Introduction to Algebraic Topology (Part - II)
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Lecture - 47
Proofs - Continued

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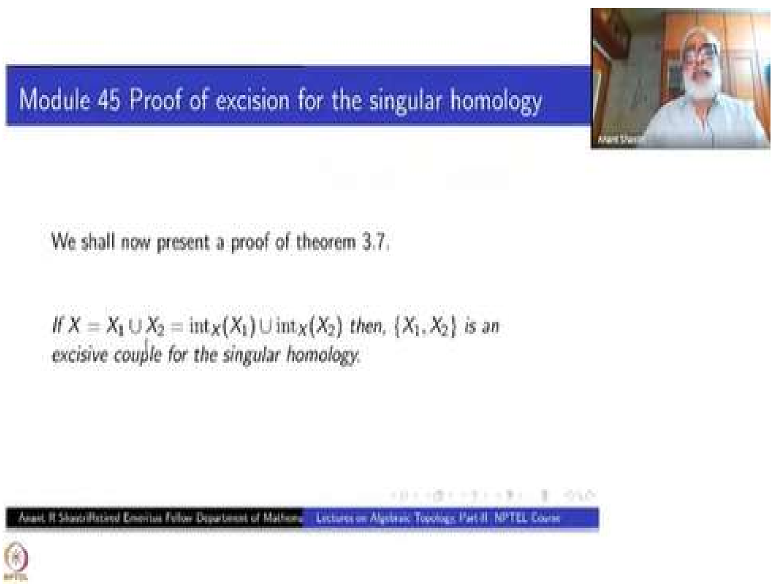

We continuing with the proofs of various statements that we have postponed. Let us take them one by one. This is one of the important results, namely, the excision theorem for singular homology. So, this is central to our theme. So, what is excision? If X is a union of two subspaces X_1 and X_2 such that interior of X_1 and interior of X_2 cover the whole space, then $\{X_1, X_2\}$ is an excisive couple for the singular homology.

In particular, if X_1 and X_2 are open, then this condition will be automatically satisfied. That is obvious. Often, what you may get is not exactly open subsets but their interiors themselves covering the space. That is good enough and is is a very, very useful statement.

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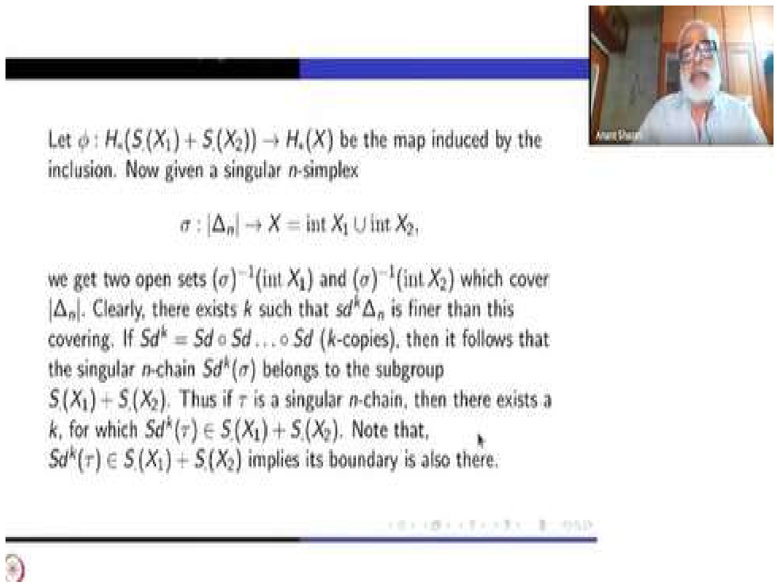

So, what is the meaning of saying that a pair is excisive? The inclusion map of the subchain complex $S.(X_1) + S.(X_2)$ into $S.(X)$ induces an isomorphism of the homology modules. That is what we want to show.

Given any singular n-simplex one of the generators, say σ from $|\Delta_n|$ to X. Remember X is written as the interior of X_1 union interior of X_2. Take the inverse image of these open subsets, under σ. They will give you two open subsets in $|\Delta_n|$ which will cover $|\Delta_n|$. $|\Delta_n|$ is a compact space. Whenever you have an open cover of a compact metric space, you have your Lebesgue number and once you have Lebesgue number you can cut down this Δ_n and iterated barycentric subdivisions as often as you want. So, that the mesh of final complex will be as small as you want, viz., smaller than any pre-chosen positive number. So, I am just recalling the proofs of various statements here in short. What you get is an iterated subdivision $sd^k(\Delta_n)$ such that the open star of each vertex will be contained in one of the two open sets. That is the meaning of saying that $sd^k(K)$ is finer than this cover. Now, remember that we have already introduced this subdivision chain map Sd composed with itself k times is also chain map Sd^k on $S.$ induces the identity isomorphism in the homology. same identity isomorphism.

It follows that that the singular n-chain $Sd^k(\sigma)$ belongs to the subgroup $S.(X_1) + S.(X_2)$, it is the sum of singular simplexes each piece is contained either in $\text{int}(X_1)$ or in $\text{int}(X_2)$. The original σ you know may be partly in X_1 and partly in X_2. Now, if τ is a singular n-chain, being a finite $\sum n_i\sigma_i$, we get integers k_i for each σ_i as above, take k equal to the maximum for all these k_i's. Then $Sd^k(\tau)$ will have the same property. The same will be true of the boundary of τ as well.

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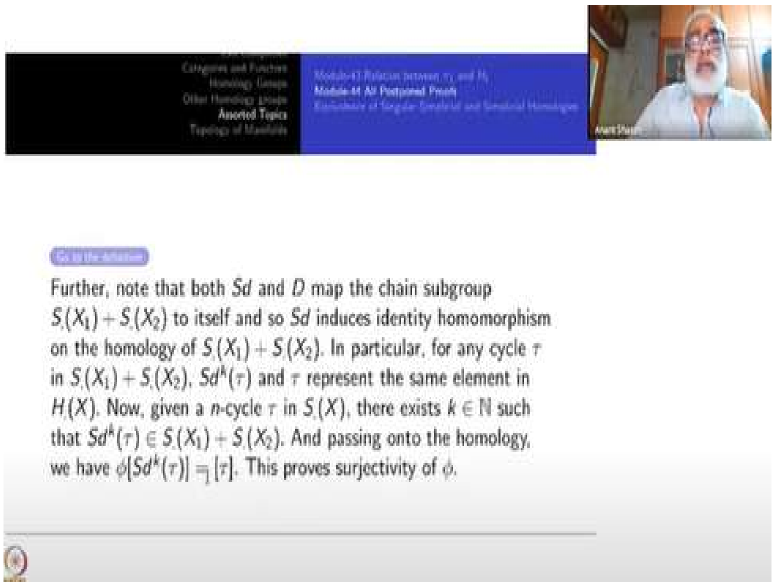

Therefore, given a n-cycle τ in $S_n(X)$, there is an integer k such that such $Sd^k(\tau)$ is in $S_n(X_1) + S_n(X_2)$. This much we have seen. Now pass on to the homology. If ϕ from $S.(X_1) + S.(X_2)$ to $S.(X)$ is the inclusion map, it follows that $\phi_*(Sd^k(\tau)) = (\tau)$. This proves the surjectivity. All this was similar to the subdivision of singular 1-simplexes which are nothing but paths.

Write a composite of two of its paths, the sum as a chain is homologous to the original 1-simplex. This is what we have seen long, long back. So, that theme is used here in a much stronger sense. So, these are all technical details here, that is all.

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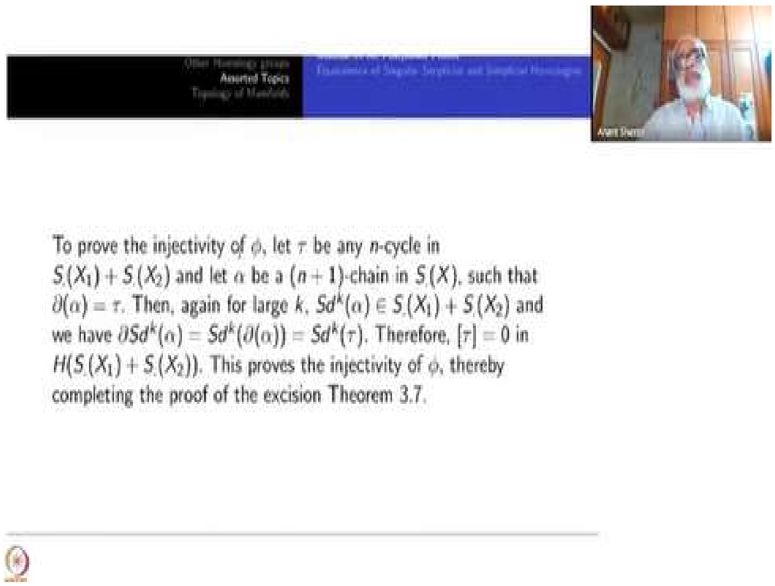

Now to prove the injectivity of ϕ_*, which is similar: Suppose τ is any n-cycle in the $S_n(X_1) + S_n(X_2)$ itself and α is a $(n+1)$-chain in $S_n(X)$ such that the boundary of α is τ. What do I mean by this one? This just means $\phi_*(\tau) = 0$ in $H_n(X)$.

Apply the above argument to α to get an integer k such that $Sd^k(\alpha)$ is in $S.(X_1) + S.(X_2)$. But then $\partial(Sd^k(\alpha)) = Sd^k(\partial(\alpha)) = Sd^k(\tau)$. This just means that $(\tau) = 0$ in $S.(X_1) + S.(X_2)$. This proves the injectivity.

Thus, we have to developed these terminologies and technicalities centred around concepts which are is very simple. So, this completes the proofs of some of the big theorems. Generally, people are either afraid of going into the technicalities or they say it is very simple and brush it of. So, the next thing that we consider is not all that simple, little more technical details are there.

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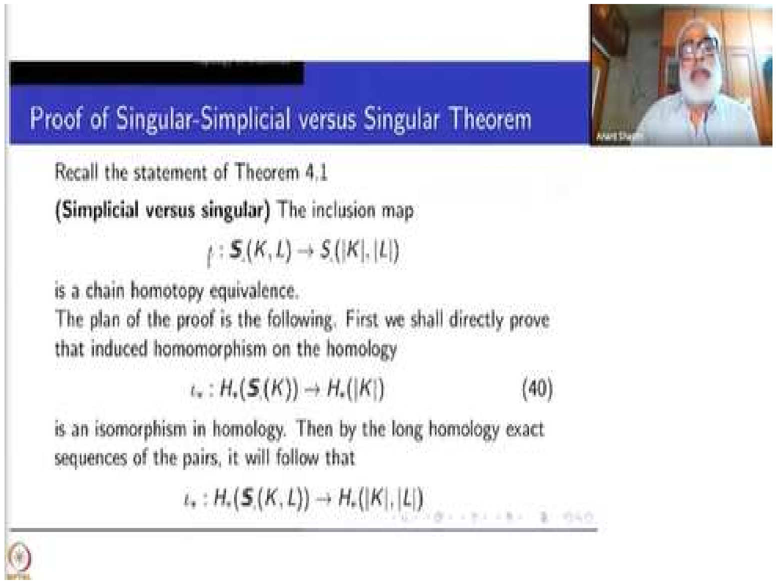

The next thing is: take a simplicial complex K and sub complex L. Take the relative singular chain complex $S.(|K|, |L|)$. Take the relative singular simplicial chain complex, that is a sub complex, inclusion map double $S.(K, L)$ into $S.(|K|, |L|)$ is a chain homotopy equivalence. That is the statement of theorem 4.1.

We are supposed to prove this. We shall not prove this statement directly, but slightly weaker looking statement which will be the more useful for us and the more central to us. Namely that induced the map on the homology is an isomorphism. We shall be satisfied with the proof of this statement for the time being.

The proof will come in two steps, namely, first, instead of pairs (K, L), we will prove the statement for the absolute case. Once you have done that, you can use the long homology exact sequences associated to the chain complex double of $S.(K, L)$ and $S.(|K|, |L|)$ to obtain a ladder of these long exact homology sequences with two consecutive arrows being isomorphisms missing the third one and so. From the five lemma the all the third arrow will be isomorphisms. These middle ones are precisely the inclusion induced morphisms of the theorem.

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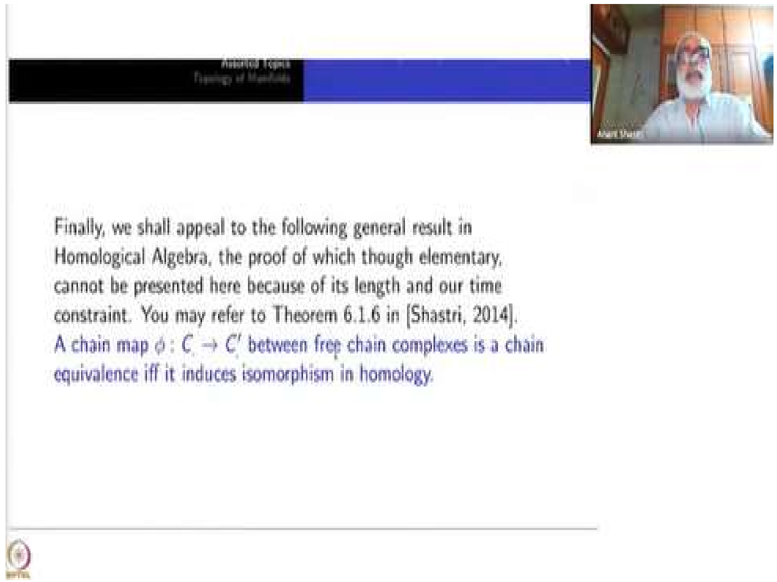

After that, the original statement 4.1 about homology equivalence comes by another general statement in homological algebra that on arbitrary chain map of free chain complexes induces isomorphism in homology iff it is a chain equivalence. That statement is purely homological algebra; it is not at all difficult but time consuming. Therefore, I am skipping it here. So, in principle, I am only going to prove 4.1 partially.

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So, another simplification: you want to prove homology isomorphism. You can do that by first proving it only for the finite K. Then usually you can take a direct limit argument and arrive at the proof for the general case. Indeed, you do not need a direct limit as strong as you want if you do not understand it. It is very simple, just like what we did for barycentre and so on.

Suppose you want to prove the surjectivity. Take a cycle here representing an element in the homology. Every chain in $|K|$ is supported on a finite simplicial subcomplex K'. So, instead of K you use that subcomplex K', then from $S.(K')$. You have already proved surjective there.

Because everything is induced by inclusion maps here, surjectiveity of the original map follows. Similarly, you can prove the injectivity also in the general case. This is essentially what happens in the direct limit argument, which is much more general. Homology commutes with direct limits. That gives a proof directly.

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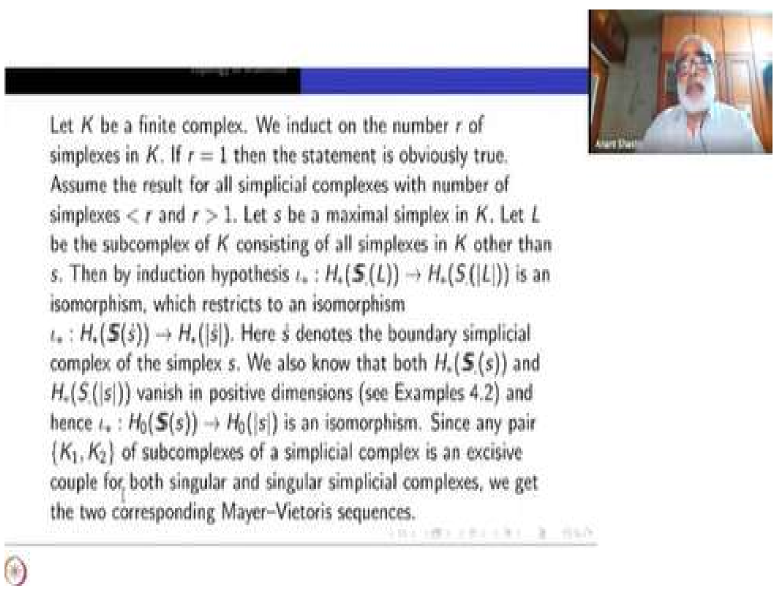

Now, so how do we prove it for a finite simplicial complexes. Here we use induction, on the total number of simplexes inside K. If the number is one, what is K? K has to be a singleton vertex. That is all. For a singleton vertex, both double $S.$ and $S.$ are the same. Every map from Δ_n to a single vertex is simplicial being a constant.

Now you assume that it is true for all simplicial complexes with total number of simplexes in K less than r, where r is some positive number. Let K be a finite simplicial complex with the number of simplexes equal to r. In any finite compelx, there is always a maximal simplex, maximal means what? It is not contained in another larger simplex. So, let s be a maximum simplex in K. Now delete that maximal simplex only, suppose $s = \{v_0, v_1, \ldots, v_n\}$.

Deleting s means that the rest of them you have to keep, namely, all the smaller faces of s as well as those which are disjoint from s. So, only one simplex you will be deleting to obtain a sub complex L of K consisting of all simplexes in K other than s itself.

By induction hypothesis what happens now? The inclusion map i_* from $H_*(\text{Double } S.(L))$ to $H_*(|L|)$ is an isomorphism, because L has a smaller number of simplices. Not only that, the same inclusion map if you restrict it to the $S.(\dot{s})$ of the boundary of s which is a subcomplex of L, that restriction also induces isomorphism in homology. Here $\dot{s}$ denotes the boundary subcomplex of the simplex s.

Since s is a simplex, both homologies, $H_*(\text{Double } S.(L))$ and the singular homology vanishes in positive dimensions, we have directly proved it in example 4.1 and this inclusion induced morphism in H_0 is the isomorphism of infinite cyclic groups. Since any pair $\{K_1, K_2\}$ of subcomplexes of a simplicial complex $K = K_1 \cup K_2$ is excessive, both for singular simplicial chain complex and singular chain complex of the underlying topological spaces, we can apply Mayer Vietoris sequences for both of them.

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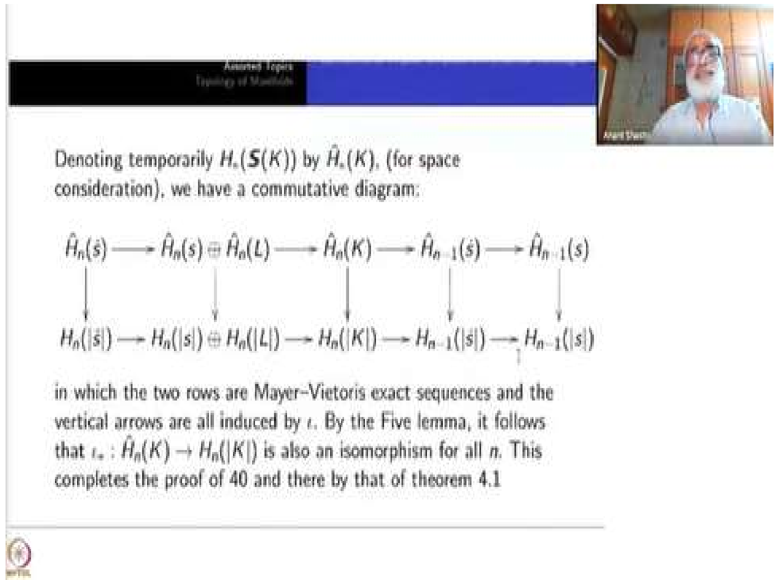

So, take $K_1 = L$ and $K_2 = s$ and use a temporary short notation $\hat{H}$ for the homology of of double S., (this is not a twiddle, this hat is just a notation, temporary notation because it is too large and will go out of the slide).

So, this is the first row here for a simplicial singular chain complex. The second row is for singular chain complex. The first term is for the intersection, this one is the direct sum and the third term is for the union K, followed by the connecting homomorphism back to the intersection. The vertical arrows are all inclusion induced. The entire ladder is commutative. By induction hypothesis, the first, second, fourth and the fifth vertical arrows are isomorphisms. By the five lemma, the third arrow is also an isomorphism. With that we can say that the proof of theorem 4.1 is complete.

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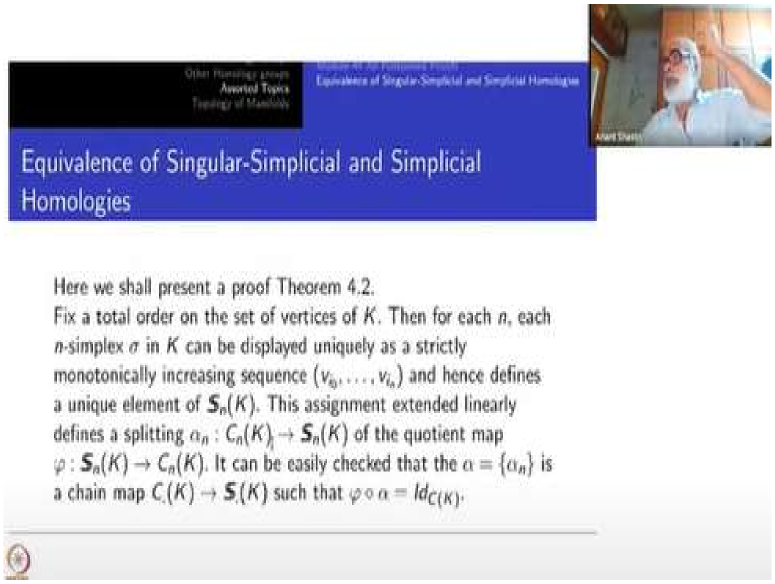

So, we have one more thing to do here: Equivalence of singular simplicial and simplicial homology, double $S.$ and $C.$ This time we do not have an inclusion map but we have a quotient map, φ from double $S.$ to $C.$ We shall present a proof of 4.2. Just for the proof of that we fix a total order on the vertices of K.

Then for each n, each n-simplex σ, we can display it by writing $\sigma = [v_{i_0}, \dots, v_{i_n}]$ where $(i_0, i_1, \dots)$ is an increasing sequence, with respect to the total order that we have fixed on the vertices of K. So, that would define a unique element of a double $S_n(K)$. This assignment extends linearly and defines a splitting α_n from $C_n(K)$ to double $S_n(K)$, of the quotient morphism φ from double $S_n(K)$ to $C_n(K)$. Splitting means what? You come back by quotient map, its identity. If you put all these αn's together, that is the totality of these, that is a chain map α from $C.(K)$ to double $S.(K)$, follows because the face operators preserve the order. So, we have a splitting of φ, which is a chain map, so that is the first thing to observe.

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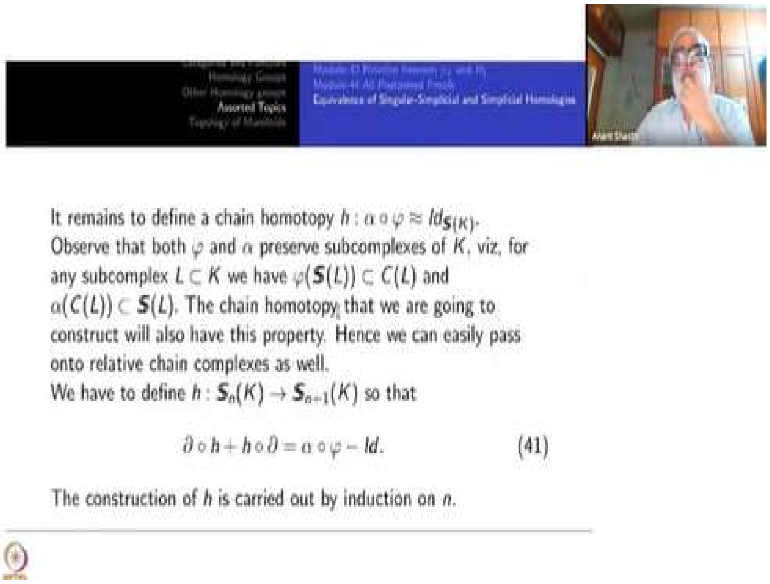

Now, it remains to define a chain homotopy on the opposite side. We already have $\varphi \circ \alpha$ is identity of C. However, $\alpha \circ \varphi$ is not identity in general. But I will now show that it is chain homotopic to identity of double S. It will then follow that φ_* is an isomorphism in the homology, so, the proof will be over. So, how to define a chain homotopy from $\alpha \circ \varphi$ to identity of double of S.

We observe that both φ and α preserve sub complexes of K. (Note that φ is functorial and hence in particular it will preserve subcomplexes. However, α is far from being a functor. It depends on the total order we have chosen.) In other words, if we have a subcomplex L of K, then $\varphi(\text{double } S.(L))$ will go inside $C.(L)$. Similarly, $\alpha(C.(L))$ will go inside double $S.(L)$. To that extent, α is canonical and this property is easy to verify.

The chain homotopy that we are going to construct will also have this property, that it will preserve the sub complexes. Hence, we can easily pass onto relative chain complexes. So, a relative version will also come automatically. That is what I wanted to emphasize here.

Now what is a chain homotopy h? It is a collection $\{h_n\}$ of morphisms h_n from double S_n to double S_{n+1} such that boundary composite h + h composite boundary is the difference

$\alpha \circ \phi - identity$. This is the way h has to be defined. As usual in many other cases we have seen, we will do this by induction on n.

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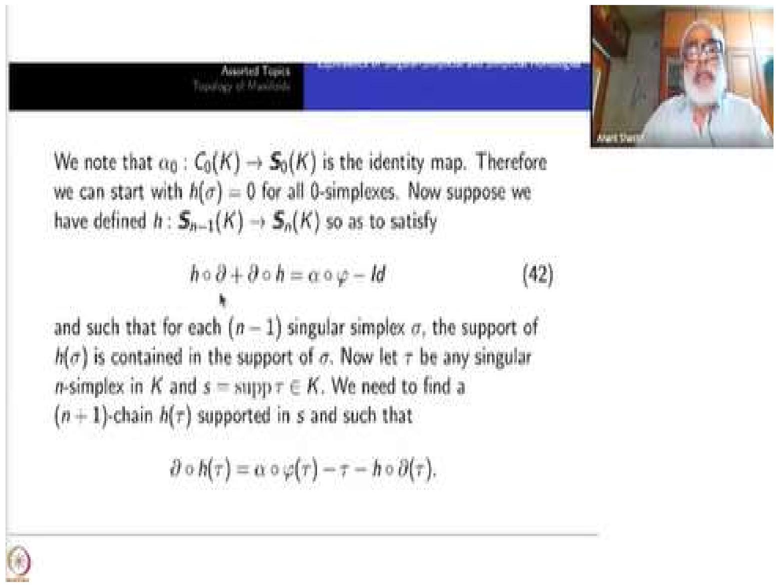

The first thing is to note that α_0 from $C_0(K)$ to double $S_0(K)$ is the identity map. What is double $S_0(K)$, free group generated on simplicial maps from Δ_0 to K. For each vertex, there is only a unique map taking Δ_0 to the vertex. That is all. Therefore the quotient map φ itself is the identity map. Therefore, there is no problem at this level. We can start with $h_0(\sigma) = 0$ for all 0-simplexes.

Now suppose we have defined h_i for $i < n$ so as to satisfy (42) and in addition such that support of $h(\sigma)$ is contained in the support of σ for each singular simplicial simplex σ. Then we would like to define h_n also to have the same property in the next stage. Why I am saying that? I want to construct it this way, that is a part of the induction hypothesis, which is automatically satisfied at the 0-level. This additional condition will help me to prove the next step. So, I will assume that I have constructed it in this way, namely, for all the singular $(n-1)$-simplexes σ, we have support of $h(\sigma)$ is contained in the support of σ.

Remember we need to define h only the generators here, and we take extend it linearly over the whole group. Both sides here are linear, therefore, if you verify this identity over generators then it will be verified for all of them. So, we are talking only about what is happening on the generators.

So, let τ be any singular, simplicial simplex in K and s equal to support of τ. We need to find an $(n+1)$-chain $h(\tau)$ supported on the same s and so that $\partial(h(\sigma))$ is given by (42), viz., $\alpha \circ \varphi(\tau) - \tau - h(\partial(\tau))$. Note that $\partial(\tau)$ is a $(n-1)$-chain and hence h of that is already defined. And we want to define $h(\tau)$ so that its boundary is this one. So, this is possible only if this entire thing is first of all is a cycle, i.e., its own image under ∂ must be 0.

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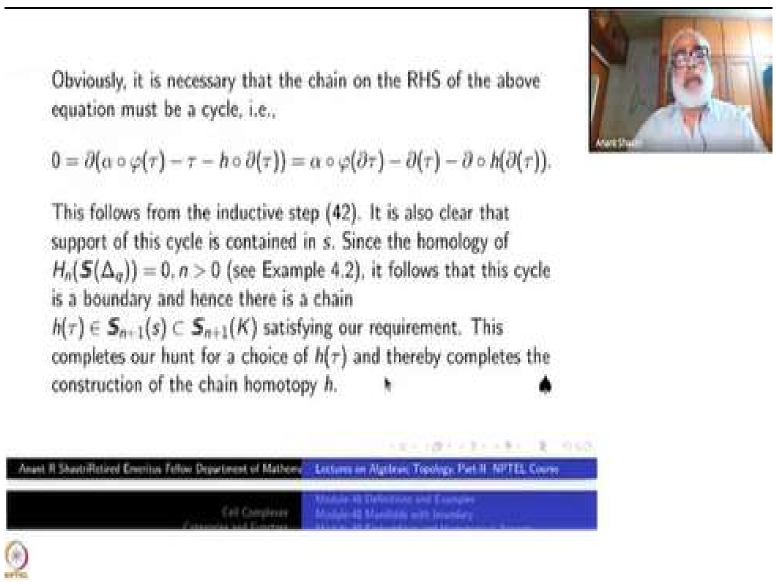

Look at the boundary of $(\alpha \circ \varphi(\tau) - \tau - h(\partial(\tau))$. It is equal to $\alpha \circ \varphi(\partial(\tau)) - \partial(\tau) - \partial(h(\partial(\tau))$. Now, why is this 0? This is by induction hypothesis, equation number (42) (subtract one more term $h(\partial^2(\tau))$ which is anyway zero.) It is also clear that the support of this cycle is contained inside s. We are not going out of the image of the simplex τ which is s.

Since the reduced homology of double $S.$ of any single simplex vanishes in all dimensions, every cycle is a boundary. So, finally this is where we are using some topology. Till then all the time we are using just some algebra in some sense. So, here we are using a result done in an example, viz., the homology of double of S of a simplex is 0 in positive dimensions. The argument is applicable in dimension one as well, by taking reduced homology. Hence there is a chain $h(\tau)$ belonging to double of $S_{n+1}(X)$, with its boundary equal to the RHS.

So, that completes the construction of h and completes the proof that the singular simplicial chain complex is equivalent to the simplicial chain complex itself. There is a pattern in the methods employed in these proofs. That pattern is generalized to what is called as `method of acyclic models', in abstract homological algbra. Something is free, something is acyclic, and these two things will be put together. My idea is to explain the rudiment of this method at the very simplest cases, how these things work, so that when you are trying to learn more algebraic topology of this sort, you will be better off even without the books or the teachers help. Often it is the case that they will have not time or patience to explain. That is why I have explained all these things separately and instead of using the high missionary right in the beginning. So, this completes one aspect of whatever you wanted to study namely, the homology.

Whatever we have aimed up till here that we have completed. We have completed all the results long back. Now whatever proofs that we had postponed, most of them we have completed, except a few things such as some homological algebra I told you just now. So, next time I will take a different topic, namely, topology of manifolds. Thank you.