with or without boundary) is homeomorphic to a closed subset of some Euclidean space  $\mathbb{R}^m$ . So, we are not bothered about how large  $m$  has to be taken and that is why the proof is very easy. Let us see how. For each  $x \in X$ , choose an open neighbourhood  $U_x$  of  $x \in X$  and a homeomorphism  $f_x$  from  $U_x$  to  $A$ , where  $A$  is the whole of  $\mathbb{R}^n$  or the whole of  $\mathbb{H}^n$ , according as  $x$  is in the interior or boundary of  $X$ . Since  $X$  is compact, there will be a finite atlas and we label them  $\{(U_i, f_i) | i = 1, 2, \dots, k\}$ .

Let  $\eta$  from  $\mathbb{R}^n$  to  $\mathbb{S}^n$  setminus the north pole be the inverse of the stereographic projection  $\phi$  from  $\mathbb{S}^n$  setminus the north pole to  $\mathbb{R}^n$ , which we know is a homeomorphism. Now look at  $\eta \circ f_i$  from  $U_i$  to  $\mathbb{S}^n$  and let  $g_i$  from  $X$  to  $\mathbb{S}^n$  be the extension of  $\eta$  composed with  $f_i$  which sends the entire of  $X \setminus U_i$  to the north pole. Arguments involving 1-pt compactification or imply using properness of homeomorphisms, you can easily check that  $g_i$  are continuous.

Put  $g$  equal to the product function  $g_1 \times \dots \times g_k$  from  $X$  to  $\mathbb{S}^n \times \dots \times \mathbb{S}^n$  ( $k$  copies), I have got some continuous functions  $g_i$ . What is the property of these continuous functions restricted to each  $U_i$ ? They are one-one mappings, they embeddings, but outside of  $U_i$ , they are constant functions. But look at  $g$ . We will see that this map is finally what we want. It is an embedding of  $X$  into  $(\mathbb{S}^n)^k$ .

It is enough to verify that  $g$  is one-one. Then since  $X$  is compact, automatically  $g$  will be a homeomorphism on its image and hence an embedding, since the codomain is Hausdorff. The proof will be over.

Verifying that  $g$  is a one-one mapping, I have left to you as an exercise in the slide. But now I will do that in a minute. So why is  $g$  a one-one map tell me. You ask why it is not. For that there must two distinct points  $x, y$  in  $X$  such that  $g(x) = g(y)$ . If both  $x$  and  $y$  happened to be inside the same  $U_i$ , then  $g_i(x)$  will be different from  $g_i(y)$ ,  $g_i$  is injective on  $U_i$ . But then  $g(x)$  not equal to  $g(y)$ . So  $x$  is in  $U_i$  and  $y$  is not in  $U_i$ . But then  $g_i(x)$  is not equal to  $g_i(y)$  which is equal to the north pole. Therefore once again  $g(x)$  not equal to  $g(y)$ .

So, now we will use this property that every manifold can be realized as a subspace of some Euclidean space, even though we proved it only for compact manifolds.

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In order to derive some homotopical and homological properties of a manifold, we need the following:

**Theorem 6.6**

Let  $X$  be a locally contractible closed subset of  $\mathbb{R}^N$  and  $U$  be an open set in  $\mathbb{R}^N$  such that  $X \subset U$ . Then there is an open set  $V$  such that  $X \subset V \subset U$  and a retraction  $r : V \rightarrow X$ .

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Cell Complexes  
Coproducts and Products  
Homology Groups  
Other Homology groups  
Associated Functors  
Topology of Manifolds

Module 40: Topology and Examples  
Module 41: Manifolds with boundary  
Module 42: Embeddings and Homotopical Aspects  
Module 50: Homotopical Aspects  
Module 51: Classification of 1-dimensional manifolds  
Module 52: Topology of Manifolds  
Module 53: Classification of Compact Surfaces  
Appendix A: Techniques of Topology of Surfaces  
Appendix B: Alexander's Lemma


Let us recall some notation and a lemma 1.3. For an integer  $k > 0$ , let  $I_k$  denote the lattice

So, in order to derive some homotopical and homological properties of manifold, the first thing we prove is the following theorem and perhaps this is the only result that we can prove today. So, have some patience because this is slightly longer.

Let  $X$  be a locally contractible, closed subset of  $\mathbb{R}^N$  and  $U$  be an open subset of  $\mathbb{R}^N$  such that  $X$  is inside  $U$ . That means this  $U$  is a neighbourhood of our subset  $X$ . What is the assumption on  $X$ ? It is a closed subset and it is locally contractible. Then there is an open subset  $V$  of  $\mathbb{R}^N$  such that  $X$  is inside  $V$  contained in  $U$  and a retraction  $r$  from  $V$  to  $X$ . In other words, every closed subset of  $\mathbb{R}^n$  which is locally contractible is a neighbourhood retract.

In fact, you can say that it is a deformation retract and so on. That is what leads to our concept of co-fibration etc. I am not trying to prove such a strong result here. Only retraction every closed subset which locally contractible is a retract of a neighbourhood and that neighbourhood can be chosen as small as you please this is the meaning of this theorem.


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Arvind Chatterjee

Let us recall some notation and a lemma 1.3. For an integer  $k \geq 0$ , let  $L_k$  denote the lattice

$$L_k := \{x \in \mathbb{R}^N : x_i = \frac{r_i}{2^k}, r_i \in \mathbb{Z}, i = 1, 2, \dots, N\}$$

Put  $L = \bigcup_{k \geq 0} L_k$ .

Let  $P_k$  denote set of all closed  $N$ -cubes  $\sigma$  of side-length  $\frac{1}{2^k}$  and with corners of  $\sigma$  inside  $L_k$ . Let  $P = \bigcup_{k \geq 0} P_k$ . We restate lemma 1.3 below for our ready-use.

Let us see how we are going to do that. So, recall our notation and lemma for the so called lattice structure in  $\mathbb{R}^N$ . Remember that, it was used to obtain examples of CW-complexes and so on and while proving CW-approximation theorem we have used that one. So, let us recall them from Module 4B.

So, this  $k$  is fixed integer,  $k \geq 0$ .  $L_k$  denotes the set of points  $x \in \mathbb{R}^N$ , all of whose coordinates  $x_i$  are rational numbers of the form an integer divided by  $2^k$ . So,  $L_0$  is the set of so called lattice points, with all coordinates as integers,  $L_2$  has points with coordinates half integers and so on. The lines planes you know drawn at length at the interval 1 at all the integer points  $N_2$  will be at half integer points.

Let  $P_k$  denote set of all closed  $N$ -cubes  $\sigma$  of side-length  $1/2^k$  with their corners inside  $L_k$  and  $P$  equal to union of all  $P_k$ 's. I am just recalling all these things which we have done earlier in Module 4B and we restate a theorem for our ready use here.

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#### Lemma 6.2

Let  $A \subset P$  such that  $\sigma_1, \sigma_2 \in A$  implies  $\sigma_1 \not\subset \sigma_2$ . Then  $W(A) = \bigcup \{ \sigma : \sigma \in A \}$  is a locally finite, countable, pure,  $N$ -dimensional CW-complex with all its  $N$ -cells being members of  $A$ .



(Editors note: The speech diverts quite a bit from the slides here, almost reproving the theorem in a slightly different way but for the proof of the theorem either of them will do. So we keep both versions for the benefit of the students.)

So a lemma: Take a subset  $A$  of  $P$  such that  $\sigma_1$  and  $\sigma_2$  belong to  $A$  implies  $\sigma_1$  is not contained in  $\sigma_2$ . (Remember that these  $\sigma_i$ 's are  $N$ -dimensional cubes of different sizes in  $\mathbb{R}^N$ . So, for instance if we have  $(1/4)$ -th size cube a  $(1/8)$ -th size cube it is likely that the second one may be a subset of the first one. That should not happen. They may intersect each other. That is allowed.) Let  $W(A)$  be the union of all the  $\sigma$ 's belonging to  $A$ . Then the subspace  $W(A)$  of  $\mathbb{R}^N$  has a locally finite, countable, pure,  $N$ -dimensional CW-complex structure with all its  $N$ -cells being precisely the members of  $A$ .

We have seen that this CW-structure actually can be further 'cut-down' into a simplicial complex by cutting each  $N$ -cube into a simplicial complex. So, that is the lemma that is relevant for now. We directly go to the proof of the theorem. I have to produce this open set  $U'$  as well a retraction  $r$ .

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**Proof:** of theorem 6.6: Let  $A_1$  be the union of all cubes in  $P_0$  contained in  $U$  and which do not meet  $X$ . Inductively, for  $k \geq 1$ , let  $A_k$  be the union of all those cubes in  $P_{1/2^{k-1}}$  which do not meet  $X$  and contained in  $U$  but are not contained in any member of  $A_{k-1}$ . Put  $A = \bigcup_{k \geq 0} A_k$ . Let  $W = W(A)$  be defined as in the lemma above. Then  $\mathbb{R}^N \setminus X = W$  and  $W$  has a CW-structure in which all these cubes of various sizes form the  $N$ -dimensional cells.

Arant Chasin

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- Appendix A: Classification of Triangulations of Surfaces
- Appendix B: Alexander's Theorem

So, let  $A_0$  be collection of all cubes in  $P_0$  ( $P_0$  is what  $N$ -cubes of side length 1, with corners as integer coordinates) contained in  $U$  and which do not meet  $X$ . This set may be empty, I do not care. Let  $A_1$  be the collection of all cubes in  $P_1$  contained in  $U$  and which do not meet  $X$  and not contained in any member of  $A_0$ . Inductively, for  $k \geq 1$ , let  $A_k$  be the collection of all those cubes in  $P_k$  contained in  $U$ , not meeting  $X$  and which are not contained in any member of  $A_i$ , for  $i < k$ . Put  $A$  equal to union of all  $A_i$ 's.

Let  $W = W(A)$  be defined as the lemma. By the very definition  $A$  will satisfy the condition of this lemma, namely,  $\sigma_1$  and  $\sigma_2$  are in  $A$ , will imply that  $\sigma_1$  is not contained in  $\sigma_2$ . Once a cube is admitted inside  $A$ , no smaller cubes inside it will be taken. So, that is why this condition is automatically satisfied.


Therefore, we can apply the lemma.  $W = W(A)$  will be defined as the union of all  $\sigma \in A$ . Clearly,  $W$  is a subset of  $U \setminus X$ .  $W$  has a CW structure in which all these  $N$ -cubes of various sizes in  $A$ . Given any point  $x \in U \setminus X$ , which is open in  $\mathbb{R}^N$ , we can choose  $k$  sufficiently large so that there is a member  $\sigma$  of  $P_k$  which contains  $x$  and  $\sigma$  contained in  $U \setminus X$ . Therefore,  $\sigma$  belongs to  $A_k$  and  $x$  belongs to  $W$ . Therefore,  $W = U \setminus X$ .


Put  $W_k$  equal to the union of all cubes  $A_i$  for  $i < k + 1$ . Then each  $W_k$  will be a closed subset of  $\mathbb{R}^N$ , being a union of a locally finitely family of closed sets. Also  $W = \text{union of all } W_k$ . So the funny thing here is that to begin with we are not doing anything to  $X$  but the complement of  $X$  in  $U$  has been given a nice CW structure, these cubes of various sizes all of them  $N$ -dimension cells, smaller and smaller maybe smaller which keep coming nearer and nearer  $X$ .



but will not intersect it.

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<ul style="list-style-type: none"> <li>Cell Complexes</li> <li>Categories and Functors</li> <li>Homology Groups</li> <li>Other Homology groups</li> <li>Assorted Topics</li> <li>Topology of Manifolds</li> </ul>	<ul style="list-style-type: none"> <li>Module-46: Definitions and Examples</li> <li>Module-47: Manifolds with Boundary</li> <li>Module-48: Embeddings and Homotopical Aspects</li> <li>Module-50: Homotopical Aspects</li> <li>Module-51: Classification of 1-dimensional manifolds</li> <li>Module-52: Triangulation of Manifolds</li> <li>Module-53: Classification of Compact Surfaces</li> <li>Appendix A: Existence of Triangulations of Surfaces</li> <li>Appendix (B): Alexander's Homotopy Lemma</li> </ul>	

We shall construct a subcomplex  $V$  of  $W$  and a function  $r : X \cup V \rightarrow X$  such that  $r(x) = x \mid x \in X$ . We shall then show that  $X \cup V$  contains a neighbourhood of  $X$  and  $r$  is continuous on this neighbourhood, which will complete the proof of the lemma. First, the constructions of  $V$  and  $r$  will be done simultaneously and inductively. Later, we shall check the continuity of  $r$  so defined.

So, that is what it is now. Let us see how we shall construct an open neighbourhood  $U'$  of  $X$  in  $U$  and a retraction  $r$  on  $U'$  which will complete the proof of the theorem 6.6. First we are actually going to construct a subcomplex  $K$  of  $W$ , and a retraction  $r$  from  $X \cup K$  to  $X$ . (Here we take the liberty to write the same symbol  $K$  for the complex  $K$  as well as the underlying subspace  $|K|$ ).

We shall then show that  $X \cup K$  contains an open set  $U'$  which contains  $X$  and  $r$  is continuous on  $U'$ . That may be a wishful thinking and it takes sometime to prove each of them. Let us go ahead with this. First the constructions of  $K$  and  $r$  will be done simultaneously and inductively, keeping in mind that  $r$  should be continuous on a suitable subset.

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Of course, we start with  $r(x) = x$ ,  $x \in X$ . Take the 0-skeleton of  $V$  to be  $W^{(0)}$ , i.e., all vertices of  $W$ . Given  $\sigma \in V^{(0)}$ , choose any point  $r(\sigma) \in X$  such that

$$d(\sigma, r(\sigma)) < 2d(\sigma, X).$$

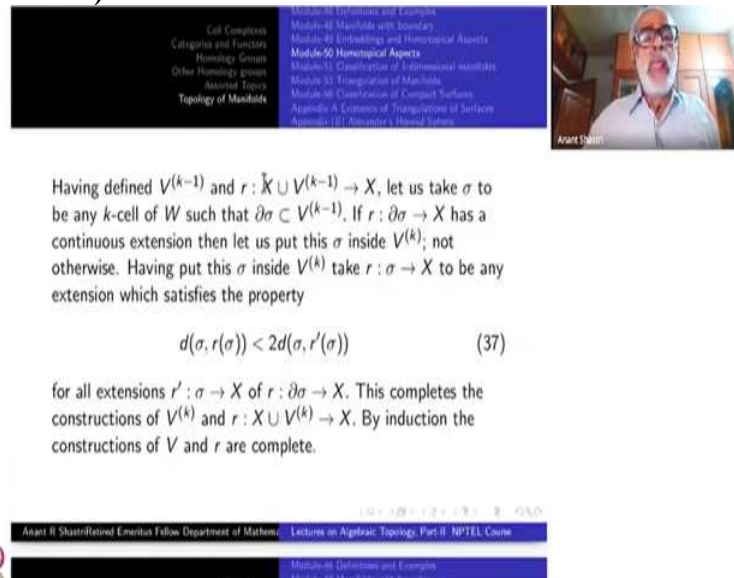
(Here  $d(F, G)$  denotes the infimum of the Euclidean distances  $d(x, y)$ , where  $x \in F, y \in G$ .) This completes the constructions of  $V^{(0)}$  and  $r : X \cup V^{(0)} \rightarrow X$ .



Of course, we start with  $r(x) = x$ , because there is no other choice, since  $r$  has to be retraction onto  $X$ . So,  $r(x) = x$ . Take the 0-skeleton of  $K$  to be  $W^{(0)}$  (Do not confuse it for  $W_0$ .) We are going to define a subcomplex of  $K$  of  $W$ . So, take all the 0-cells of  $W$ . These are some lattice points, away from  $X$ .

Given a 0-cell  $\sigma$ , choose  $r(\sigma)$  to be any point in  $X$  such that the distance between  $\sigma$  and  $r(\sigma)$  is less than twice the distance between  $X$  and  $\sigma$ . Remember  $\sigma$  in  $W$  does not intersect  $X$  which is a closed set so the distance is positive. Here the distance between two subsets  $A, B$  of  $\mathbb{R}^N$  is nothing but the infimum of all  $d(a, b)$  where  $a$  ranges over  $A$  and  $b$  ranges over  $B$ . Therefore such a point  $r(\sigma)$  exists.

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Having defined  $V^{(k-1)}$  and  $r : X \cup V^{(k-1)} \rightarrow X$ , let us take  $\sigma$  to be any  $k$ -cell of  $W$  such that  $\partial\sigma \subset V^{(k-1)}$ . If  $r : \partial\sigma \rightarrow X$  has a continuous extension then let us put this  $\sigma$  inside  $V^{(k)}$ ; not otherwise. Having put this  $\sigma$  inside  $V^{(k)}$  take  $r : \sigma \rightarrow X$  to be any extension which satisfies the property

$$d(\sigma, r(\sigma)) < 2d(\sigma, r'(\sigma)) \quad (37)$$

for all extensions  $r' : \sigma \rightarrow X$  of  $r : \partial\sigma \rightarrow X$ . This completes the constructions of  $V^{(k)}$  and  $r : X \cup V^{(k)} \rightarrow X$ . By induction the constructions of  $V$  and  $r$  are complete.

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The construction of  $K^{(0)}$  and  $r$  from  $X \cup K^{(0)}$  to  $X$  is over. Actually, here you can easily see that this  $r$  has to be continuous function because on  $X$  it is identity and  $K^{(0)}$  is discrete. But let us not discuss continuity right now. Let us go ahead. Having defined  $K^{(k-1)}$  and  $r$  from  $X \cup K^{(k-1)}$  to  $X$ , first of all, let us take  $\sigma$  to be any  $k$ -cell in  $W$  such that its entire boundary is contained in  $K^{(k-1)}$ . You know  $\sigma$  is a  $k$ -dimensional box, its boundary consists of  $(k-1)$ -dimensional boxes. (For instance, for  $k=1$ , you could select any 1-cell in  $W$ , because you have admitted all the 0-cells of  $W$  inside  $K^{(0)}$ . But wait.)

Now look at  $r$  restricted to  $\partial(\sigma)$ , which is already defined by the induction hypothesis. If this extends continuously to a function from  $\sigma$  to  $X$ , then and then only admit  $\sigma$  inside  $K^{(k)}$ . In general, we do not know if there is such a continuous extension. So this is a non-vacuous

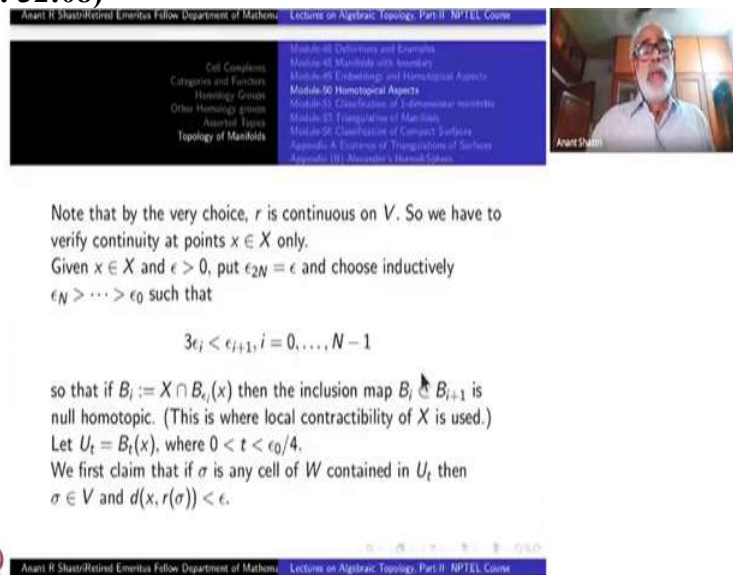
condition. If there is no such extension, do not take this cell  $\sigma$  in  $K^{(k)}$ .

Now, choose  $r$  from  $\sigma$  to  $X$  to be any continuous extension which satisfies the property that amongst all such continuous extensions  $r'$ , you must take one such that distance between  $\sigma$  and  $r(\sigma)$  is less than twice the distance between  $\sigma$  and  $r'(\sigma)$  for all such extensions  $r'$ .

You see if you take the set of all distances between  $\sigma$  and  $r'(\sigma)$  where  $r'$  ranges over the non empty set of continuous extensions of  $r$  restricted to  $\partial(\sigma)$ , and look at its infimum, it will be positive being bigger than distance between  $\sigma$  and  $X$ . However, we do not know whether this infimum can be attained. So, I have doubled these distances so that one of them among all extensions  $r'$  from  $\sigma$  to  $X$ , will definitely satisfy this condition, because a positive number cannot be smaller than twice itself. So, there must be one  $r$  which satisfies this property. We can now extend  $r$  over the whole of  $K^{(k)}$  continuously. By induction, the construction of  $K$  which is equal to union of all its skeletons  $K^{(k)}$  and  $r$  is over.

It remains to prove a number of things, that  $X \cup K$  contains a neighbourhood  $U'$  of  $X$  and  $r$  is continuous on  $U'$ . Once you have done that, the proof is over. So, let us do that.

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Anant P. Shahi/Retired Emeritus Fellow Department of Mathematics Cell Complexes Categories and Functors Homology Groups Other Homology groups Algebraic Topology Topology of Manifolds	Lectures on Algebraic Topology, Part II: NPTEL Course Module-01: Definitions and Examples Module-02: Manifolds with Boundary Module-03: Embeddings and Homotopy Groups Module-04: Homotopy Groups Module-05: Classification of Manifolds Module-06: Classification of Manifolds Module-07: Classification of Manifolds Module-08: Classification of Manifolds Appendix A: Examples of Transitions of Surfaces Appendix B: Examples of Transitions of Surfaces
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Anant Shahi

Note that by the very choice,  $r$  is continuous on  $V$ . So we have to verify continuity at points  $x \in X$  only.  
 Given  $x \in X$  and  $\epsilon > 0$ , put  $\epsilon_{2N} = \epsilon$  and choose inductively  $\epsilon_N > \dots > \epsilon_0$  such that

$$3\epsilon_i < \epsilon_{i+1}, i = 0, \dots, N-1$$

so that if  $B_i := X \cap B_{\epsilon_i}(x)$  then the inclusion map  $B_i \hookrightarrow B_{i+1}$  is null homotopic. (This is where local contractibility of  $X$  is used.)  
 Let  $U_t = B_t(x)$ , where  $0 < t < \epsilon_0/4$ .  
 We first claim that if  $\sigma$  is any cell of  $W$  contained in  $U_t$  then  $\sigma \in V$  and  $d(x, r(\sigma)) < \epsilon$ .

Note that by the very choice  $r$  is continuous on  $K$  because for any CW complex a function is continuous iff restricted to each cells it is continuous. So, we have to verify continuity at points of  $X$ . You see  $r$  restricted to  $X$  is identity. That it is continuous on  $X$ . However, that does not mean that as a function from  $X \cup K$  to  $X$ , it is continuous at  $x$ . So many people make this mistake, since  $r$  restricted to  $X$  being identity continuous. (The trouble is because

$X$  is not open in  $X \cup K$ .)

But we do not need  $r$  to be continuous on the whole of  $X \cup K$ . So, we must achieve continuity at points of  $X$  but we may cut down the space  $X \cup K$ . Remember that. So, given  $x$  belonging to  $X$  and  $\epsilon$  positive, let us put  $\epsilon_{2N}$  equal to  $\epsilon$ . So go on choosing  $\epsilon_i > 0$  in the reverse order, for  $i = 2n - 1, 2n - 2, \dots$  till you hit  $i = 0$ , such that  $3\epsilon_i$  is less than  $\epsilon_{i+1}$ .

Put  $B_i$  equal to  $X \cap B_{\epsilon_i}(x)$ , where  $B_\epsilon(x)$  is the the standard open ball in  $\mathbb{R}^N$ . Further, we need  $\epsilon_i$  to satisfy the condition that inclusion map  $B_i$  to  $B_{i+1}$  is null homotopic. So, how do you ensure this? This is where local contractibility of  $X$  has to be used. What is the meaning of local contractibility: given any neighbourhood of any point in  $X$ , we must have a smaller neighbourhood such that the inclusion map of the smaller in the larger one is null homotopic. So, I do not say that this  $\epsilon_i$  is exactly one third of  $\epsilon_{i+1}$ . No, it is possible to choose  $\epsilon_i$  smaller than that so as to satisfy this homotopy condition also.

Let  $U_t$  be  $B_t(x)$ , where  $0 < t < \epsilon_0/4$ . After choosing the last  $\epsilon_0$ , I am going to further divided it 4 and take a positive  $t$  smaller than that. As a step toward continuity, we claim that if  $\sigma$  is any cell of  $W$ , contained in this  $U_t$ , then  $\sigma$  will be automatically in  $V$ , and the distance between  $x$  and  $r(\sigma)$  (remember that  $r(\sigma)$  was some point of  $X$ ) will be less than  $\epsilon$ . The original  $\epsilon$  looks very big now, because we have come down to  $\epsilon_0/4$ . We shall prove this by induction on the dimension of the  $\sigma$ , by proving slightly stronger claims in the inductive steps.

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Suppose  $a$  is a 0-cell and  $a \in U_t$ . Then

$$d(x, r(a)) \leq d(x, a) + d(a, r(a)) < t + 2d(a, x) < 3t < \epsilon_0.$$

Now suppose for all  $(k-1)$ -cells  $\tau$  of  $V$  contained in  $U_t$  we have  $d(x, r(\tau)) < \epsilon_{2k-2}$ . Let  $\sigma$  be a  $k$ -cell of  $W$  contained in  $U_t$ . Then  $r(\partial\sigma) \subset B_{\epsilon_{2k-2}}$  and the inclusion  $B_{\epsilon_{2k-2}} \subset B_{\epsilon_{2k-1}}$  is null homotopic. So, there exist extensions of  $r : \partial(\sigma) \rightarrow B_i$  inside  $B_{i+1}$ . From (37), it follows that  $d(\sigma, r(\sigma)) < 2\epsilon_{2k-1}$  and therefore,

$$d(x, r(\sigma)) \leq d(x, \sigma) + d(\sigma, r(\sigma)) < t + 2\epsilon_{2k-1} < \epsilon_{2k}.$$

Therefore, by induction, we have,  $r(U_t \cap V) \subset U_\epsilon$ .



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Module-46: Cell Complexes and Topology  
Module-47: Manifolds and Homotopy

Suppose  $a$  is a 0-cell in  $W$  and  $U_t$ . Already by definition  $K^{(0)} = W^{(0)}$  and so  $a$  belongs to  $V$ . Now the distance between  $x$  and  $r(a)$  is less than the distance between  $x$  and  $a$  plus the distance between  $a$  and  $r(a)$ . Since  $x$  and  $a$  are inside  $U_t$ , the first term is less than  $t$  and second one is, by the definition of  $r$ , less than twice distance between  $a$  and  $x$ . So the sum is less than  $3t < \epsilon_0$ , because  $t < \epsilon/4$ . Of course  $\epsilon_0$  is less than  $\epsilon$ . So, the inductive step for 0 is proved.

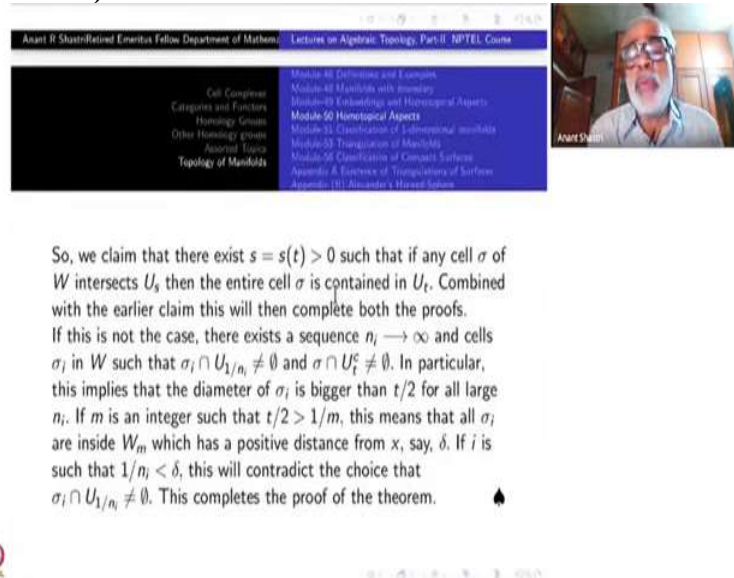
Now, suppose for all  $(k-1)$ -cells  $\tau$  of  $W$  contained in  $U_t$ , we have proved that  $\tau$  is in  $K$  and that distance between  $x$  and  $r(\tau)$  is less than  $\epsilon_{2k-2}$ . We want to prove this to be less than  $\epsilon$  but we need to have this stronger inductive hypothesis to go to the next step. Any way we have proved it for 0-cells. Let now  $\sigma$  be a  $k$ -cell of  $W$  contained in  $U_t$ . Then boundary of  $\sigma$  will be contained inside  $B_{\epsilon_{2k-2}}$  from the induction hypothesis since the boundary of a  $k$ -cell in  $W$  is the finite union of  $(k-1)$ -cells  $W$ . Also, we have the inclusion map  $B_i$  to  $B_{i+1}$  is null homotopic. (That is important now.) Therefore, you can always extend  $r$  which is defined on  $\partial(\sigma)$  to a map on  $\sigma$  into  $B_{\epsilon_{2k-1}}$  continuously. So, first of all  $\sigma$  belongs to  $K$ . Moreover, from (43), it follows that the distance between  $\sigma$  and  $r(\sigma)$  will be less than  $2\epsilon_{2k-1}$ . Therefore, once again, the same thing happens here, viz.,  $d(x, r(\sigma))$  is less than  $d(x, \sigma) + d(\sigma, r(\sigma))$ . This is always less than  $t$  and the second term is less than  $2\epsilon_{2k-1}$  so the sum is less than  $\epsilon_{2k}$ . Therefore, by induction, we have proved the claim, because  $\epsilon_{2k}$  is always less than  $\epsilon$  for all  $k = 0, \dots, N$ .

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Note that this however does not complete the proof of continuity of  $r$  at  $x$ . Nor does this

prove that  $X \cup K$  contains a neighbourhood of  $X$  in  $\mathbb{R}^N$ . But this is a good step toward continuity. We have to prove one more step for we have not yet proved that  $U_t$  itself is contained inside  $K$ . Since  $U_t$  may have many points belonging to the cells  $\sigma$  in  $K$  where  $\sigma$  itself is not completely inside  $U_t$ .

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Cell Complexes	Module 45: Definitions and Examples
Categories and Functors	Module 46: Manifolds with Boundary
Homology Groups	Module 47: Manifolds and Homotopy
Other Homology Groups	Module 48: Homotopy and Homological Aspects
Assorted Topics	Module 49: Classification of Manifolds
Topology of Manifolds	Module 50: Transgression of Manifolds
	Module 51: Classification of Compact Surfaces
	Appendix A: Examples of Transgressions of Surfaces
	Appendix B: Alexander's Horned Sphere

Asmit Shastri

So, we claim that there exist  $s = s(t) > 0$  such that if any cell  $\sigma$  of  $W$  intersects  $U_s$  then the entire cell  $\sigma$  is contained in  $U_t$ . Combined with the earlier claim this will then complete both the proofs.


If this is not the case, there exists a sequence  $n_i \rightarrow \infty$  and cells  $\sigma_i$  in  $W$  such that  $\sigma_i \cap U_{1/n_i} \neq \emptyset$  and  $\sigma_i \cap U_{1/n_i}^c \neq \emptyset$ . In particular, this implies that the diameter of  $\sigma_i$  is bigger than  $t/2$  for all large  $n_i$ . If  $m$  is an integer such that  $t/2 > 1/m$ , this means that all  $\sigma_i$  are inside  $W_m$  which has a positive distance from  $x$ , say,  $\delta$ . If  $i$  is such that  $1/n_i < \delta$ , this will contradict the choice that  $\sigma_i \cap U_{1/n_i} \neq \emptyset$ . This completes the proof of the theorem. ♣

So, we claim that there exists  $0 < s = s(x, t) < t$  (depending upon  $x$  as well as  $t$ ) such that if any cell  $\sigma$  of  $W$  intersects  $U_s$ , then the entire cell  $\sigma$  is contained in  $U_t$ . This would clearly imply that  $r(U_s)$  is contained in  $U_\epsilon$ , since we have proved earlier that  $r(U_t \cap K)$  is contained in  $U_\epsilon$ . Therefore the continuity at the point  $x$  for the function  $r$  from  $X \cup K$  to  $X$  follows.

By choosing  $\epsilon$  to be any fixed number, say  $\epsilon = 1$  for all  $x \in X$ , and taking  $U' = \text{union of all } U_{s(x, \epsilon_0/5)}$ , we get an open subset of  $X \cup K$  which is a subset of  $U$  and contains  $X$ . This will then complete the proof. So, it remains to prove the existence of  $s(x, t)$  as claimed.

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So, we claim that there exist  $s = s(t) > 0$  such that if any cell  $\sigma$  of  $W$  intersects  $U_s$  then the entire cell  $\sigma$  is contained in  $U_t$ . Combined with the earlier claim this will then complete both the proofs. If this is not the case, there exists a sequence  $n_i \rightarrow \infty$  and cells  $\sigma_i$  in  $W$  such that  $\sigma_i \cap U_{1/n_i} \neq \emptyset$  and  $\sigma_i \cap U_t^c \neq \emptyset$ . In particular, this implies that the diameter of  $\sigma_i$  is bigger than  $t/2$  for all large  $n_i$ . If  $m$  is an integer such that  $t/2 > 1/m$ , this means that all  $\sigma_i$  are inside  $W_m$  which has a positive distance from  $x$ , say,  $\delta$ . If  $i$  is such that  $1/n_i < \delta$ , this will contradict the choice that  $\sigma_i \cap U_{1/n_i} \neq \emptyset$ . This completes the proof of the theorem. 



So, if this is not the case what happens? That means that there exists a sequence  $n_i$  converging into infinity such that none of the numbers  $s_i = 1/n_i$  satisfy this property, meaning, there are cells  $\sigma_i$  in  $W$  such that  $\sigma$  intersects both  $U_{s_i}$  as well as  $U_t^c$ . In particular, this implies that the diameter of  $\sigma_i$  is bigger than  $t/2$  for large  $i$ . viz., for all  $i$  such that  $1/n_i < t/2$ . Note that the diameter of any  $N$ -cells in  $P_k$  is equal to  $\sqrt{N}/2^k$ . So choose an integer  $k$  such that diameter of any  $N$ -cell in  $P_k$  is smaller than  $t/2$ . Then none of these  $\sigma_i$  are in  $P_m$  for any  $m < k$  and hence they are contained in  $W_k$ . But  $W_k$  has a positive distance say,  $\delta(x)$  from  $x$ . Therefore if  $1/n_i < \delta(x)$ , it is a contradiction to the fact that  $\sigma_i$  intersects  $U_{1/n_i}$ . This proves the theorem 6.6.

Using this theorem, next time we shall prove one of the very important things answering a question that we had raised earlier, namely, whether the fundamental group of any topological manifold is countable.

That is the homotopical aspect of this one that is what retraction is already some kind of homotopy though we have not proved that it is a deformation retraction, that is not needed. So, let us stop here. Thank you.