Introduction to Algebraic Topology (Part – II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

Lecture – 46
Definitions and Examples

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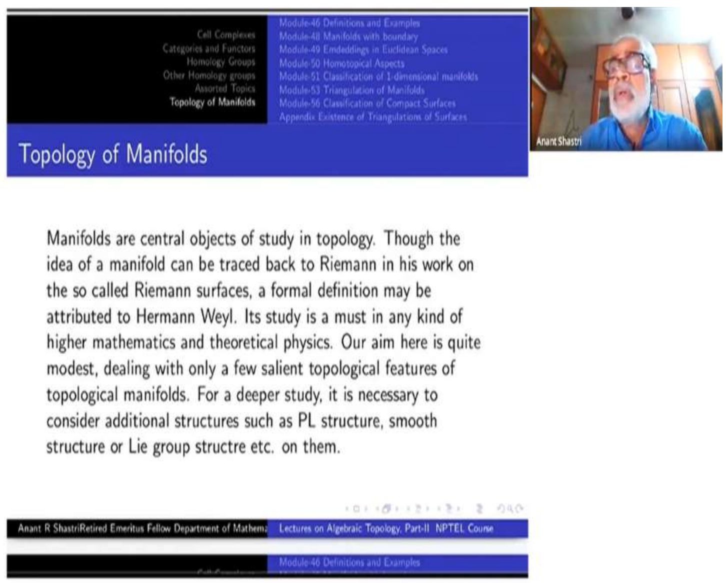

So, today we will start the last topic of the course `topology of manifolds'. As I told you several times, manifolds are the central objects of study in topology. The idea of manifold can be traced back to Riemann. The formal definition as we use today is perhaps due to Hermann Weyl. Its study is a must for any kind of higher Mathematics and Theoretical Physics. Our aim here quite modest, dealing with only few salient topological features of the topological manifolds.

Actually as you keep putting more and more structures, like differentiable manifolds, and PL manifolds, Lie groups and so on, the study becomes more and more interesting and more and more concentrated also. We are not going to do all that. Normally, the topological manifolds are quite difficult to handle because you do not have much structure on them. On the other hand some of these things are neglected in the other studies and taken for granted.

Suppose you are studying Lie groups you just assume whatever is happening to the underlying topological manifolds quite often. I mean that is what happens with the books and teachers. So, I thought of giving a good treatment for just topological manifolds here.

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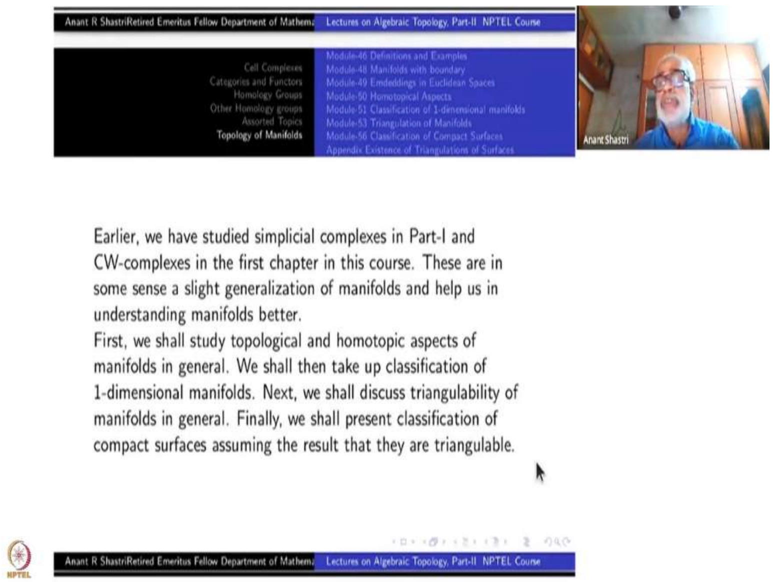

Earlier we studied simplicial complexes in part I and then CW complexes in the very first chapter of this part. They are in some sense, a slight generalization of manifolds and help us in understanding manifolds better. So, we will bring them in study of topological manifolds also, the CW-structures as well as simplicial structures okay.

First we shall study purely topological and a bit homotopical aspects of general topological manifold. We will then take up a classification of one dimensional manifolds. Next we shall discuss triangulability of manifolds in general. Finally, we will end up these talks with classification of compact surfaces. By surfaces, I mean 2-dimensional manifolds. Classification will be done assuming that they are triangulable. But of course, because of lack of time we are not able to present the proof of the triangulability of surfaces. So, this is just the gist of what is going to come in another 15 modules or so, okay?

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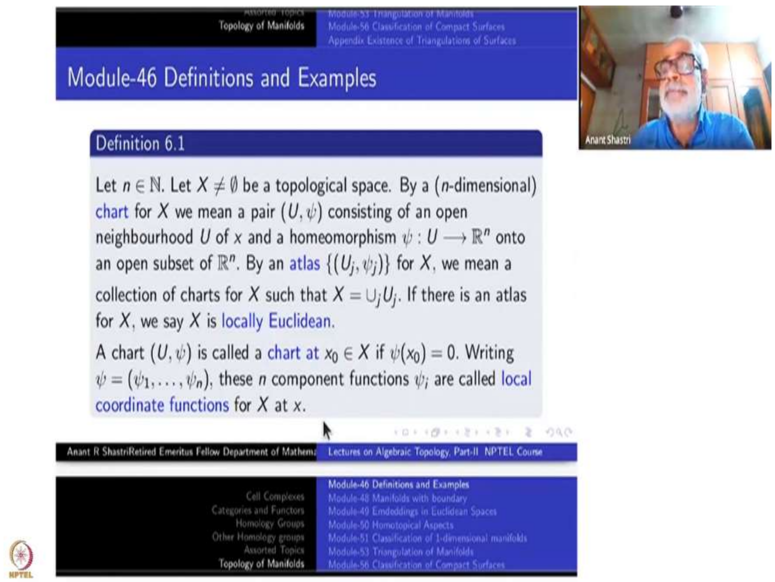

So, today we will just see the basic working definitions and examples. Fixed a positive integer n. Let X be a non empty topological space by an n-dimensional chart for X, we mean a pair (U, ψ) consisting of an open subset U of X and a homeomorphism ψ from U onto an open subset of $\mathbb{R}^n$ okay. By an atlas for X, we mean a collection of charts $\{(U_j, \psi_j)\}$ for X, such that the domains of these charts we will cover the whole of X. So, X is a union of U_j's.

If there is an atlas for X, we say X is locally Euclidean. The `locally Euclidean' just means that each point has a neighbourhood which just looks like, what it looks? It is homeomorphic to an open subset of $\mathbb{R}^n$, okay? In fact one can assume that this open set is the whole of $\mathbb{R}^n$, no problem.

A chart (U, ψ) is called a chart at x_0, if $\psi(x_0) = 0$ okay? This is a special word I am just using. You may not find it in the literature elsewhere. This is my own convention. By just saying that psi is a chart x_0, I want to convey the meaning that x_0 is in U and $\psi(x_0) = 0$. We can write $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$, in terms of its n-coordinate functions, because ψ takes values in $\mathbb{R}^n$. Often, we refer to this by saying that ψ_i are local coordinate functions for X at x_0. That is the reason why I am putting this x_0 going to 0, so that x_0 corresponds to the origin in $\mathbb{R}^n$, and these functions can be thought of as x_1- coordinate, x_2-coordinate respectively.

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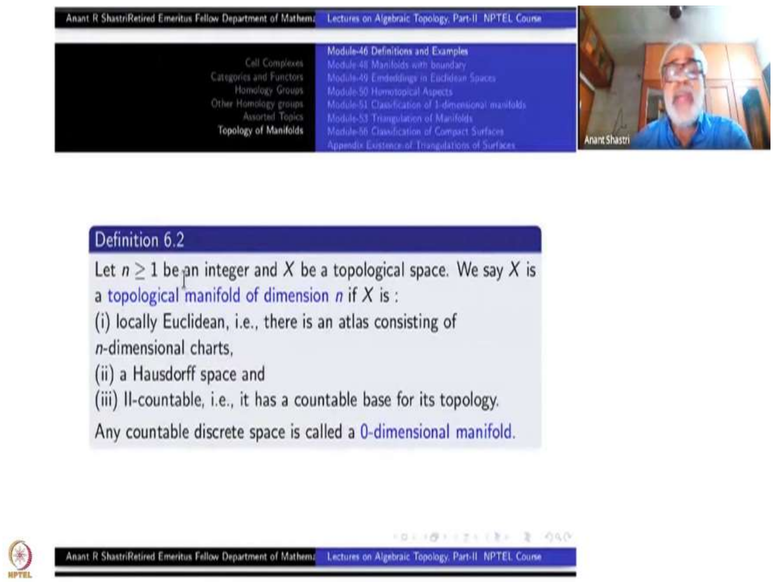

Let X is a topological space with an n-dimensional chart. We call such a space `locally Euclidean'. This means that there is a collection of charts, and each member of this collection is an n-dimensional chart where n is fixed. That is the first condition. We put two more topological conditions, namely, that X must be a Hausdorff space and it must be II-countable. Recall that II-countable means that the topology on X has a countable base okay? With these three conditions, on X, X is called a topological manifold of dimension n.

So that finishes the definition of a topological manifold. In all this, we seem to have assumed that $n > 0$. So, I would like to make the definition explicitly for the case $n = 0$. Any countable discrete space is called a 0-dimensional topological manifold. Here, local Euclideanness follows because when $n = 0$, $\mathbb{R}^0$ is just a singleton space. Hausdorffness follows because every discrete space is Hausdorff.

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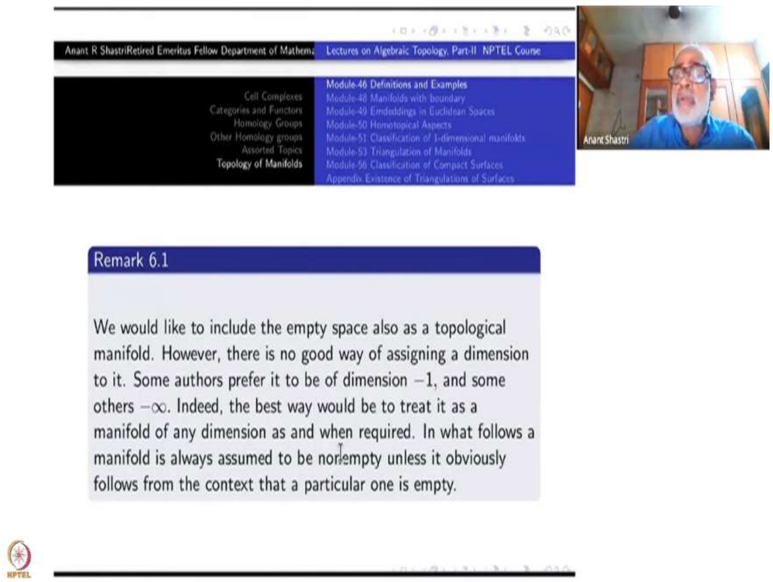

Finally, we would like to include the empty space also among manifolds. So far, we have assume that X is with non empty topological space. It is convenient to include the empty space also as a topological manifold. However there will be always the problem of assigning a dimension n to the empty space. What should be n? $0, 1, 2, 3, \ldots$ What should we assign? It makes sense to have n any of non negative integer. Since there is clash we cleverly choose to fix dimension of the empty space to be -1.

It should be noted however, that at variously contexts in algebra and geometry, it is convenient to assign $-\infty$. But you will see that the best way is to not assign any specific dimension to 0. That is just like in the case of 0-polynomial, the degree can be taken to be any integer. Similarly, the empty manifold can be given any dimension depending on the context, you can choose it conveniently. Anyway, these things do not matter right now for us but when you do more complicated mathematical theories, you will see the importance of this remark.

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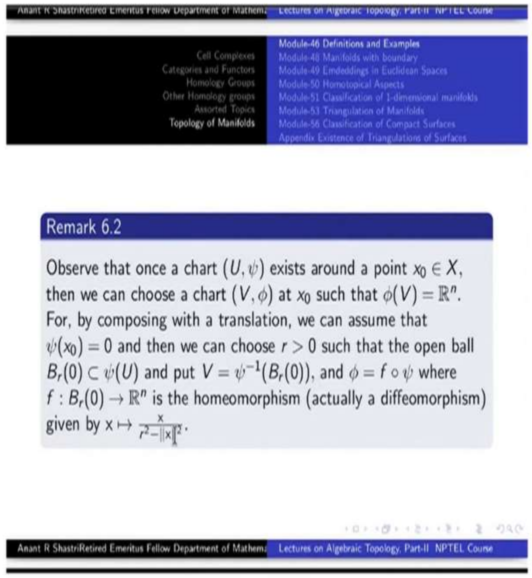

Observe that given a chart (U, ψ) and a point x_0 in U, okay, we can choose a chart (V, ϕ) at x_0 such that $\phi(V)$ equal to the whole of $\mathbb{R}^n$ okay? Because given any open subset of $\mathbb{R}^n$, and any point whatever in it, you can choose a round disc okay? The set of all points which are at a distance from the point strictly less than some suitable chosen positive ϵ will be contained inside the open set. And such an open disc is homeomorphic to the whole of $\mathbb{R}^n$. So, you can compose it with the restriction of ψ and take that to be ϕ okay. So, this is the advantage of topological manifolds. It is true for $\mathbb{C}^\infty$ manifolds also but when you go to analytic manifolds, like complex manifolds and so on, this is not possible. So, you have to be careful about this. By composing with a translation, we can assume that $\psi(x_0)$ is 0 also. Okay these are not restrictions these are always possible.

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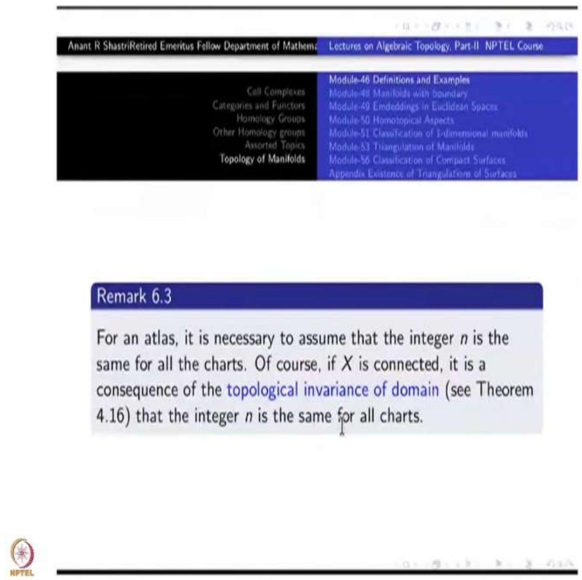

For an atlas, it is necessary to assume that integer n is the same for all the charts. Of course, if you assume X is connected, it easy consequence of topological invariance of domain. If (U, ψ), (V, ϕ) are charts and $U \cap V$ is non empty, then $U \cap V$ will be homeomorphic to an open subset of both $\psi(U) \subset \mathbb{R}^n$ and $\phi(V) \subset \mathbb{R}^m$. Therefore $m = n$. That proves that dimension is a locally constant function on a locally Euclidean space. Hence, if X is connected then it must be a constant.

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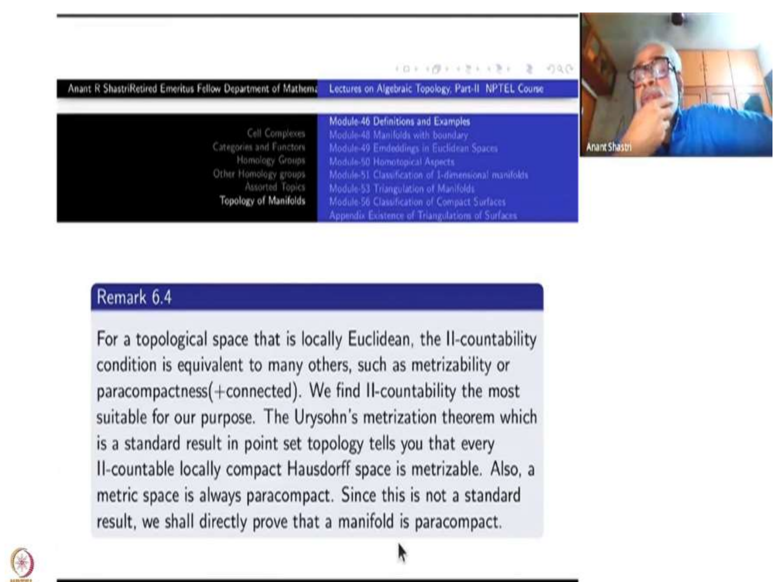

In any case, since we do not allow disjoint union of two manifolds of different dimensions, as manifold, why bother about all this, just fix the integer n right in the beginning as we have done.

For a topological space that is locally Euclidean we have put two more conditions, Hausdorffness and II-countability, to be a manifold. The II-countability condition is equivalent to many other conditions such as metrizability or (paracompactness plus connectedness of course), okay? So, depending upon a particular author's fancy, you may find different definitions. But II-countability is most popular and easier to handle. The point is that under slightly suitable conditions like Hausdorff and so on, they are all equivalent. So do not bother if someone has taken a different condition than II-countability in the definition.

However, if you ignore Hausdorffness, there will be chaos. There are people who study locally Euclidean spaces which are not Hausdorff, okay? That is one thing we are NOT going to do here okay?

So, we find second countability most suitable for you our purpose. The Uryson's meterzation theorem which is a standard result in point-set-topology tells you that every second countable, locally compact Hausdorff space is metrizable, you can embed it inside the Hilbert cube, okay and then take the induced metric. Also a metric space is always paracompact okay? Since this is not a standard result here we shall directly prove that a manifold is paracompact.

If you have studied it in your topology course or not then there is no problem okay. So, here you will see that anyway. just assume $1, 2, 3$, namely, locally Euclidean, II-countable and Hausdorffness will show that it is paracompact. Indeed, a more suitable conclusion that we are interested in is that a manifold admits partition of infinity. So, this proof will be much simpler than the proof of partition of infinity for CW-complexes okay? So that is one of the aims here. That we will do next time.

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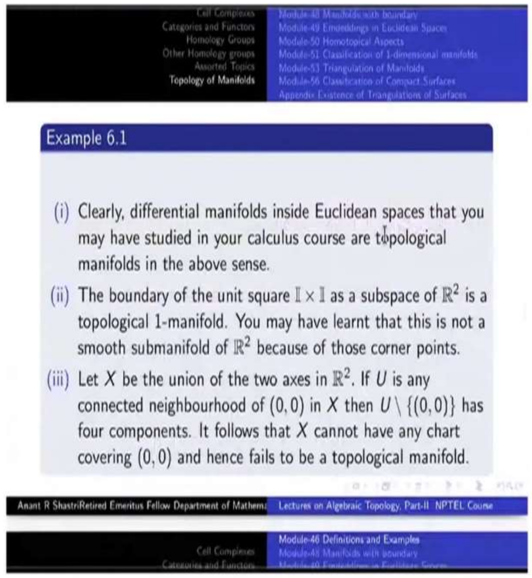

Clearly, the differentiable manifolds inside Euclidean spaces okay, that you may have studied in your calculus courses while studying Stokes theorem and so on, they are topological manifolds in the sense that each point has a neighborhood which is diffeomorphic to some open subset of $\mathbb{R}^n$. So ,since diffeomorphisms are homomorphisms, automatically our conditions of a topological manifold are satisfied okay. The boundary of the unit square $\mathbb{I} \times \mathbb{I}$, as a subspace of $\mathbb{R}^2$ is a topological 1-dimensional manifold. It is a simple example okay. $\mathbb{R}^n$ itself for all n, is n-dimensional manifold. Any non empty open subset of $\mathbb{R}^n$ is n-dimensional manifold. These are easy examples.

Inside $\mathbb{R}$, only open subsets, all of them are 1-dimensional manifolds. Other than those, if you get out, say, in $\mathbb{R}^2$, the circle is a nice (smooth) manifold, (differentiable manifold of dimension 1, okay? But here $\mathbb{I} \times \mathbb{I}$ okay has corners. Yet it is a topological manifold of dimension 1. You may have learned that this is NOT a smooth manifold inside $\mathbb{R}^2$, because there are corners. So, I just wanted to give this example just to contrast between topological manifolds and smooth manifolds.

Take the union two axes, x-axis and y-axis in $\mathbb{R}^2$, okay? This is not a manifold. If it were a manifold it would have been a 1-dimensional manifold right? Because you take any point in it other than $(0,0)$, it has neighborhood which is homeomorphic to an open interval right? At $(0,0)$, there is a problem, everywhere else there is no problem. If you take a neighborhood U of this origin, okay, no matter how small it is the complement of the origin, i.e., if you remove the $(0,0)$ from U, then there will be at least four connected components. When you remove a point from an interval, you will get at most two components. Therefore, there cannot be any homeomorphism of U to an open interval okay? So, this argument gives you many examples of graphs with such forks which are not manifolds right?

So, among conic sections, intersecting pair of straight lines is not a manifold. All other conics in $\mathbb{R}^2$ are 1-dimensional manifolds, connected or not.

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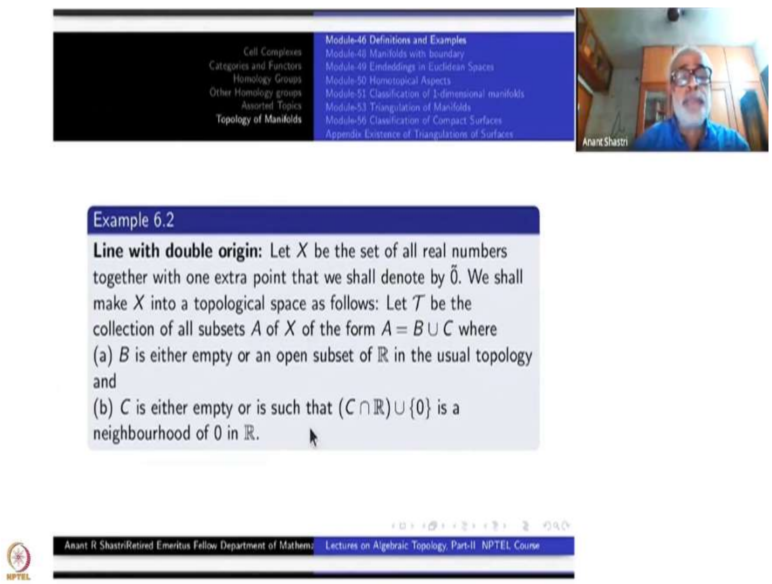

So, here is an example of a locally Euclidean space which is II-countable also but not Hausdorff. It is very easy to construct such things, namely, all that you have to do is to take two lines, okay, and identify all the points from one line to the other line except the origin, origin here and origin there standout separately. So the quotient space so obtained is called a line with a double origin. What happens here. Say in the two line denote the two origins by $0'$ and $0''$ say.

And neighborhood of $0''$ will be an open interval that open interval will intersect every open interval containing the $0'$ in the quotient space. So, the quotient space cannot be Hausdorff okay? Other two conditions are satisfied here okay. So, you can do many such simple examples wherein Hausdorffness does not follow from the local Euclideanness and/or II countability.

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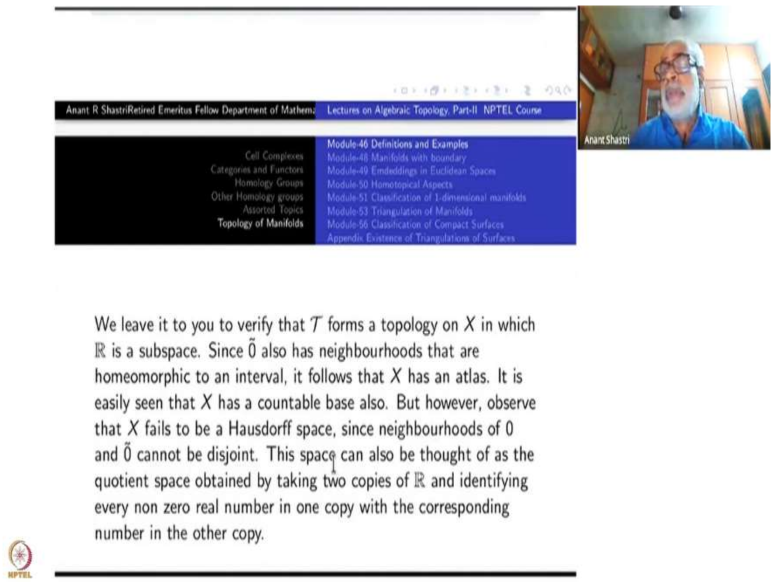

Similarly you can construct lines with triple origins also, and so on. But the purpose of introducing such examples is just to illustrate the fact that of Hausdorffness is not a consequence of the local Euclideanness. We are not interested in this kind of spaces okay? We do not call them manifolds.

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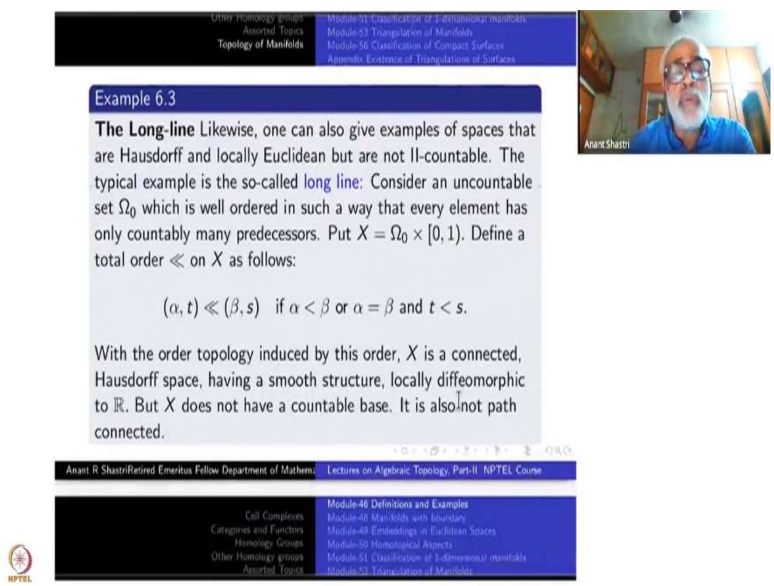

Similarly, there is something called the `long line'. It has all the features of the real line but it is `too long' to be II-countable. To understand this one what you need is to understand the ordinal topology, ordinal numbers. Ordinals are first of all well ordered sets okay? And then you take Ω to be the set of all the countable ordinals which itself is an uncountable set with a total order. Now take its product with the closed interval $[0,1]$ and take the lexicographic order, viz.,

$(\alpha, t) \preceq (\beta, t')$ iff α is less than or equal to β and if $\alpha = \beta$ then $t \leq t'$. Check that this makes $\Omega \times [0,1]$ into a totally ordered set. Take the corresponding order topology on it. If $\tilde{0}$ is the least element of Ω, then throw away the point $\tilde{0} \times 0$ from $\Omega \times [0,1]$. The resulting space $\mathcal{L}$ is called the long line. It is not hard to check that this $\mathcal{L}$ is locally Euclidean of dimension 1. All ordered topologies are anyway Hausdorff.

However, $\mathcal{L}$ is not II-countable okay? We have no time to recall ordinal topology and so on. So, you have to assume that. If you want to read more about it, well there are a lot of sources. One source is KD Joshi's book set topology okay? Another is my NPTEL course on Point Set Topology Part-II. This space is path connected 1-dimensional locally Euclidean but not a manifold. Something strange is happening here, okay

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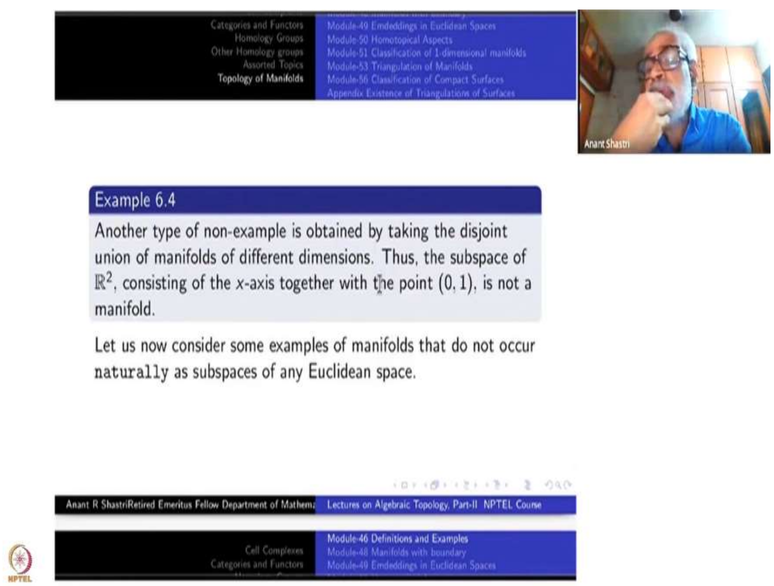

Another type of non example is obtained by taking disjoint union of manifolds of different dimensions. Take a circle and take its union with a discrete point. We can do that kind of things in a CW complex but this will not be a manifold. Or take a circle and take its union with disjoint 2-sphere, etc.

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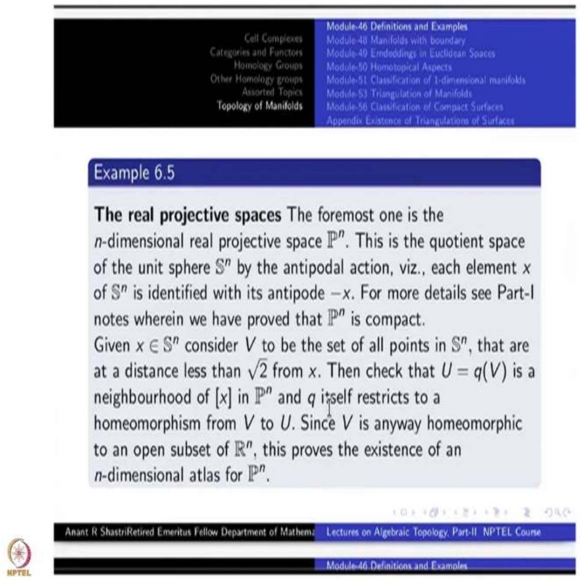

So, we have quite a few elementary kind of examples and non examples. But now let us come to more serious and more useful and central examples and the first one is of course the sphere, okay, parabola, hyperbolas, conic sections except a pair of intersecting st. lines, spheres of all dimensions, product of finitely many of these, open subsets of any of them etc, all these things are standard right?

One important non trivial example is the projective space, which does not come as a subspace of $\mathbb{R}^n$. It does not occur like that but later on we can find models for $t_i \in R^N$, that is different matter. But as a definition it is not taken as a subspace of $\mathbb{R}^n$, because it is occurring as a quotient space right? The n-dimensional real projective space $\mathbb{P}^n$ (or you can take the complex projected space $\mathbb{CP}^n$ also exactly similarly), is the quotient space of the unit sphere $\mathbb{S}^n$ by the antipodal action of $\mathbb{Z}/2\mathbb{Z}$, okay? Every point x in $\mathbb{S}^n$ is identified with $-x$. Or you can take $\mathbb{R}^{n+1}$ setminus the origin and then take quotient by the action of $\mathbb{R}$ set minus 0) via scalar multiplication. That gives you another picture of the projective space viz, the space of all lines inside $\mathbb{R}^{n+1}$ passing to the origin okay? We done all this in more details earlier.

So, given x belong to $\mathbb{S}^n$, consider V_x to be the set of all points in $\mathbb{S}^n$ which are at a distance less than $\sqrt{2}$ from x. Look at the open ball of radius $\sqrt{2}$ around that point and intersect it with the sphere okay. If you restrict the quotient map to V_x that will be a homeomorphism. There is no identifications there okay? And this open subset V_x is definitely homeomorphic to an open subset in $\mathbb{R}^n$. That will tell you that the projective space is locally Euclidean of dimension n. okay? The only thing that you have to verify is that it is Hausdorff.

The point is that this being a quotient of a sphere is compact space and every compact space is II-countable automatically okay?

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So, why this is Hausdorff? That also follows easily. Take any two points $x, y \in \mathbb{S}^n$. Depending upon the distance between them, okay? You can choose $\epsilon > 0$ such that an ϵ neighborhood around each of four points $x, y, -x, -y$ are disjoint. Now when you go to the quotient space $\mathbb{P}^n$, you will see that you get image of these open subsets will form disjoint neighbourhoods of $[x]$ and $[y]$.

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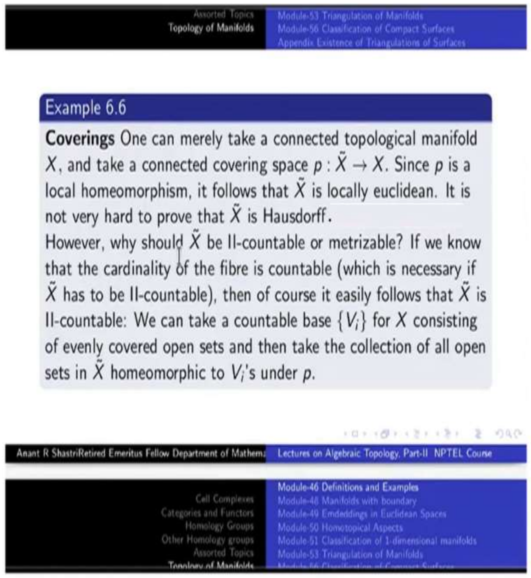

Now I want to give you some more theoretical examples namely when you take a covering projection, okay? I would like to have a picture like this. If X is a manifold and $\tilde{X}$ to X is a covering projection, then $\tilde{X}$ will be also manifold. Or the other way round, viz., if $\tilde{X}$ is manifold, we may anticipate that X is also a manifold? It happens to be true, okay? But we have to be careful here.

So, let consider only connected spaces, and not bother about too many connected components. Take a connected space X and take a connected covering space $\tilde{X}$. Since every covering projection p is a local homeomorphism, it follows that $\tilde{X}$ is locally Euclidean iff X is locally Euclidean (of the same dimension). Moreover, we have seen in part I that X is Hausdorff iff $\tilde{X}$ is. We leave this to you as a not so difficult exercise.

So, the only problem is about II-contability. If $\tilde{X}$ is II-countable being a quotient of $\tilde{X}$, X will be also II-countable. It is not difficult. But the converse is not clear. Why II-countability of X implies that of $\tilde{X}$. So, earlier I have said that the II-countability is easy to handle but now we have a problem here. On the other hand, if try use other equivalent conditions, such a metrizability or paracompactness etc. you will have more problems okay. So, problem is there, how to prove that a connected covering space of a topological manifold is II-countable. In fact it will not be true if this covering has uncountable sheets. Why? Because an uncountable sheeted covering means that the inverse image of a point under p is an uncountable discrete subset. And if you have an uncountable discrete subset of a topological space, then the space cannot be II-countable. So, the only hope is that any connected covering of a manifold has at most countably many sheets, finite case is okay.

Then fixing a countable base $\{V_i\}$ of evenly covered open sets for X, we can take the collection of all open sets $\{U_j\}$ in $\tilde{X}$ such that p restricted to U_j is a homeomorphism onto some V_i. And check that this collection forms a countable base for $\tilde{X}$. But then we know that the number of points in the fiber i.e., the number of sheets of a connected covering is closely related to the cardinality of the fundamental group of the base space X, viz, it can be identified as a quotient set of $\pi_1(X)$. Indeed, if the cardinality of the fundamental group of X is uncountable, then you can take the simply connected covering of X, that will not be II-countable okay? Luckily we can prove that the fundamental group of a topological manifold is countable. So that will be one of the central results here okay? So, once we prove that, it will follow that a connected covering spaces of a topological manifold is a manifold okay.

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So, this is one of the fundamental problems. There are other kinds of problems with covering spaces which have easy affirmative answers. Like, if you have a differentiable $\mathbb{C}^{\infty}$, Or analytic, etc., manifold then automatically $\tilde{X}$ will be differentiable (respectively, $\mathbb{C}^{\infty}$ or analytic). If it is Lie group the covering will be Lie group. II-countability of the covering is not all that obvious.

Likewise, paracomapctness of a manifold is an important result. That is the first thing that we want to prove is this. We will do it next time. Thank you.