Introduction to Algebraic Topology (Part – II)
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Lecture - 47
Paracompactness

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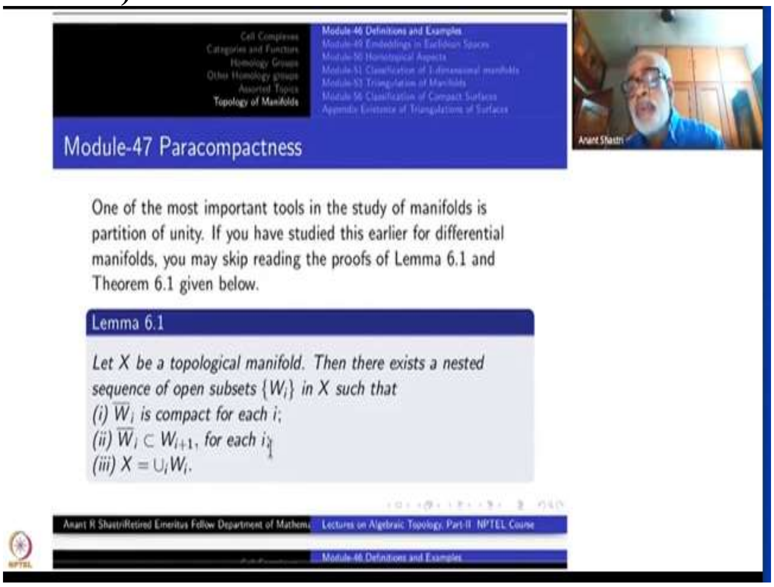

Continuing with the study of topological aspects of topological manifolds, so today we shall study paracompactness. Essentially, we are more interested in the partition of unity part of paracompactness rather than all those topological conditions of locally finite open refinement and so on. So we will directly verify the existence of partition of unity. Together with Hausdorffness, paracompactness is equivalent to the existence of partition of unity. This is a general result. So we will not lose anything okay?

You may have studied in calculus or in differential topology course, that every subset of $\mathbb{R}^n$ is paracompact. The proof is much simpler here. There, perhaps you have to bother about smoothness of the functions, in getting a smooth partition of unity. So the proof here is less difficult, okay? Then the proof of the existence of smooth partition of unity for subsets of $\mathbb{R}^n$. Indeed more or less, we repeat the steps that you have to go through in the case of a subset of $\mathbb{R}^n$. So let us carry on. The first step is the following lemma.

Start with a topological manifold. The statement of this lemma is that there exists a nested sequence of open subsets W_i in X such that each W_i closure is compact, W_i closure will be contained inside W_{i+1}, so on and X itself is the union of W_i's.

The proof of this lemma would have been totally easy in the case of $X = \mathbb{R}^n$, because then you can take W_i to be just the open ball with centre at the origin and radius equal to i. But of course, if X is any subset of $\mathbb{R}^n$, then you need to slightly modify these subsets W_i. There are many different ways of doing that. What we are doing here is select one method which easily generalizes to any manifold. So I am going to do that proof here okay.

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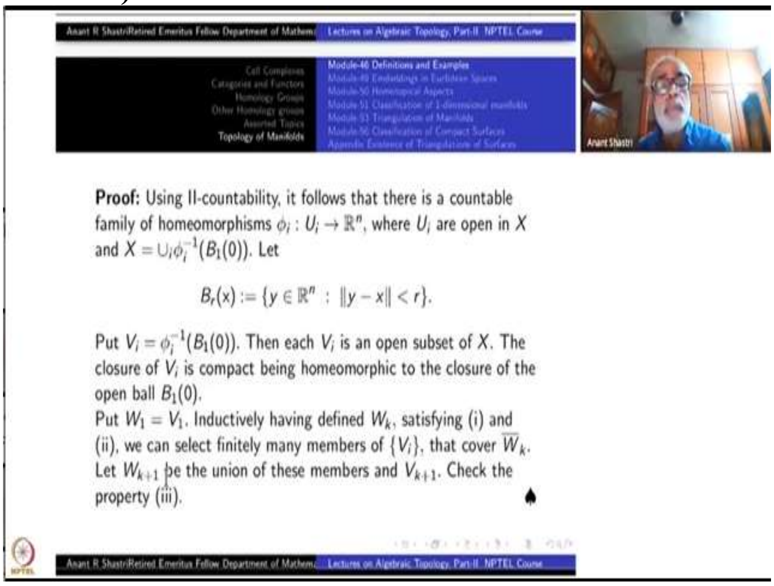

So what we are using here is the II-countability of X, okay? It follows that there is a countable family of homeomorphisms ϕ_i from open subsets U_i of X to $\mathbb{R}^n$, such that X is the union of $\phi_i^{-1}(B_1(0))$. There is always such an atlas for X. Out of which I can take a countable cover because every II-countable space is Lindelof. So whenever you have an open cover, it admits a countable sub cover, that is the Lindelof property okay.

Now, let us have this notation more generally. $B_r(x)$ is the set of all $y \in R^n$ such that $\|y - x\| < r$, an open ball of radius r centered at x. Okay? So put $V_i = \phi_i^{-1}(B_1(0))$, okay? Each V_i is an open subset of X. The closure of V_i will be ϕ_i^{-1} of the closure of $B_1(0)$, the closed ball okay. So that is compact so inverse image under the homeomorphism is compact. Okay?

So now start with $W_1 = V_1$, $\bar{V_1}$ is compact okay? Inductively suppose we have defined W_k to satisfy the property (i) and (ii) okay? Suppose you have constructed $W_1, W_2, \ldots, W_k$ with those properties. We select finitely many members of $\{V_i\}$ that cover $\bar{W_k}$, because $\bar{W_k}$ is covered by all the V_i's and $\bar{W_k}$ is compact. Therefore there are finitely many of them which cover $\bar{W_k}$. Let W_{k+1} be the union of these members and also V_{k+1}.

In other words, to begin with we have $W_1 = V_1$. At the second stage, W_2 contains V_2 and so on at the $(k+1)$-th stage with W_{k+1} contains V_{k+1}. It follows that property (iii) is satisfied. Obviously by the very definition, each W_k is open and its closure is contained in W_{k+1}. The closure of W_k being a finite union of compact sets, is compact. So all the properties (i), (ii) and (iii) are satisfied.

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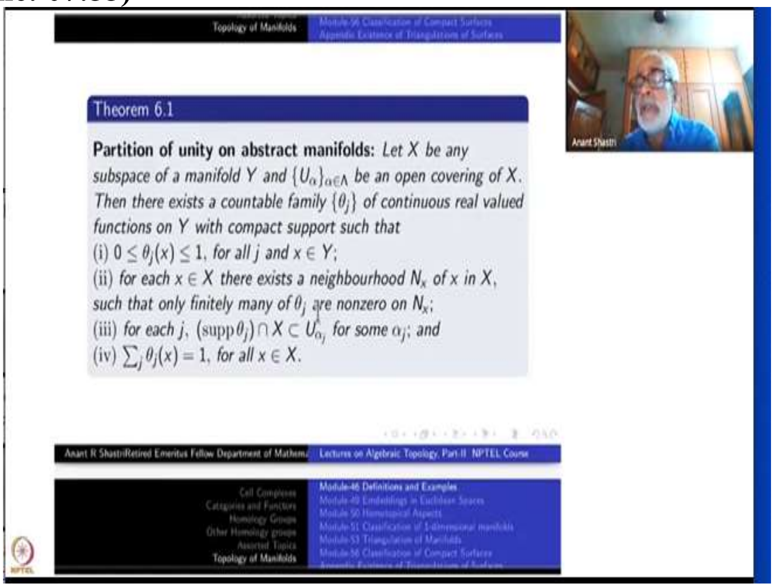

Having done that, we will now attack partition of unity. So let recall the definition here. Okay? Let X be any subspace of a manifold Y. So we are going to prove that every subspace is paracompact which is a stronger result than saying that the manifold Y is paracompact. Okay? Let X be any subspace of a manifold Y and $\{U_\alpha\}$ be an open covering for X. Then there exists a countable family $\{\theta_j\}$ of continuous real valued functions on Y with compact supports, (all this is part of the definition of a partition of unity, okay?) such that the following conditions hold:

(i) all θ_j are taking values between in the closed interval $[0, 1]$? And they are all defined on the whole of Y, the domain of each θ_j is the whole of Y, okay?

(ii) For each $x \in X$, there exists a neighbourhood N_x of x such that only finitely many θ_j's are nonzero on N_x. So this is called the locally finiteness of the family $\{\theta_j\}$ at each point of X. I am not claiming the local finiteness of the family on the entire of Y; that is very important, note it down carefully okay?

(iii) The third condition is that for each j, support of $\theta j \cap X$ is contained in one of the members of the open cover, say, U_{α_j}. This condition relates the family of fucntions with the given covering. support of $\theta j \cap X$ is contained inside one of the U_α.

(iv) The fourth condition is that the sum the total of θ_j's at any given point $x \in X$ is 1. That is why the name `partition of unity'.

The third condition has a name viz., the family $\{\theta_j\}$ is subordinate to the cover $\{U_\alpha\}$. I am just recalling this one here. I presume that you know paracompactness and also you know partition of unity. At least in part I, we have done all this.

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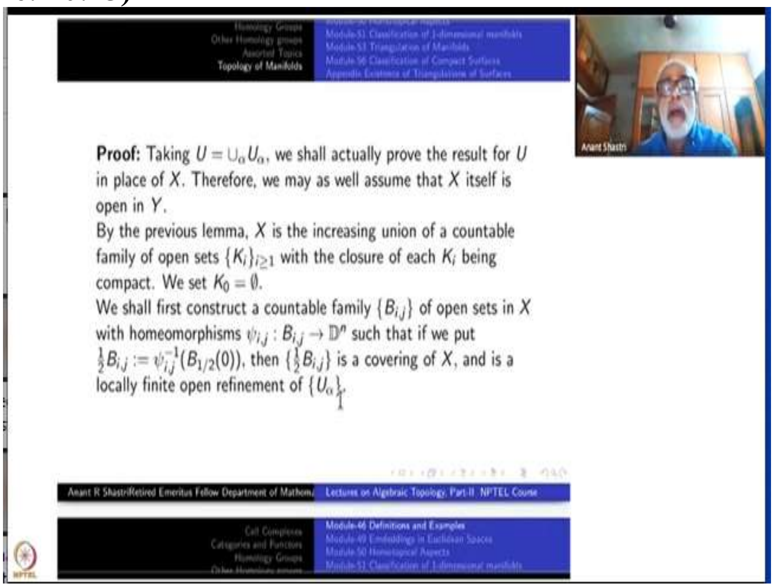

Take $U = \cup U_\alpha$ that will be an open set containing contain X okay? Instead of doing these things on X, I will do it on U itself, then condition all the conditions especially (ii), (iii) and (iv) will get verified for points of X as well. Because functions are going to be defined on the whole of Y any way. So it is enough to prove the theorem for open subsets of Y. Okay? So that is a simplification first. Therefore, we may assume that X itself is open in Y.

By the previous lemma, X is the increasing union of a countable family of open subsets $\{K_i\}$ (i ranging over natural numbers) with the closure of each K_i being compact and contained in K_{i+1}. So I am using the lemma here okay, every open subset of a manifold is a manifold and any manifold can be written like this is the previous lemma. So I can write X as an increasing union of open subsets such that the closure of each of them is compact and contained inside the next open set.

Just for the sake of logical simplification, $U_i = \emptyset$ for $i = 0$ and -1. Okay? We shall first construct a family $\{B_{i,j}\}$ double-indexed family, of open subsets of X homeomorphisms $\psi_{i,j}$ to the open unit ball $\mathbb{D}^n$ such that if we put $1/2B_{i,j}$= the inverse image of open ball with radius $1/2$ and centre 0, (this is just a convenient notation), then the family $\{1/2B_{i,j}\}$ themselves cover X and is a locally finite open refinement of the family $\{U_\alpha\}$. So this is what we are going to do. We have not yet done it, okay? Refinement means what? It means that each $B_{i,j}$ is contained in some U_α. Local finiteness means what? At each point $x \in X$, there is a neighbourhood N_x which will intersect only finitely many members of $\{B_{i,j}\}$. This will be done by induction on i. For $i = 0$, there is nothing to construct.

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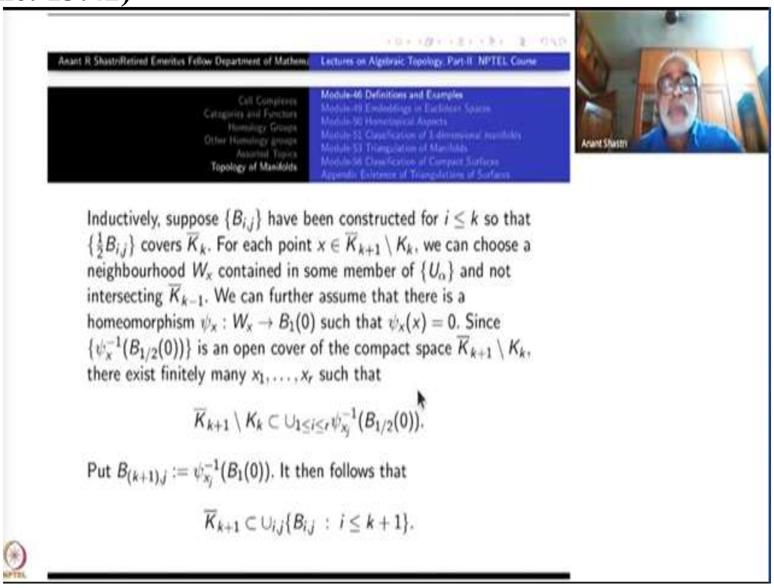

Inductively, suppose $B_{i,j}$ have been constructed for $i \leq k$. Okay, For each point x in the complement of K_k inside $\overline{K_{k+1}}$, (K_k is an open subset and $\overline{K_{k+1}}$ is a compact subset, so this complement is a closed subset of a compact and hence compact), you can choose a neighbourhood W_x contained in some member of U_α because U_α cover them and such that W_x is not intersecting $\overline{K_{k-1}}$. Okay? K_k contains $\overline{K_{k-1}}$. So I have thrown away K_k itself which contains $\overline{K_{k-1}}$ which is a closed set In a Euclidean space, you used the distance function here, but you do not need all that. You just use the fact that you are working on a compact set $\overline{K_{k+1}} \setminus K_k$ and then inside the open set U_α intersected with the complement of $\overline{K_{k-1}}$.

We can further assume that there is the homeomorphism ψ_x from W_x to $\mathbb{D}^n$ okay? By taking a smaller open subset if necessary. You can also assume that $\psi_x(x)$ goes to 0. Now

$\{\psi_x^{-1}(1/2\mathbb{D}^n)_x$ the family of inverse image of half open disc is an open cover for the compact set $\overline{K_{k+1}} \setminus K_{k-1}$.

So I get finitely many $x_1, \ldots, x_r$ such that the members $\psi_{x_i}^{-1}$ cover this compact set. Label them as $B_{k+1,j}, j = 1, 2, \ldots, r$. Of course, r depends on k but we do not need to bring in more elaborate notation. Notice that, we have not used any induction hypothesis here So, the above step is valid simultaneously to all k.

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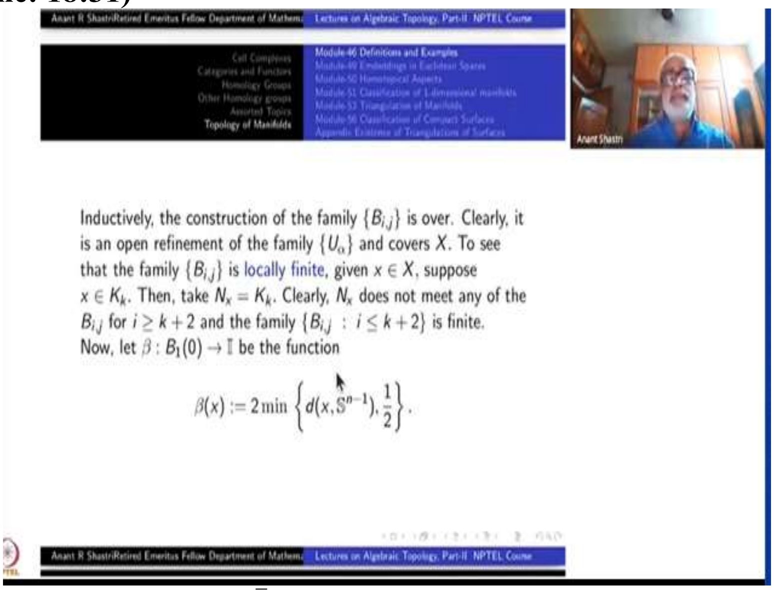

It follows that $B_{i,j}$ for $i \leq k$ cover $\bar{K_k}$, for each k, so therefore $\{B_{i,j}\}$ form a countable open cover for X. We have selected these open subsets which are smaller than one of the members of $\{U_\alpha\}$ and therefore, this family is a refinement of the original covering $\{U_\alpha\}$.

Given $x \in X$, say x is in K_k. Take $N_x = K_k$ itself. That is an open neighbourhood of x which does not meet any of the $B_{i,j}$ if $i > k+2$. Remember B_{k+1}, j were chosen not to intersect K_{k-1}. In particular, $B_{k+2,j}$ will not intersect K_k. The same will be true for all $B_{i,j}$ for $i > k+2$ as well. So how many of $B_{i,j}$'s will be there which may intersect K_k? You have to take i only upto $k+1$. For each i, there are only finitely many j. So there are only finitely many these balls which will intersect $N_x = K_k$, okay? So this proves the local finiteness of the family $\{B_{i,j}\}$. Alright?

Finally, let β from $\mathbb{D}^n$ to $\mathbb{I}$ be this function (In the case of $\mathbb{R}^n$, you can actually get smooth maps which are the called bump functions. Here I am not interested in smooth maps so I am just taking this.) viz, twice the minimum of the distance between x and $\mathbb{S}^{n-1}$ and $1/2$. So β takes values always between 0 and 1 and is a continuous function okay? You could have taken also such smooth functions but here continuous functions are enough. The important property of β that we are interested in is that β takes the constant value 1 on the closed half ball and as the point moves toward the boundary it decreases to 0 and is actually 0 on the boundary. We do not need β to be smooth because anyway when we compose it with the homeomorphism ψ, there is no notion of smoothness. So do not worry about that.

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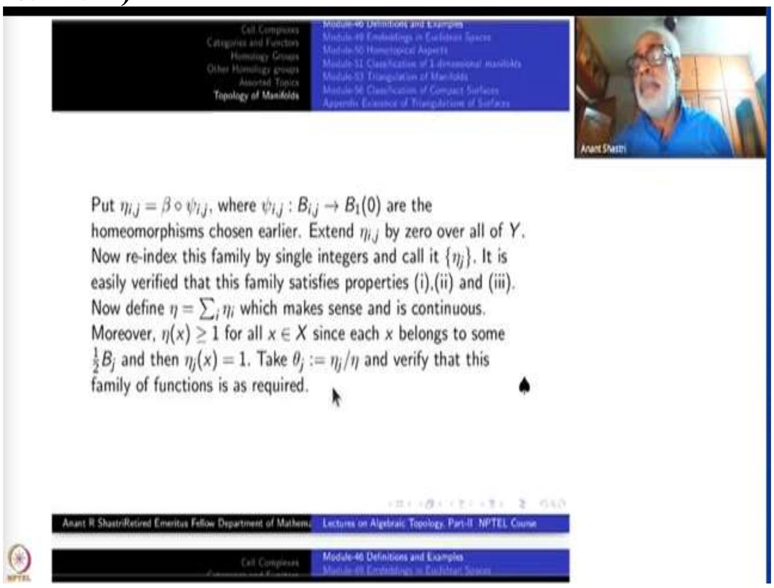

So put $\eta_{i,j} = \beta \circ \psi_{i,j}$ where $\psi_{i,j}$ is homeomorphism from $B_{i,j}$ to $B_1(0)$ are chosen as before. So first takes $\phi_{i,j}$, it will come in to $B_1(0)$ and then compose it with β. Extend $\eta_{i,j}$ by 0 over all of Y. This is possible because $\eta_{i,j}$ are such that on the boundary, they are identically 0. So, beyond that you can take their value to be the 0 vector to get continuous functions $\eta_{i,j}$ from $Y_t[0,1]$. Okay? Now, reindex this family by single integers, why bother about double indexing it. Right in the beginning itself, I could have done that because anyway for each i fixed i, there are only finitely many j's, so it is just a countable family. Re-index it say by $\{\eta_j\}$. It is easily verified that this family satisfies properties (i), (ii) and (iii) for θ_i's. First one is that the values are between 0 and 1 that is what I have told you because they are got by composing with β whose codomain is $[0,1]$, okay?

We verified that $B_{i,j}$ is locally finite and outside $B_{i,j}, \eta_{i,j}$ is 0. For the same reason, (ii) and (iii) are also automatic. Okay. Now the fourth condition. We do not know anything about that yet. Okay? So take η to be the sum total of all η_j, why this makes sense? Because it is a locally finite family. In a neighbourhood of a point, only finitely many of these things will survive and hence the total sum restricted to that open neighbourhood is finite sum finite sum of continuous functions. So η is continuous on that open set. Since it is continuous at every point of Y, it is continuous on the whole of Y.

But also at each point there is at least one j such that $\eta_j(x)$ is positive. Since you are taking sum of non negative functions, so value of the total sum is positive. Indeed, it is always bigger than or equal to 1, because each $x \in \psi_j^{-1}$ of some half ball and η_j takes the value 1 there.

Now, take $\theta = \eta_j/\eta$ okay? Dividing by it makes sense because this is nonzero continuous function okay? So I can divide by η and verify that this the new family $\{\theta_j\}$ satisfies the last condition also, namely, now the sum of all θ_j will be always equal to 1, because the numerator and the denominator are the same. That completes the proof of the theorem.

So let us look at a few remarks following this theorem. If we begin with a smooth manifold Y , then θ_j can be chosen to be smooth. How? First of all the β can be chosen to be smooth. Then all these local homeomorphisms, the charts can be chosen to be smooth. So this gives smooth of partition of unity for smooth manifolds.

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So let us look at a few remarks following this theorem. If we begin with a smooth manifold Y then θ_j can be chosen to be smooth. How? First of all the β can be chosen to be smooth. Then all these local homeomorphisms, the charts can be chosen to be smooth. So this gives smooth of partition of unity for smooth manifolds.

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Naturally, there are many consequences of existence of countable partition of unity. For instance, you do not have to prove Urysohn's lemma separately for manifolds Okay? So all that I told you, you could have done independently, normality etc. II countability together with local compactness give quite a few other topological properties. Those things you can do just by using partition unity. For instance, Tietze's extension theorem can proved by using partition of unity okay?

For example, here is some thing new. If $\{\theta_j\}$ is some partition of unity on X, then define f from X to $[0,\infty)$ by the formula $f(x) = \sum k\theta_k$. okay? Then f will be a proper mapping of X into $[0,\infty)$. Proper map means what? inverse image of compact sets are compact okay? So this I will leave it to you as an exercise, easy exercise.

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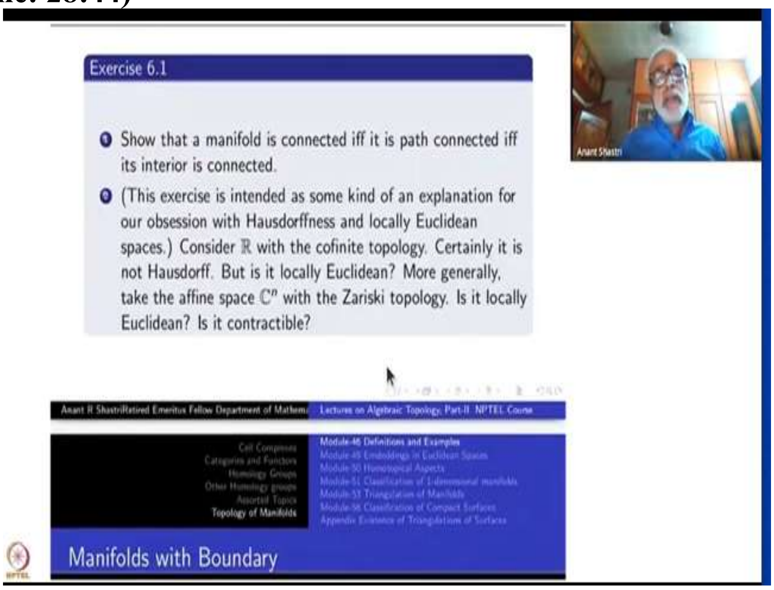

So I have given a few more exercises here, which are not all that difficult.

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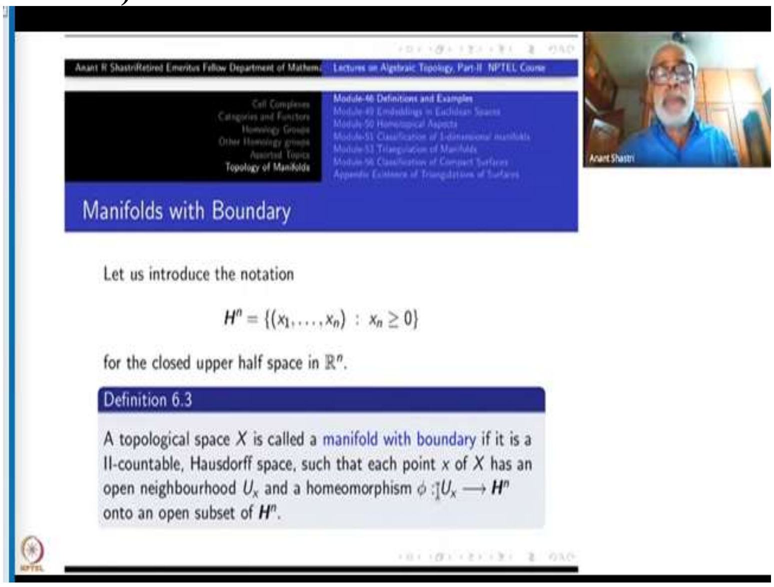

I will just start the next topic of interest now. Today, but we will not go very much deeper into it. So that you will have some time to get familiar with this definition. Namely, start with this definition of half space $\mathbb{H}^n$ inside $\mathbb{R}^n$. What is this? Set of all those points $(x_1, \ldots, x_n)$ with the n-th coordinate $x_n \geq 0$. So this is a subspace of $\mathbb{R}^n$. I call it the upper half-space.

A topological space X will be called a manifold with boundary, if... (I am going to define the entire phrase `manifold with boundary' a new nomenclature here, Now I am going to extend the definition of local Euclideaness okay? To cover things like closed intervals closed disc etc. Our original definition includes open intervals and open discs but a closed interval was not a manifold. So I want to include them now. So this is the mechanism. The conditions of II-countability and Hausdorffness they are there. However, local Euclideanness changes. How?)... that each point of x of X has an open neighbourhood U_x and a homeomorphism ϕ from ϕ (now use $\mathbb{H}^n$ instead of $\mathbb{R}^n$ okay) onto an open subset of $\mathbb{H}^n$, Okay? Think of the half closed interval $[0, \infty)$, okay. That is the upper half space $\mathbb{H}^1$ in $\mathbb{R}$. What are its open subsets? $[0, 1)$ is an open subset whereas $(0, 1]$ is not open. All open intervals contained in $\mathbb{H}^1$ are also open. So the half space has more open sets than the full space okay? You have this swallow this one. The half space has more open sets than the full space upto homeomorphism. Therefore, this is an extension of the old definition of locally Euclideanness, namely you are allowed to have an open subset which is completely in the open part of the half space it does not touch the boundary of $\mathbb{H}^n$ in $\mathbb{R}^n$. (Boundary is what? Points with the last coordinate equal to 0 okay?

Because if you take x_n positive here okay that gives interior points of $\mathbb{H}^n$, they form an open subset of $\mathbb{R}^n$, okay? Oo our old definition of Local Euclideanness is included also. So, more manifolds will come now Okay? So let us start from here next time. Thank you.