Introduction to Algebraic Topology (Part – II)
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Lecture – 4B
More examples

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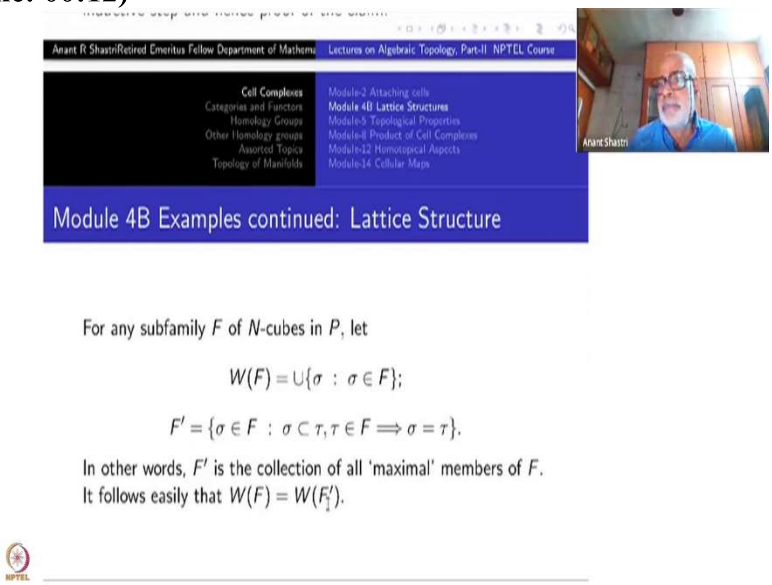

So, now, we carry on with more examples of CW complexes. For any subfamily F of N-cubes in P, not necessarily finite, we have this notation $W(F)$ equal to the union of all N-cubes sigma which are present in F. We also have this notation: F' is the subfamily of all those σ inside F which are maximal; what is the meaning of that? If σ is contained in τ and τ is also in F that means σ is equal to τ. That is the meaning of maximal; only the maximal elements of F are taken in F'. Then clearly the union of all the members of F is equal to union of all the maximal members of F.

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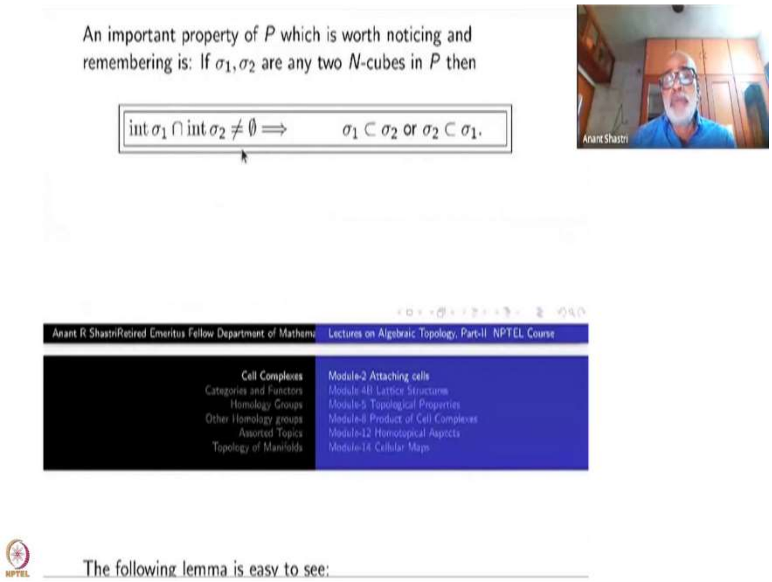

So, this is where we use this fact. Remember that we have seen that interior of σ_1 intersection with interior of σ_2 is non-empty implies either σ_1 contains σ_2 or σ_2 contains σ_1. Either they are non-overlapping completely or one of them is contained in another. Now we can take only the maximal ones to take care of the entire union.

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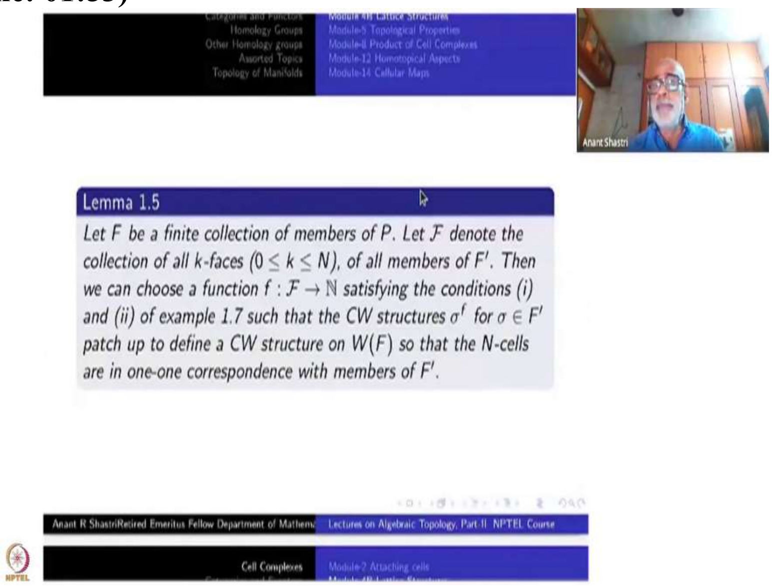

Now, here is the extension of the earlier example for infinite case. So, earlier we had this example only for one σ. Now, I am taking any finite number of members of P. Let $\mathcal{F}$ denote the collection of all k-faces for $0 \leq k \leq N$, of members of F'. (Earlier there was only one σ. Now, I am taking a finite collection F.) Then we can choose a function f on this family $\mathcal{F}$ which is an integer valued function satisfying condition (i) and (ii) of example 1.7., properly modified.

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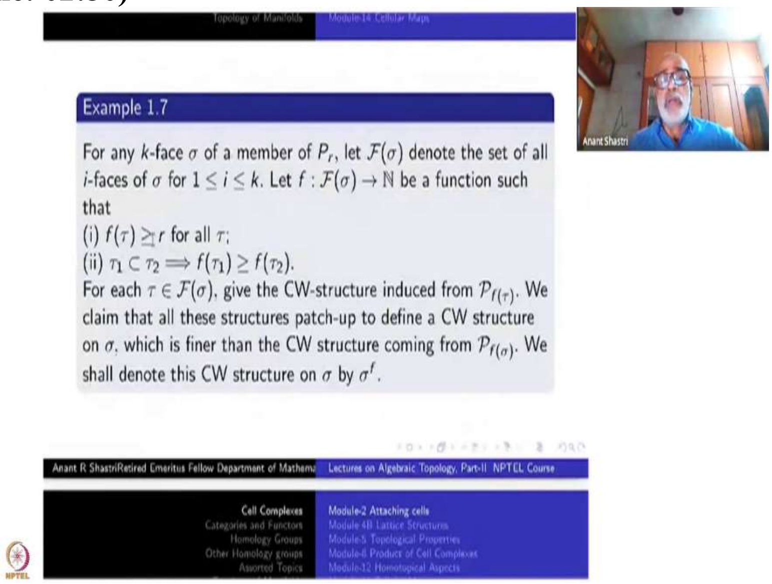

Let me recall what are these conditions:

(i) $f(\tau)$ is greater than equal to r for all τ, whenever τ is a member of P_r;

(ii) τ_1 is contained in τ_2 implies $f(\tau_1) \geq f(\tau_2)$.

So, this must be true for all members of these $\mathcal{F}$. So, those are the condition (i) and (ii). Then the CW structures σ^f coming from example 1.7 patch-up to define a CW structure on $W(F)$, so, that the N-cells are in one-one correspondence with members of F'. This is the statement of the lemma.

There in that example, we started with a function f with these two properties. But we did not say anything about whether there is one such and so on. In fact there are many such functions, but here I say we can choose a function f satisfying these two conditions. I am not saying that this will be so for any arbitrary function. Anyway condition (i) and (ii) are necessary even to have a CW-structure on each σ, viz., the structure σ^f.

So, once you have such that function, on each k-face of $\mathcal{F}$, we will have a CW-structure. Now, the claim is that if f is chosen properly, then these σ^f. structures on each of σ will patch up to define a CW-structure on $W(F)$ itself. So this is the claim. Not only that, the second part says that the N-cells of this structure are precisely the members of this F', the maximal members of F. So, if you understood the σ^f. in the earlier example, this will be an easy consequence of that.

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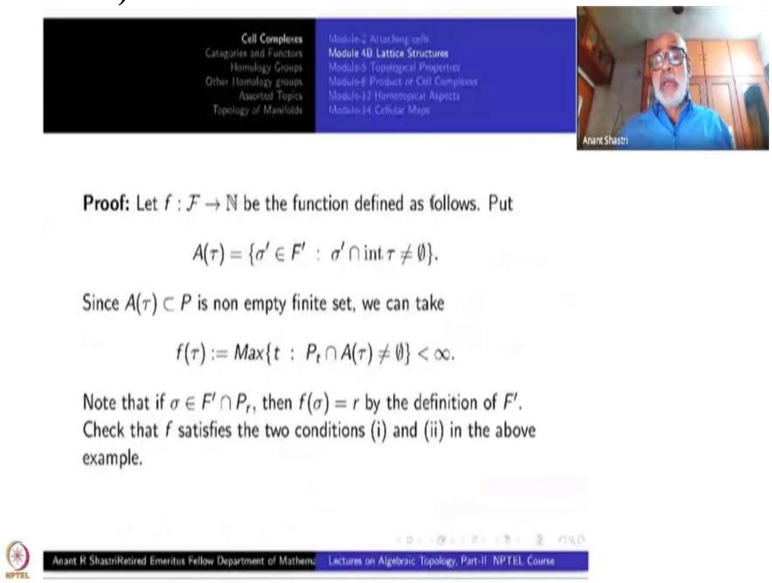

So, let me write down the formula, what is the function f? Given τ in $\mathcal{F}$, let f from $\mathcal{F}$ to the set of natural numbers be the function defined as follows: take $A(\tau)$ to be the set of all those σ' in F', the maximal members of F, such that they will intersect the interior of τ. There is at least one such σ' because each τ is the face of some member σ of F and hence σ intersects interior of τ. But then σ is contained in some member σ' of F'.

So, this set is non-empty. Of course, it is finite also. Thus the set $A(\tau)$, first of all is contained in P and is a non-empty finite set. Therefore, we can take $f(\tau)$ to be the maximum of all t such that $P_t \cap A(\tau)$ is not empty. Each member of $A(\tau)$ belongs to some P_t, put all such t in this set if P_t intersect of $A(\tau)$ is non-empty. That set is also nonempty and finite. So, take the maximum of this set, it is some positive integer. This defines f.

As τ varies, this $f(\tau)$ will take different values. That does not matter. What you have to see is that it satisfies the condition (i) and (ii). Note that if τ is equal to σ, a member of some P_r as well as it is in F', then $f(\sigma) = r$ because then the whole set on which we are taking the maximum itself is just the singleton $\{r\}$.

By the definition of F', σ is a maximal one. So, the set $A(\sigma)$ will contain only one member viz., σ. So, that is the only member and hence $f(\sigma)$ will be r. We have to see that this f satisfies the two conditions (i) and (ii) in the above example.

The point here is that I have put here the condition that the interior of τ intersects this σ'. That is a trick here. So, if you take σ' intersection interior of τ non-empty and say τ' is a face of τ, this will be true for τ' also. So, all those cells which are present in $A(\tau)$ will be there in $A(\tau')$. Therefore, the set will be bigger set for τ' than τ which means that the maximum is bigger. So, τ_1 contained τ_2 will imply $f(\tau_1)$ is bigger than equal to $f(\tau_2)$. You have to verify this fully. Hopefully you will be able to do it on your own now. Any way I have explained the second condition also, you can see what happens.

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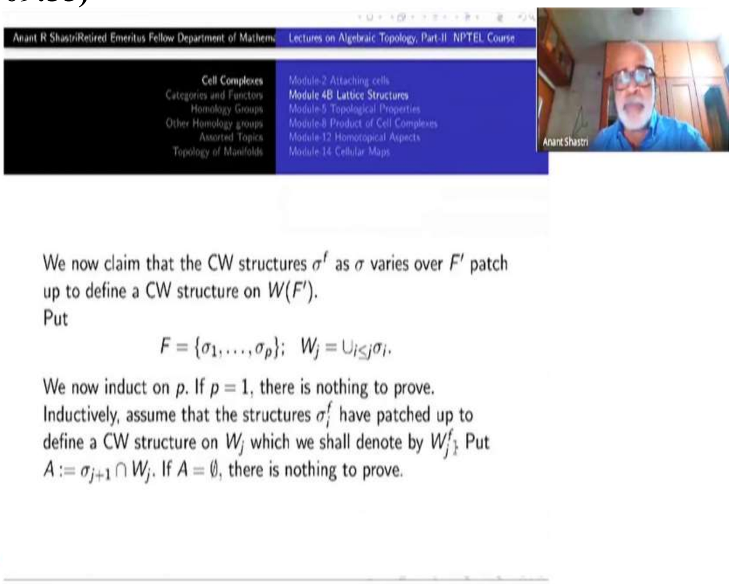

So, now the crucial thing in the lemma namely you claim that the CW-structures σ^f as σ varies over F', patch up to define a CW structure on $W(F)$. So, we want to do this by induction. Suppose F' has only one member that is already covered in the example. So, therefore, we just want to do induction here.

Let F' equal to $\sigma_1, \sigma_2, \dots, \sigma_p$ be an enumeration of members of F'. F' is a finite family. Let us set the notation W_j equal to union of those σ_i for $i \le j$. W_1 is just σ_1, W_2 is $\sigma_1 \cup \sigma_2$ and so on. So, we induct on this number p. If this p is 1, we know the result already, there is nothing to prove nothing to prove means what we have discussed this in the example 1.7.

Inductively, assume that the structure σ^f's patch up to define a CW structure on W_j, and that structure is temporarily denoted by W_j^f coming from f. Put $A = W_j \cap \sigma_{j+1}$, take the next member here intersect with W_j put A equal to that. If A is empty, you have two disjoint CW structures, union is automatically a CW complex in which W_j structure will be kept as it is and σ_{j+1} structure will be kept as it is. Both of them will be subcomplexes. There is no problem. The problem comes when A is non-empty, if at all.

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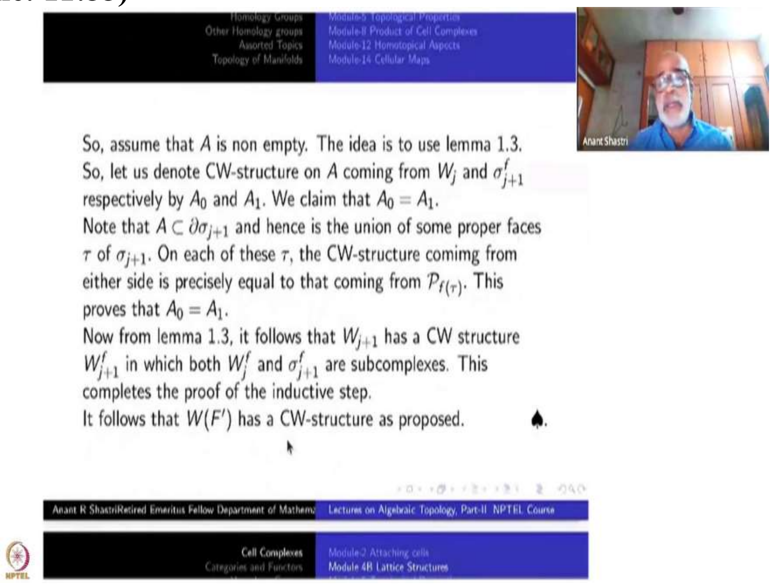

So, suppose A is non-empty. The idea is to use our earlier lemma. So, let us denote the CW structure on A coming from W_j by A_0 and the coming from σ_{j+1} by A_1. What we wanted is that either A_0 is finer than A_1 or A_1 is finer than A_0. Then we could put them together. So, that was our earlier lemma.

So, here we claim that A_0 is actually equal to A_1. So, you can patch it and then induction will take care of it. So, let us look at this. What is this A? A is the intersection of W_j with σ_{j+1}. This intersection never contains any interior of σ_{j+1}, it is only some part of the boundary because both are made up of maximal N-cells.

So, it is the boundary part of σ_{j+1}. Any part of the boundary will be made up of k-faces of σ_{j+1} for k strictly less than the dimension of σ_{j+1}, and hence is the union of some proper faces, τ of σ_{j+1}. On each of these τ, the CW structures coming from either side is precisely equal to that coming from $P_f(\tau)$.

By the very definition $f(\tau)$, the maximum of all those t, where P_t contains some σ' which intersects the interior of τ. This proves that A_0 is A_1, Because both are actually equal to the $P_f(\tau)$; they do not depend upon whether we consider A as subspace of σ_{j+1} or of W_j.

From the lemma 1.3, it follows that W_{j+1} as a CW structure, which will denote by W_{j+1}^f. Clearly, W_j^f is a sub complex. So, the structure keeps extending as you extend the space itself. So, that completes the proof that $W(F')$ has a CW structure as proposed.

This time instead of one cell σ, we have now got it on the finite union of n-cells.

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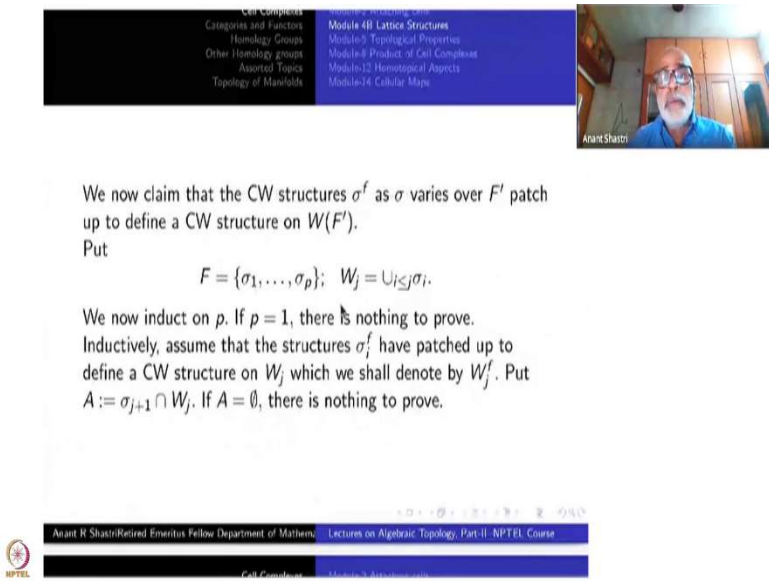

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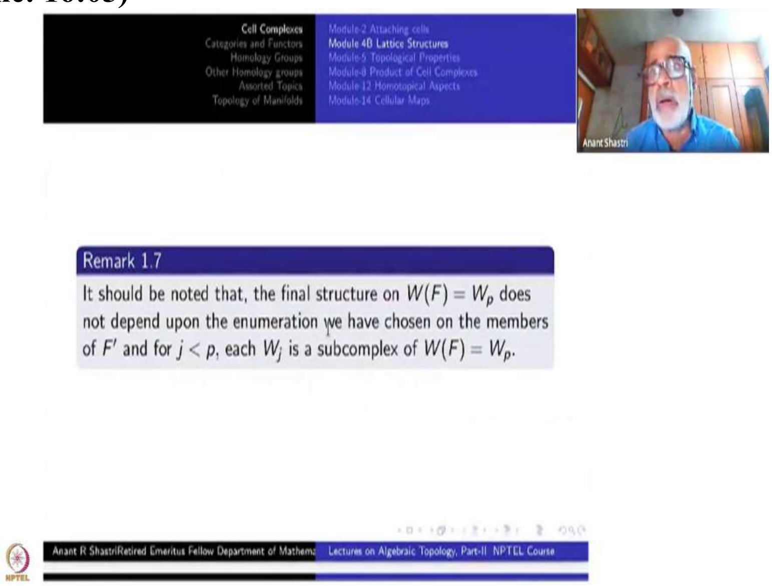

So, it should be noted that the final structure on $W(F)$ which is denoted by W_p^f does not depend upon the enumeration we have chosen on the members of F'. And for $j < p$, each W_j^f is a sub-complex. So, it depends only on the collection F. Once F is chosen no matter what, the enumeration is only for our convenience of writing down the proof.

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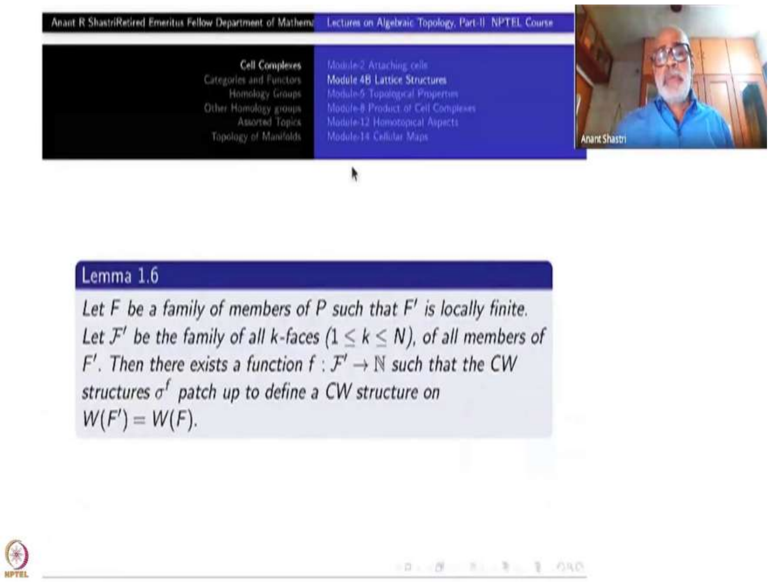

So, let us now go for the infinite case. This is our final aim. Let F be a family of members of P such that F' is locally finite. So, we are not in a completely general case. F is not any infinite family, that is not our aim, that is not possible either. So, we allow F to be infinite but F' to be locally finite. Let $\mathcal{F}$ consist of all k-faces, $1 \le k \le N$, of all members of F'.

Then there exists a function f on F' to the positive integers such that the CW structures σ^f patch up to define a CW structure on $W(F') = W(F)$. Here, I am not saying that explicitly that the function f satisfies (i) and (ii). So, it should satisfy these conditions, because σ^f is defined only when f satisfies these conditions. So, here the finiteness is not there you see, so, we have to be little more careful, but it is not at all that difficult.

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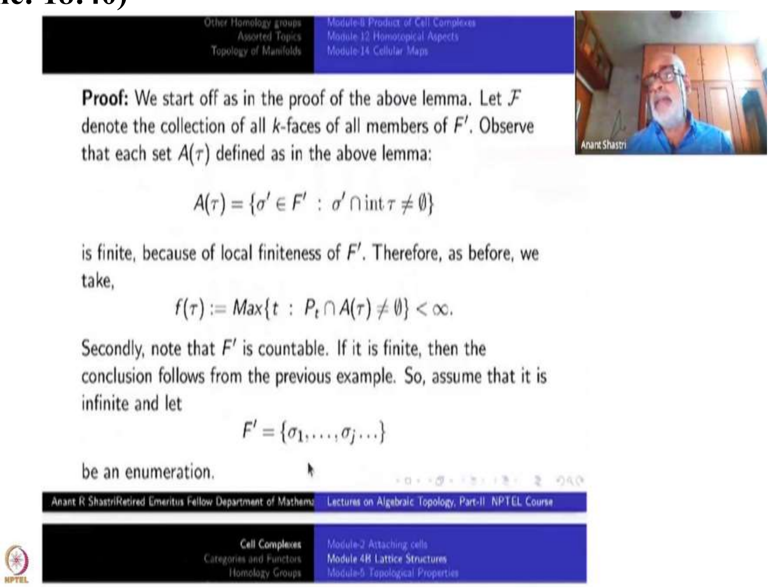

We start off as in the proof of the previous lemma. So, let $\mathcal{F}$ denote the collection of all k faces of all members of F'. Observe that each set $A(\tau)$ is defined as in the above lemma. $A(\tau)$ equals to the collection of all σ' in F' such that σ' intersection interior τ is non-empty. This $A(\tau)$ is the same thing. By local finiteness of F', (F' may not be finite)

Each $A(\tau)$ is finite. Therefore, as before we can take $f(\tau)$ to the maximum of this finite set. Therefore, we have the function f and it has satisfied the same properties (i) and (ii) no problem. The second thing is that F' may be infinite but it is countable. Why? Because P itself is countable. If it is finite then the proof is over by the previous lemma.

So, assume that F is infinite and form F' just like in the previous lemma, enumerate F', $\sigma_1, \sigma_2, \ldots, \sigma_j$ and so on, the only difference is now that this is infinite family.

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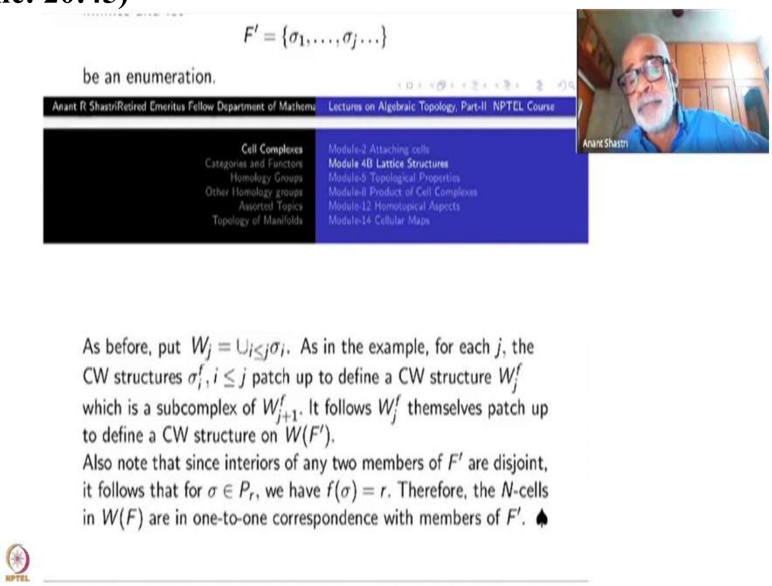

As before put W_j equal to the union of all σ_i for $i \leq j$. As in the previous example for each j, the CW structure $\sigma_i^f, i \leq j$ patch up to define a CW-structure on structure W_j which is a sub-complex of W_{j+1}^f. So, this is all in the previous lemma, because these are all finite now, there is union of finite in many of them. Now the claim s that W_j^f themselves patch up to define a CW structure on $W(F') = W(F)$.

At each finite stage you have the CW structure. $W(F')$ is union of these finite CW-structures one being the subcomplex of the next.

Look at any N-cell or any k-face of it in F', it will be present at some finite stage. There things are nice. Attaching k cells make sense. How does it makes sense? Provided its boundary is inside the $(k-1)$-skeleton. That is all we have to say. That is automatically satisfied here. Also note that since interior of any two members of F' are disjoint, it follows that for σ belonging to P_r, we have $f(\sigma) = r$, as in the previous case. Therefore, the N cells of $W(F')$ are in one-one correspondence with member of F', there is no subdivision of the N -cells. In other words, the proof of previous lemma is valid here word by word provided that $A(\tau)$ is finite for all τ. And that is ensured by local finiteness, that is all the difference.

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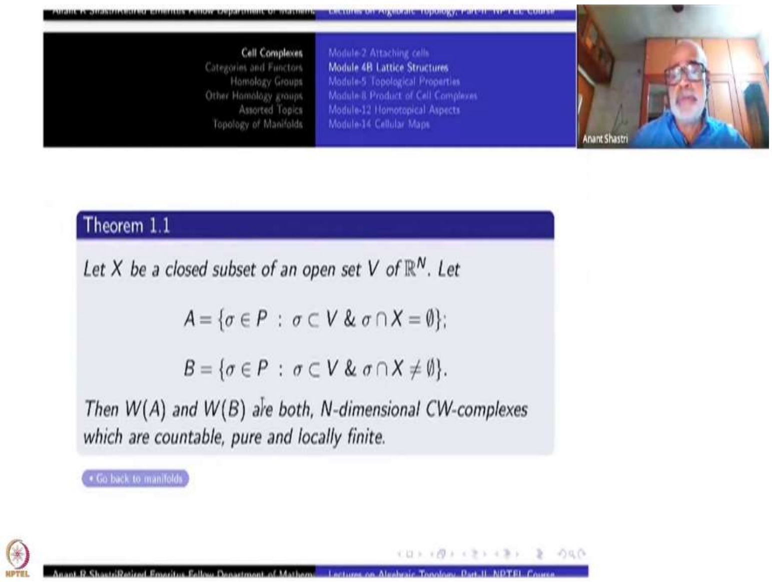

We now come to an important result which I have stated as a theorem but it is also an example. But this will be useful elsewhere. When studying manifolds we will be using this one. This was our aim of doing all this fundamental work here.

Take X to be any closed subset of an open set V inside $\mathbb{R}^N$; V is some open subset of $\mathbb{R}^N$ and X is a closed subset of $\mathbb{R}^N$. I want to tell you right in the beginning that V could be the whole of $\mathbb{R}^N$. Another extreme case allowed is when X could be empty also.

These two cases are importance also. The conclusion may be trivially true for these two extreme cases sometimes but they are of importance. You can take V to be just any open subset of $\mathbb{R}^N$ and X to be any closed subset. So, X is a close subset of open subset V of $\mathbb{R}^N$.

Let A be the collection of all those closed N-cells σ belonging to P and such that σ is contained inside V and intersection of X with σ must be empty. That is my A.

The second set is B which is the collection of all N-cells sigma in P such that again σ is contained inside V, (so that is the first condition) but the second condition is that X intersection σ is a non-empty.

So, this that is the difference. The two sets A and B are quite different, if X is non-empty then these are different. If X is the emptyset then the first condition is always true and hence B will be empty. So, these two are quite different. Let $W(A)$ be the union of all members of A, and $W(B)$ be the union of all members of B. Both are N-dimensional CW complexes which are countable, pure and locally finite. So, in the statement, I am not including the case when B may be empty and so on. Those things are obvious and you can just figure them out yourself.

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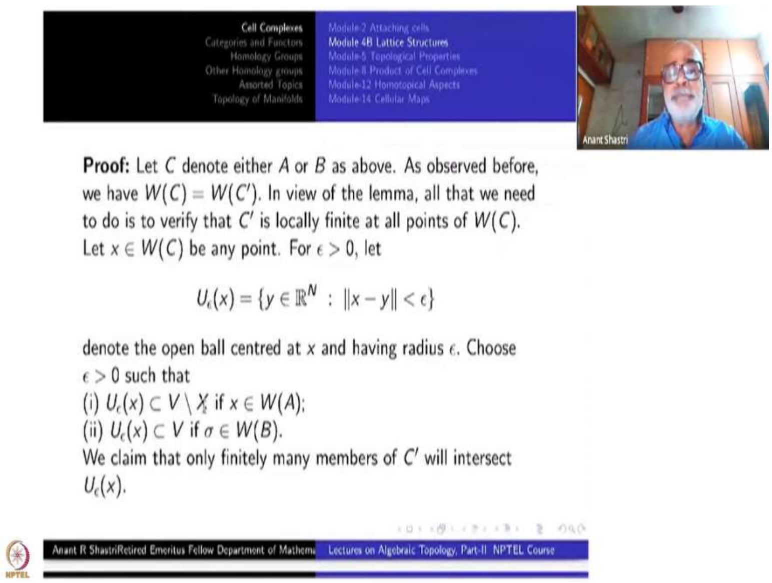

So the proof. Though there are two statements here one for A and one for B, the proofs are similar. So, I am going to put them together. Let us denote by C either A or B as above. As observed before, first of all $W(C)$ which is the union of all members of C, the same thing as $W(C')$.C' is what? collection of maximal members of C. In view of the lemma, all that we need to do is to verify that C' is locally finite.

That was our previous case if it is locally finite, then we have the required structure. So, in view of this lemma, all at we need to do is verify that C' is locally finite. At all the points of $W(C)$, this family should be locally finite. So, here is the proof. Take any point $x \in W(C)$.

For $\epsilon > 0$, let us have this notation a standard notation: $U_\epsilon(x)$ is the open ball of radius ϵ centered at x, all points y in $\mathbb{R}^N$ such that $\|x - y\| < \epsilon$. This is the notation. Now choose ϵ such that the ball $U_{2\epsilon}(x)$ is contained in

(i) the open set $V \setminus X$, if we are working with $C = A$, and $x \in W(A)$;
(it means that x is not a point of X and is, of course, a point of V. So x is in $V \setminus X$.) Since $V \setminus X$ is an open subset of $\mathbb{R}^N$, you can choose ϵ sufficiently small so that $U_{2\epsilon}(x)$ is contained inside $V \setminus X$. So that is no problem. Similarly, in the second case,

(ii) $U_{2\epsilon}(x)$ should be just inside V. This is easier.

In the first case, you have throw away X and then take that open set $V \setminus X$. Once you take $U_{2\epsilon}(x)$ inside V, this $U_\epsilon(x)$ is going to ensure local finiteness. We claim that only finitely many members of C' will intersect $U_\epsilon(x)$. The emphasis is on C'. If you ask this for members of C that may not be true. Indeed, it is definitely not true, Not true at all. Only for C', it is true. Finitely many members of C' will intersect $U_\epsilon(x)$. That is the local finiteness for C' at every point. Let us prove this one.

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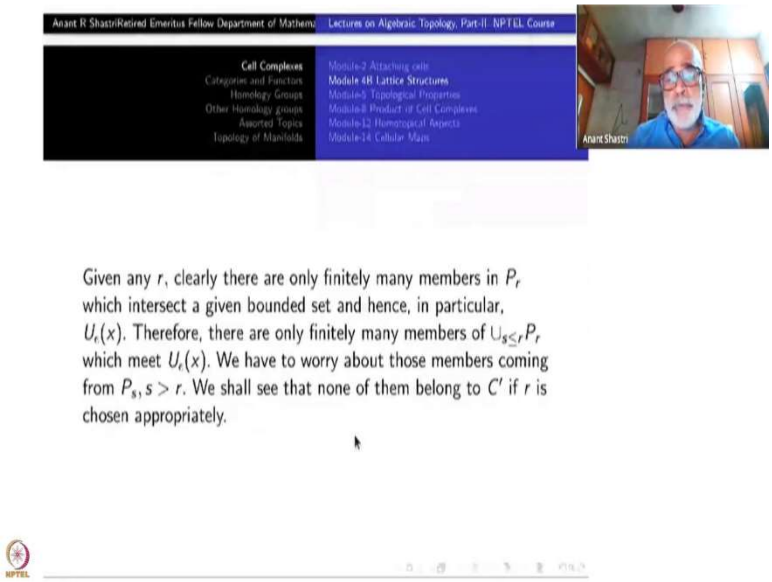

Given any r clearly there are only finitely many members of P_r which intersect a given bounded set. Take any bounded set inside $\mathbb{R}^N$. Any bounded set is inside some large cube. Then only finitely many members of P_r will be there. When r is fixed. Given any fixed r, this true.

Hence in particular how many will intersect $U_\epsilon(x)$ which is a bounded set. Only finitely many members. If you take $s \leq r$, each P_s will contribute only finitely many. The problems comes with members of P_s where $s > r$. That means these N-cells are smaller and smaller... they can create problems.

But this is the point here. You should see that none of them belong to C' if r is chosen appropriately. ϵ is already chosen. If I choose r sufficiently large, after that no member of P_s will intersect $U_\epsilon(x)$. I mean, some of them may intersect but they are not members of C'. So, once I have said this, it should be obvious to you, but let me just make it clearer.

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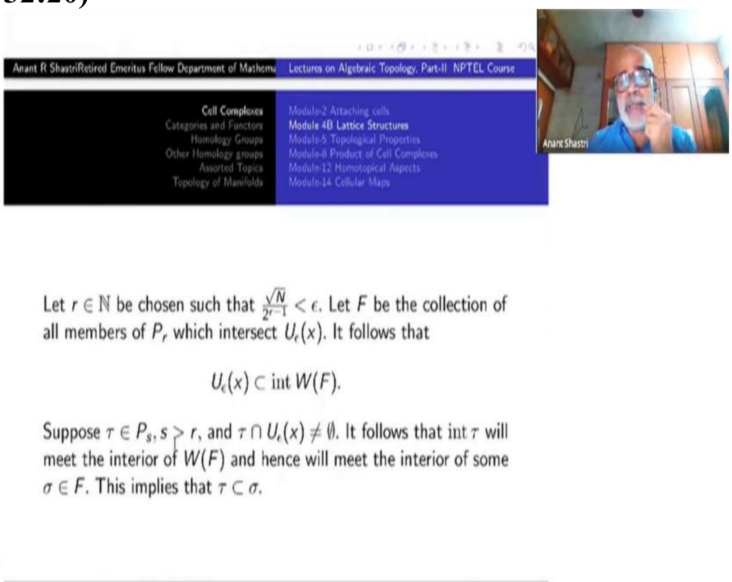

Choose the inter r such that this number $\sqrt{N}/2^{r-1}$ is less than ϵ. Recall that 2^{r-1} is the side length an N-cube in P_{r-1}. But why this $\sqrt{N}$? The $\sqrt{N}$ is the diagonal of any N-cube of side length 1. So, that is why that is $\sqrt{N}/2^{r-1}$. 1 is there the diameter of any N-cube in P_{r-1}. It should be less than ϵ. Let F be the collection of all members of P_r (now r has been chosen correctly) which intersect $U_\epsilon(x)$.

Then this $U_\epsilon(x)$ will be contained in the interior of $W(F).F$ contains all the N-cells which intersect $U_\epsilon(x)$. Therefore $U_\epsilon(x)$ is contained in $W(F)$ is obvious.Since $U_\epsilon(x)$ is an open subset of $\mathbb{R}^N$, it follows that it is contained in the interior of $W(F)$. For example, if I take all of them, it will be a bigger cube of side length double of that one 2^r around that point. Similarly, if I take an edge also that will do same thing for this maybe may not be look like that it may be brought up the but it will be some open set that open set will contain here union of cells. So, this is elementary topology which is inside $\mathbb{R}^N$.

Suppose now, τ is in P_s where $s > r$, and $\tau \cap U_\epsilon$ is non-empty. This τ some N-cube in P_s where $s > r$. It may very tiny member depending on how big s is. Its intersection with $U_\epsilon(x)$ is non-empty would mean that interior of τ intersection with interior of $W(F)$ is non empty.

That would mean interior of τ intersects interior some σ where σ is in F. That will imply that τ is contained in σ. τ is an N-cube of smaller size σ is of size $1/2^r$ and $s > r$. So, it cannot be that σ is inside τ but τ must be inside σ.

Next observe that since σ intersection $U_\epsilon(x)$ condition (i) or (ii) for the choice of ϵ imply respectively that σ is contained in $V \setminus X$ or V. We want to show that τ is never in C'. If both τ and σ are inside C, then clearly τ is not inside C'. So, when both of them are in C? That is what we have to examine. Here, the cases $A = C$ and $B = C$ to be done separately. In case $C = A$, what does this imply? We get both τ and σ belong to C why? See what the definition of definition of C here.

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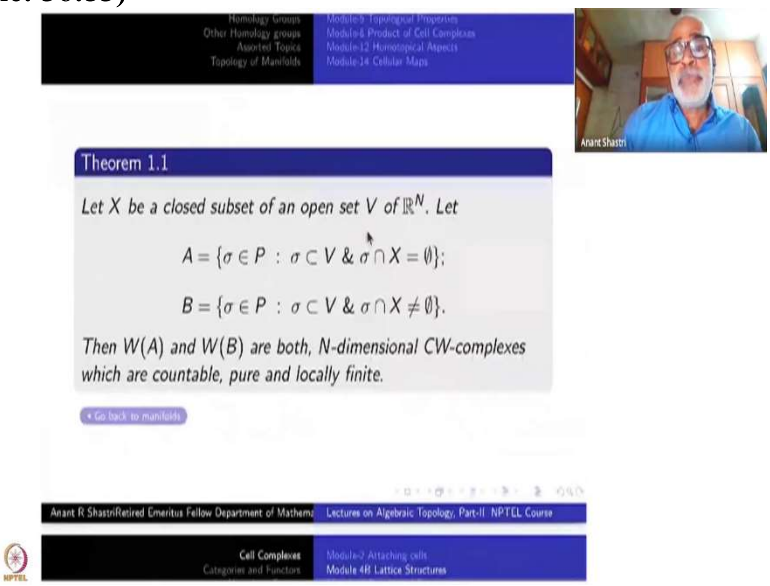

Go back definition of A: $\sigma \cap X$ is empty. Because τ is inside σ, it follows that τ intersection X is also empty. So, both of them will be inside A. That is one case in which τ is not inside A'.

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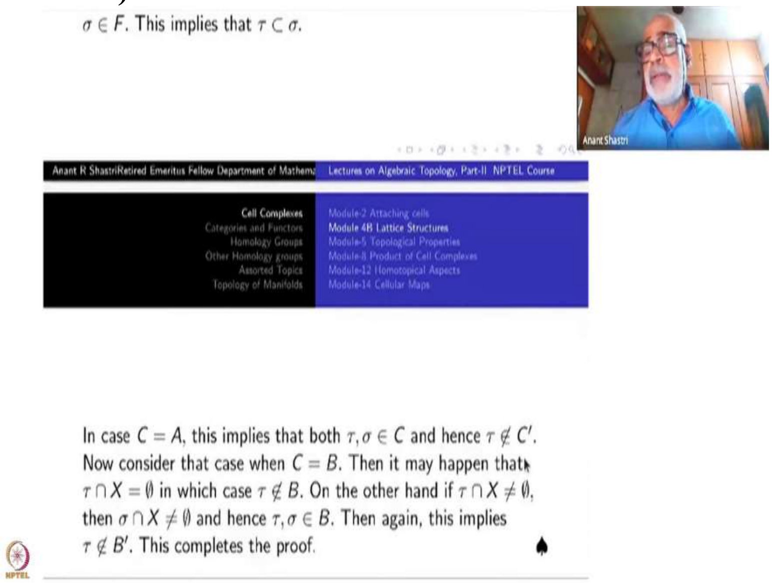

Now consider the case when $C = B$. Now it is something different. It may happen that τ intersect X is empty, but we want members of B and then τ intersection X must be non-empty. But then $\sigma \cap X$ is also non-empty and both τ and σ are in B which means τ is not in B'.

So, that completes the proof that this both A and B have a nice CW structures of dimension N, which are countable pure and local finite. You can further assume (this I am not going to use, but you can further assume) that the N-cells in each of them are in one-one correspondence with members of A' or members of B' respectively.

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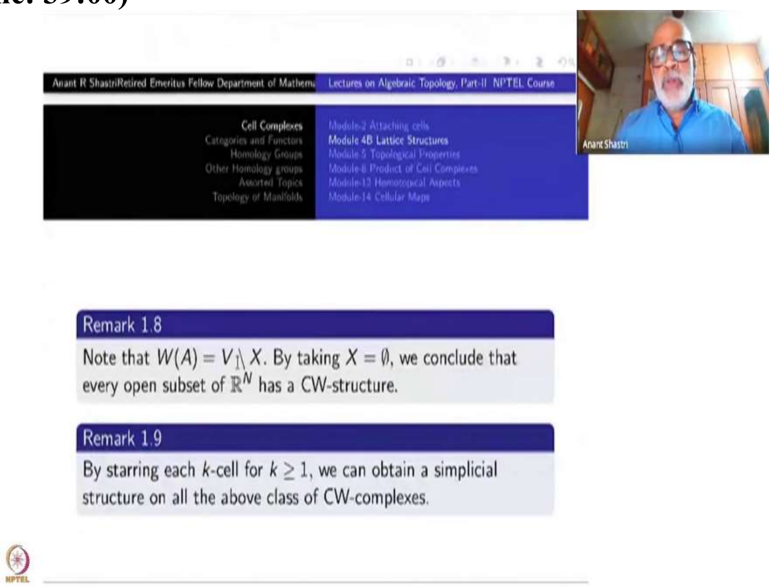

I want to make one more remark. I want to draw your attention this: $W(A)$ is exactly equal to $V \setminus X$. Take any point x in $V \setminus X$, then there is a positive distance d, of that point to the complement of $V \setminus X$ in $\mathbb{R}^N$. You can choose r large enough so that $\sqrt{N} < 2^r d$. Now there will exist some $\sigma \in P_r$ which contains x. It would follow that the entire σ is contained in $V \setminus X$.

That means $V \setminus X$ gets a CW structure. If X is empty what we conclude is that every open subset of $\mathbb{R}^N$ can be given a CW-structure, every open subset. So that is one conclusion. Further, all these CW structures are so nice they are all made up of N-cubes. Just star each face of these N cubes. That will become a simplicial complex. Therefore what we have proved here that every open subset $\mathbb{R}^N$ can be given as simplical complex structure. So we will use them in studying manifolds.

Here are some easy exercises for you. Try them; keep trying them. If you do not get them we are here to explain it to you. Our TAs will explain it to you. That is for today. Thank you.