Introduction to Algebraic Topology (Part - II)
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Lecture - 46
All Postponed Proofs

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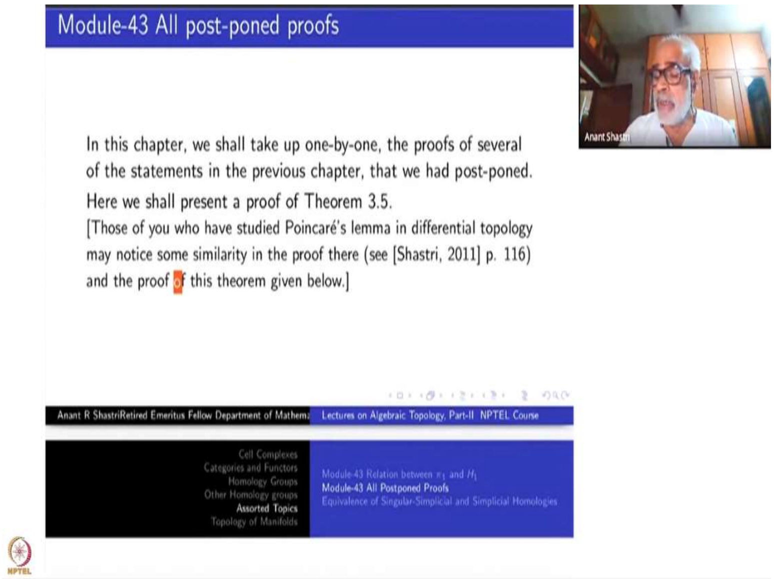

Finally, we come to the section in which we would like to complete the proofs of various statements that we have made in this chapter and have postponed their proofs. So, let us take them one by one.

The very first thing is about homotopy invariance. It would remind you of Poincare' lemma in differential topology if you studied it. Indeed, people got the idea from there and now we have completely generalization.

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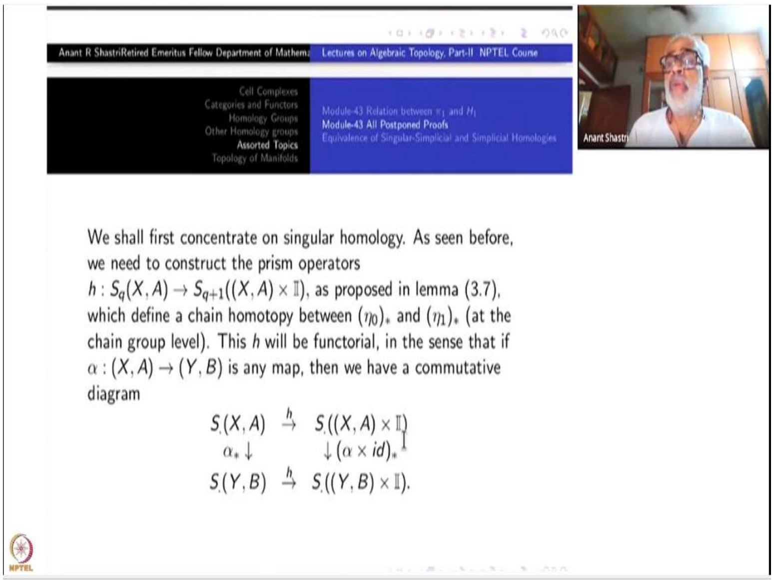

So, let us first concentrate on singular homology. The homotopy axiom will be proved by proving the lemma 3.7, in which we promised that we will construct the prism operator. This prism operator is going to be the homotopy between $(\eta_0)_*$ and $(\eta_1)_*$. h is going to be a map from $S_n(X)$ to $S_{n+1}(X \times \mathbb{I})$ and η_i are various coordinate inclusion maps from X into $X \times [0, 1]$, and $(\eta_i)_*$ are the morphisms induced at the chain level.

And the homotpy h we promised, that will be functorial. In a way, by demanding that it should be functorial actually helps us in construction of H itself. So, what is the functoriality? Suppose you have a continuous function α of the pairs from (X, A) to (Y, B), any continuous function. Then we have the homotopies h from $S.(X, A)$ to $S.((X, A) \times \mathbb{I})$ as well from $S.(Y, B)$ to $S.((Y, B) \times \mathbb{I})$).

They should be compatible under is α, viz, $h \circ \alpha_* = (\alpha \times Id)_* \circ h$, where Id denotes the identity map on the factor I. This diagram must be commutative. So, this is what we demand, even though it would not have been necessary for us just to prove the homotopy invariance theorem.

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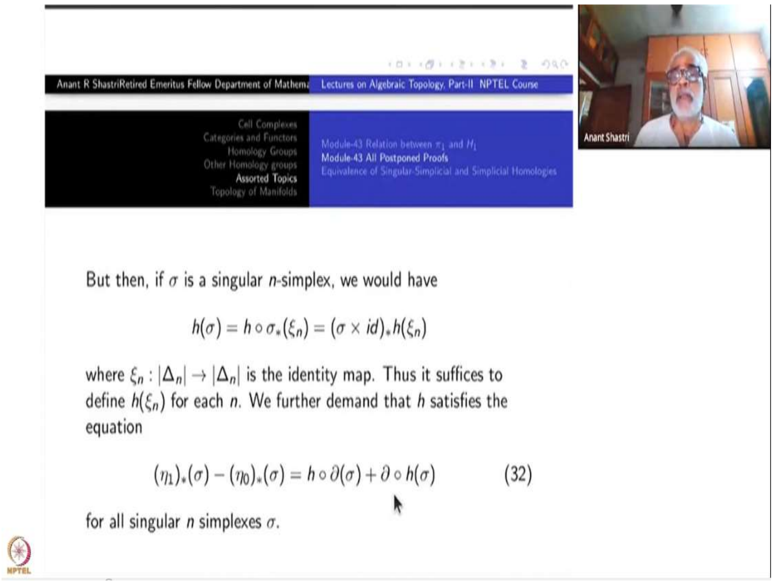

But then for any singular n-simplex can be thought of as a map from Δ_n to X. So, putting sigma in place of α here, and then what we get is that $h(\sigma) = h \circ \sigma_*(\xi_n) = (\sigma \times Id)_* h(\xi_n)$, where ξ_n denotes the identity n-simplex in $|\Delta_n|$.

Therefore, this gives us the idea that it is enough to define this h not for arbitrary σ but for just ξ_n. Once you have defined it for ξ_n, then for an arbitrary σ, $h(\sigma)$ must be given by this formula, by functoriality.

Therefore, our task of defining h on $S_n(X)$, first of all is reduced to defining it on all singular n-simplexes, because then by linearity, it is enough to define it on the generators. What are the generator? Generators are arbitrary continuous maps from Δ_n to X. Even that is not necessary. Just define it on ξ_n for each n.

Then we are done. Then I take this as a formula for $h(\sigma)$'s and extended linearly over all of $S_n(X)$. Of course, we have to verify that it is a chain homotopy. So, what is the property that will make it a chain homotopy. So, that also I have to decide for $h(x_n)$ only. Because, first of all we have to have $((\eta_1)_* - (\eta_0)_*)(\sigma) = (\partial \circ h + h \circ \partial)(\sigma)$.

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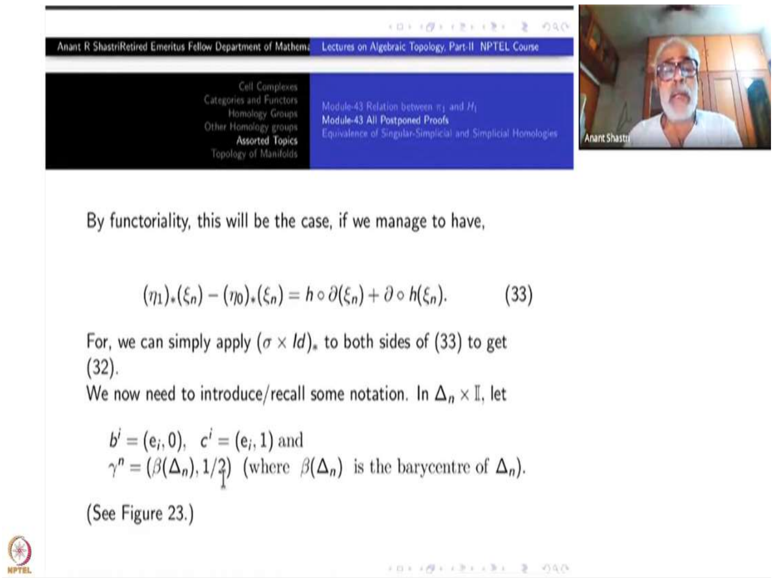

So, this is the meaning of saying that h is a chain homotopy between $(\eta_1)_*$ and $(\eta_0)_*$. If this is true for all singular simplexes then it will true for all chain as well since both sides are linear. Once again by functoriality, this will follow if we have the same condition for all ξ_i in place of σ.

Because we can apply simply $(\sigma \times Id)_*$ to both sides of (38) to get (37). So, our task is to define $h(\xi_n)$ so that it satisfies (38). Now, I have to recall the prism construction, which is a subdivision of $|\Delta_n \times \mathbb{I}|$, in order to define the prism operator. This was elaborately introduced in part one, but I should to do it here the way I want it, right now here. So, in $\Delta_n \times \mathbb{I}$, let b^i denote $(e_i, 0)$. Remember that $e_0, e_1, e_2, \ldots, e_n$ are the vertices of Δ_n. In $\Delta_n \times \mathbb{I}$, $(e_i, 0)$ are at the at 0-th level.

And let $c^i = (e_i, 1)$, the same thing at 1-level. So, similarly, you can take the barycentre of Δ_n and place it at the level $1/2$-level, a notation for that being $\gamma^n = (\beta(\Delta_n), 1/2)$.

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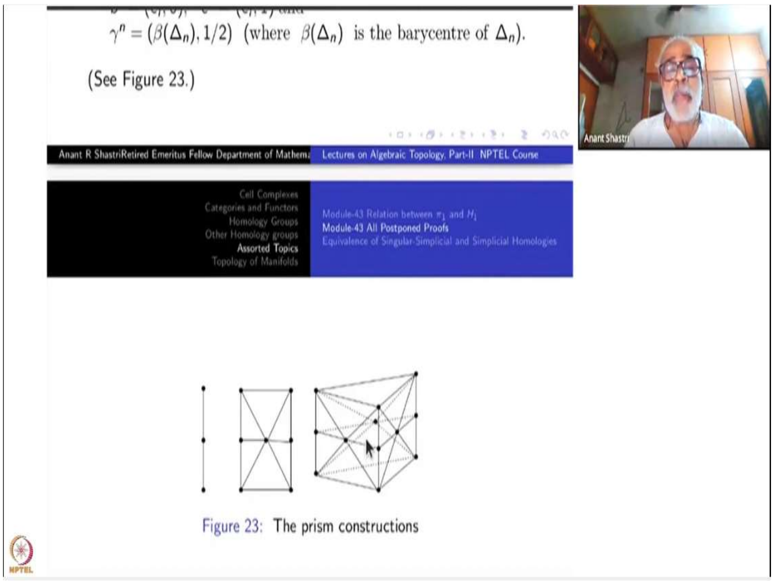

So, this figure is just for $n = 0$, $\Delta_0 \times \mathbb{I}$ is this one. This will be your b^0 and this will be c^0 and this will be γ^0. Here $n = 1$, this is $\Delta_1 \times \mathbb{I}$, these are is b^0 and b^1, and these are c^0 and c^1. These can be labelled by γ_0^1, γ_1^1 and γ^1, which are respectively, $(e_0, 1/2), (e_1, 1/2)$ and $(\beta(\Delta_1), 1/2)$. Similarly, for Δ_3 and Δ_4 and so on. You have to do this inductively. First define in each vertex $v \in \Delta_n$, introduce the extra vertex $(v, 1/2)$ in $v \times \mathbb{I}$.

The construction for the 0-skeleton of Δ_n is over. Now for each edge $\sigma \in \Delta_n$ introduce the additional vertex $(\beta(\sigma), 1/2)$, Look at the boundary of $\sigma \times \mathbb{I}$, which already has been given a simplicial structure. Use the come construction to extend it to a simplicial structure on $\sigma \times \mathbb{I}$, with $(\beta(\sigma), 1/2)$ as the apex of the cone. That complete the construction for 1-skeleton of $\Delta_n \times \mathbb{I}$. Proceed exactly, this way to complete the prism construction on the entire of $|\Delta_n| \times \mathbb{I}$. So, this simplicial structure is somewhat different than the simpler one viz, the barycentric subdivision. Note that the top and the bottom faces do not get subdivided here.

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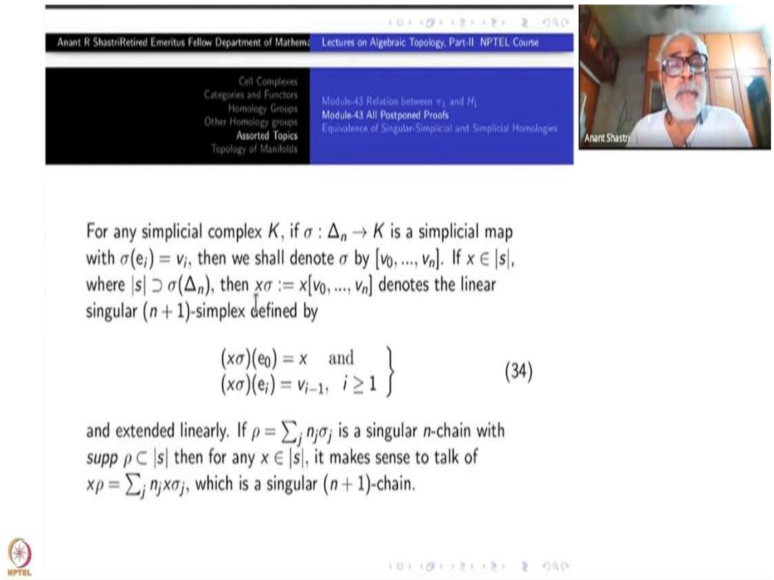

Next, recall, for any simplicial complex K, if σ from Δ_n to K is a simplicial map with $\sigma(e_i) = v_i$, we had introduced the notation $\sigma = [v_0, v_1, \ldots, v_n]$. If x belongs to $|s|$, where s is a simplex in K and such that $|s|$ contains $\sigma(\Delta_n)$, then $x\sigma = x[v_0, \ldots, v_n]$ denotes the linear singular $(n+1)$-simplex given by $x\sigma(e_0) = x, x\sigma(v_i) = v_{i-1}$, the first vertex is x and the remaining vertices are got by shifting σ. The affine linear extension makes sense within the convex set $|s|$ subset of $|K|$.

Clearly, support of $x\sigma$ is contained inside $|s|$. Similarly, if $\rho = \sum \sigma_i$ is a singular n-chain such that support of rho is contained in $|s|$ for some s, then for any $x \in |s|$, it makes sense to talk about $x\rho = \sum n_i x\sigma_i$ as an $(n+1)$-chain.

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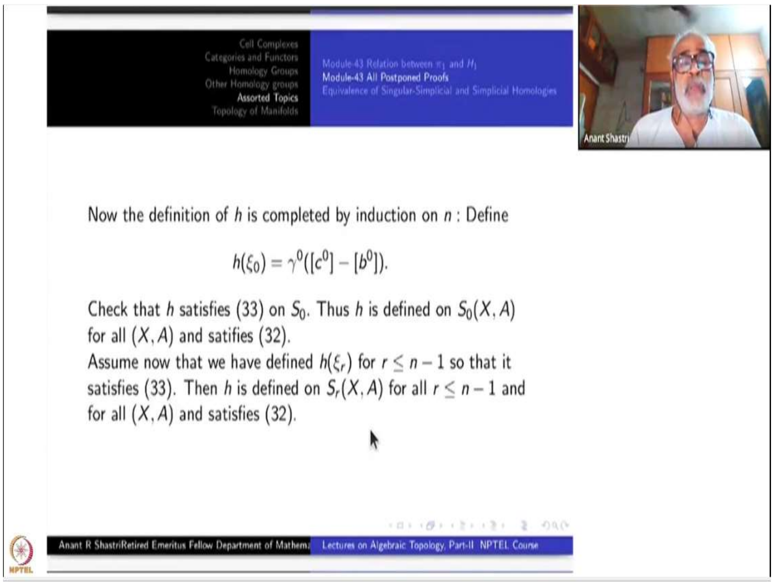

So, now the definition of h is completed by induction again. The first thing is $h(\xi_0)$ is taken to be $\gamma^0([c^0]-[b^0])$, the extension of $[c^0]-[b^0]$ by γ^0. Thus, $h(\xi^0)$ is 1-chain in $\Delta_0 \times \mathbb{I}$, i.e., $[\gamma^0, c^0]-[\gamma^0, b^0]$. So the two 1-simplexes of the prism are directed toward the barycenter. Check that it satisfies (38) on ξ_0. That completes the definition of h on $S_0(X)$ for all spaces X. $\partial(\xi_0)=0, \partial h(\xi_0)=[c^0]-[b^0]$, since the term γ^0 cancels out. And c^0-b^0 is precisely equal to $((\eta_1)_*-(\eta_0)_*)(\xi_0)$. Now inductively assume that you have defined h on $S_r(X)$ for $r<n$ so as to satisfy (37).

Then h is defined on S_n by only defining $h(\xi_n)$, which we know is equivalent to defining h for (ξ_r) only so as to satisfy (38). What is then $h(\xi_n)$? So, take $h(\xi_n)$ to be $\gamma^n([c^0,\ldots,c^n]-[b^0,\ldots,b^n]-h(\partial(\xi_n)))$, the extension by γ^n of the n-chain ξ_n shifted to the top $-\xi_n$ shifted at the bottom minus an error term, so that it is equal its boundary is equal to $((\eta_1)_*-(\eta_0)_*)$ operating on ξ_n. That is as dictated by the condition (38) as you will notice soon. That is why you are taking the error term $-h$ of boundary of ξ_n. Boundary of ξ_n is an $(n-1)$-chain, h of that is an n-chain, so, I can subtract that one. That is a troublemaking term, you delete that one.

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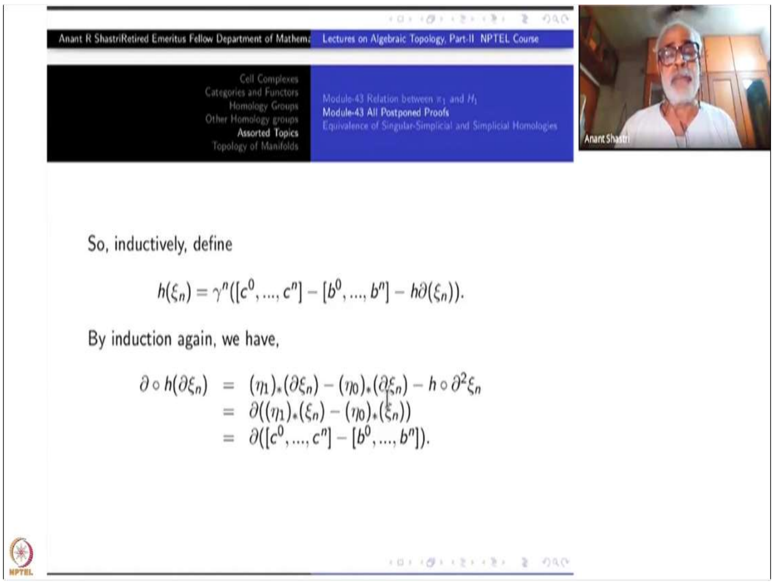

Any way, let us verify (38) now. By induction again, (38) holds for $\partial(\xi_n)$ and hence first of all, $h((\xi_n) = (\eta_1)_*(\partial(\xi_n))-(\eta_0)_*(\partial(\xi_n))-h(\partial^2(\xi_n))$, last term drops out and hence, this is equal to boundary of $(\eta_1)_*-(\eta_0)_*(\xi_n)$ which is equal to boundary of $([c^0,c^1,\ldots c^n]-[v^0,v^1,\ldots,v^n])$.

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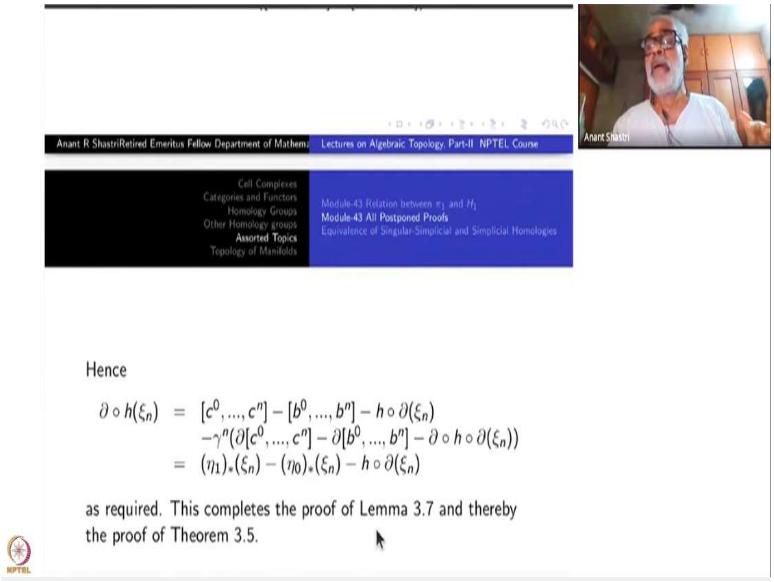

Therefore, boundary of $h(\xi_n)$ is (the boundary of γ^0 of some chain in the bracket, so, first you delete γ^n and write the chain bracket term and then $-\gamma^n$ of the boundary of the chain in the bracket) equal to $[c^0, \ldots, c^n] - [b^0, \ldots, b^n] - h(\partial(xi_n)) - \gamma^n(\partial([c^0, ..., c^n] - [b^0, ..., b^n] - h(\partial(x_{in})))$.

From the previous paragraph, whatever is in the bracket is just 0. So, you get only the first line. The first term is $(\eta_1)_*(\xi_n)$ and the second one is $-(\eta_0)_*(\xi_n)$ and the third one is $-h$ of boundary of ξ_n. You take this last term on the left-hand side then what you get is this formula (38), that we wanted. So, the construction of this prism operator is over. Remember once this is done the homotopy axiom is completely proved.

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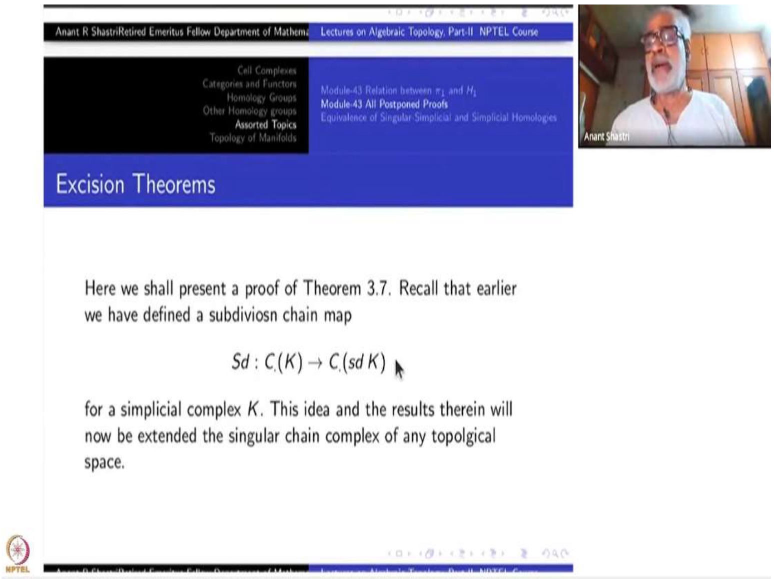

So, let us go to the excision now. Here we shall present the proof of the excision Theorem. Recall that we have defined a subdivision chain map Sd from $C.(K)$ to $C.(sdK)$, sdK denoting the barycentric subdivision of K, where K is a simplicial complex. This idea will be now completely generalized here. This was used to prove the simiplicial approximation theorem. But the idea is very good so, we are going to extend it to the entire singular chain complex now.

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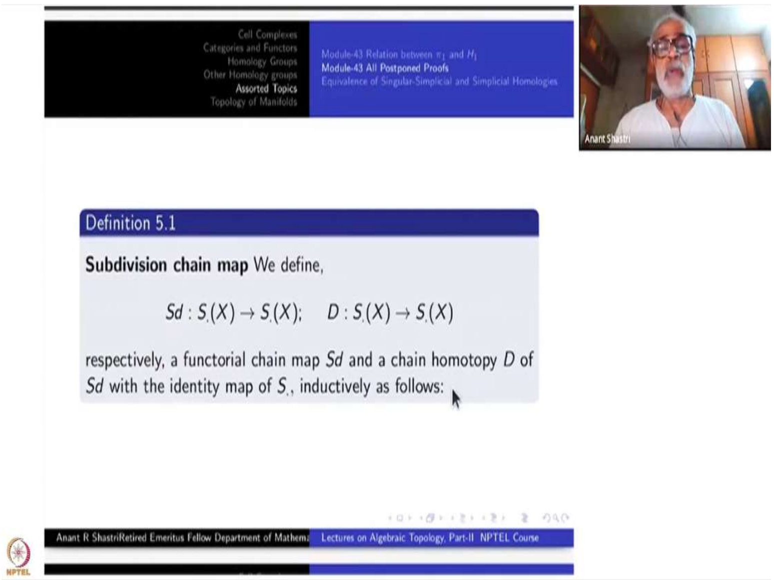

So, the subdivision chain map of the singular chain complex itself and a chain homotopy D from $S.(X)$ to $S.(X)$, of the chain map Sd with the identity map, simultaneously, these two things will be defined.

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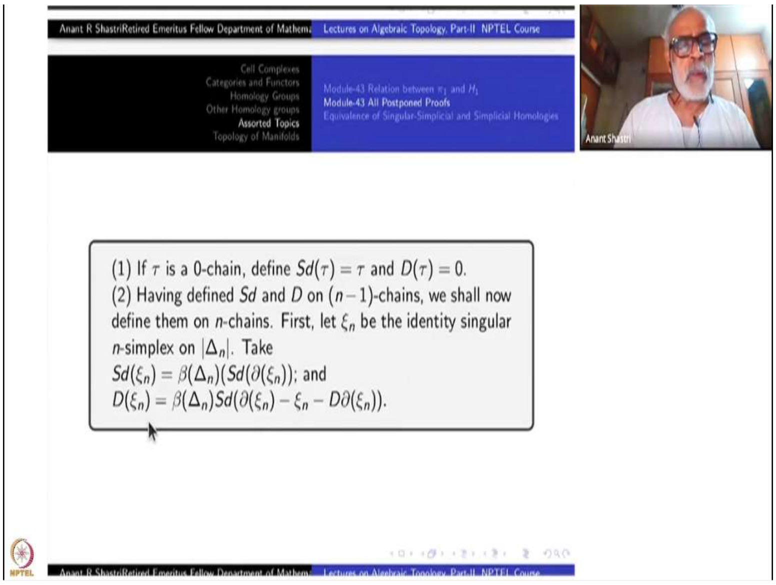

So, again this will be done inductively. First of all, if τ is a 0-chain, then $Sd(\tau)$ is taken as τ itself. That means each point is taken to itself; this was the case with Sd on simplicial complex also. The Sd is identity on the original vertices and the extra vertices were some other vertices, but now we do not have to bother about them. So, it is in some way simpler here. Take $D(\tau) = 0$.

So, inductive construction starts at 0-level. Having defined Sd from S_{n-1} to S_{n-1} and D from S_{n-1} to S_n, we shall now define them on n-chains. First let ξ_n be the identity singular n-simplex in Δ_n, just like we did for the prism operator. On Δ_n, let us first define $Sd(\xi_n)$ (now you see, this is a game that we repeat again, the kind of thing that we did in prism operator, though here we are doing similar thing with in Δ_n) to be the cone construction of $Sd(\partial\xi_i)$ with apex $\beta(\Delta_n)$, i.e., the n-chain $\beta(\Delta_n)(Sd(\partial(\xi_n))$. And take $D(\xi_n)$ to be $\beta(\Delta_n)(Sd(\xi_n)$–$\xi_n - D(\partial(\xi_n)))$.

See we are first taking $Sd(\xi_n) - \xi_n - D(\partial(\xi_n))$ which defined by induction as a n-chain and then taking the cone over the entire thing.

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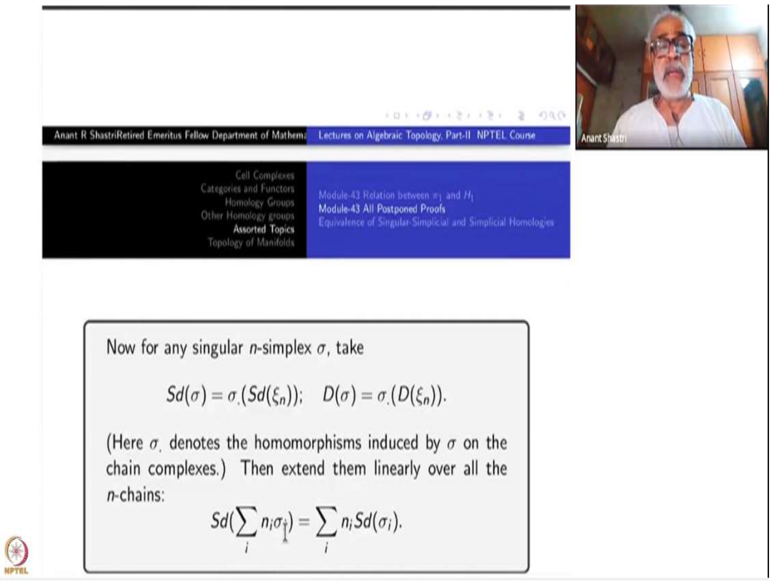

So, let us let us go ahead now. For any n-simplex σ in X, $Sd(\sigma)$ will be $\sigma.$ of $Sd(\xi_n)$. $\sigma.$ is the chain map from $S.(\Delta_n)$ to $S.(X)$, induced at the chain level. So, $\sigma.(Sd(\xi_n))$ makes sense in $S_n(X)$. So, this idea is similar to what we did in homotopy theory, using functoriality. After defining the value of the function on ξ_n, for arbitrary σ, $\sigma.$ out that is taken as the definition of the function on σ. Similarly, take $D(\sigma) = \sigma.(D(\xi_n))$.

So, then extend both of them linearly over all the chains. $Sd(\sum n_i\sigma_i)$ is the $\sum n_i Sd(\sigma_i)$; D (summation) is the summation of D's.

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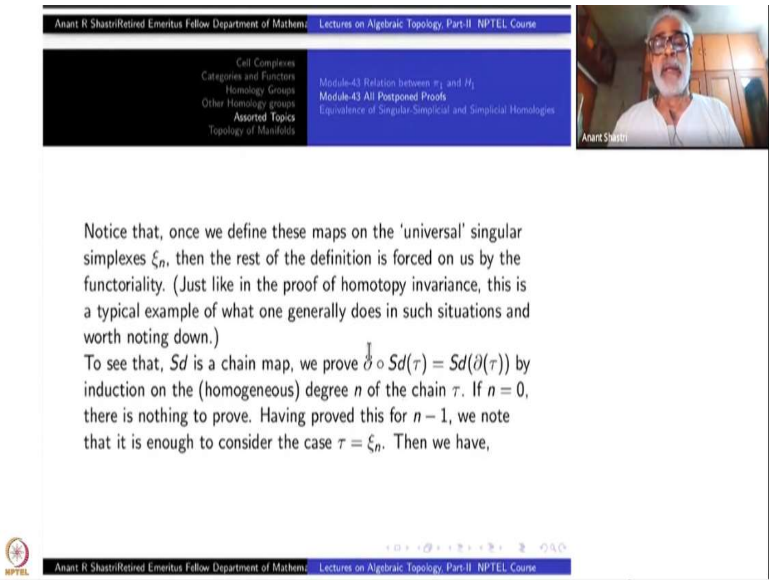

Notice that once we define these maps on the universal singular simplexes namely ξ_n, then the rest of the definition is forced on us by the functoriality and linearity. So, that helps us to define the whole thing, just like in the proof of homotopy invariance. This is a typical example of what one generally does in such situations and worth noting now.

So, once again I keep telling you that if you have to do some existence theorems, construction and so on you better understand how far it is free, and how much is forced on us. For example, in the existence of solutions of differential equation, there is a uniqueness part. The uniqueness part actually tells you what could be the map what could be the solution. Here is a case wherein because it has to be like this, because functoriality, that restriction helps us to cut down our work, and define it for only for ξ_n.

Now to see that Sd is a chain map. Chain map means what? Sd should commute with ∂. $\partial \circ Sd(\tau)$ must be $Sd(\partial(\tau))$. This is what we have to prove. Again, we can do it by induction. By linearity, we need to verify it for singular simplexes τ only instead of chains.

If $n = 0$, there is nothing to prove. Assume that it is valid for $n-1, n > 0$, we should prove it for a n-simplex τ. We note that instead of arbitrary τ here, it is enough to prove it for $\tau = \xi_n$, again by functoriality.

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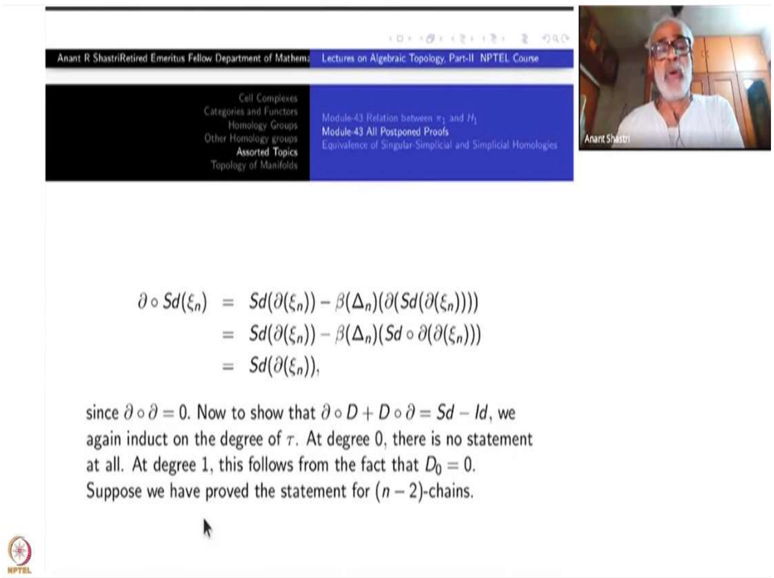

But then we have $\partial(Sd(\xi_n)) = Sd(\partial(\xi_n)) - \beta(\Delta_n)\partial(Sd(\partial(\xi_n)))$. By induction hypothesis, the second term is equal to $\beta(\Delta_n)Sd(\partial^2(\xi_n))$ and hence is zero. Similarly, now I have to show finally that D is a chain homotopy, $D \circ \partial + \partial \circ D = Sd - identity$. So, again you do this by induction. The induction start at level 1. Here it follows because D_0 has been chosen to be 0. So, suppose we have proved the statement for n-chains, $n > 1$.

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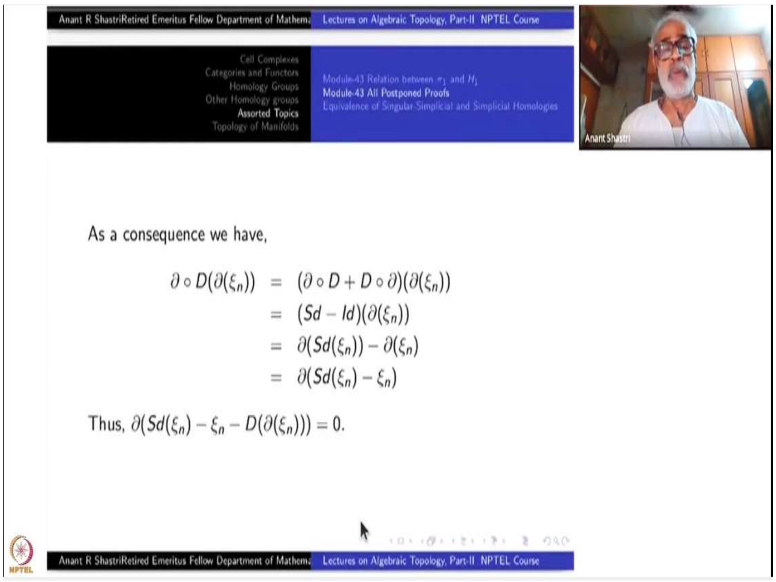

As a consequence, we have boundary of $D(\partial \xi_n)$ is equal to $(\partial \circ D + D \circ \partial)(\partial(\xi_n))$ (you ca just add the zero term ∂^2), and this is equal to $(Sd - Id)(\partial(x_i))$, by induction. So, it follows that $\partial(Sd(\xi_n) - \xi_n - D \circ \partial(\xi_n)) = 0$, by taking all the terms to one side. This was an elementary computation.

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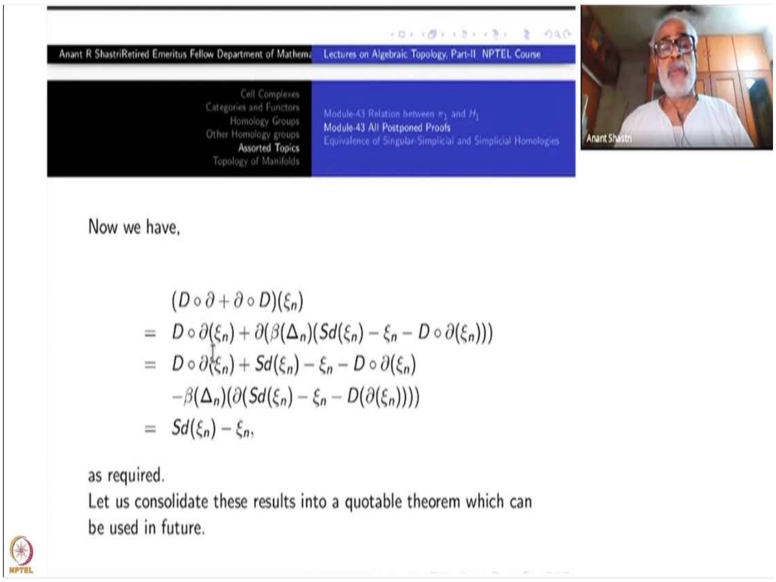

Therefore, now, $(D \circ \partial + \partial \circ D)(\xi_n) = D \circ \partial(\xi_n)$, keep the term as it is) plus second term is the boundary $D(\xi_n)$, in which you write the full definition of $D(\xi_n)$. So the boundary of $D(\xi_n)$ is a summation in which the first term is obtained by dropping $\beta(\Delta_n)$ and just writing the term in the bracket, and then subtracting $\beta(\Delta_n)$ of ∂ of the bracketed term. Using the result in the paragraph above, it follows that this is equal to $Sd(\xi_n) - \xi_n$ is left out. So, we have proved the chain homotopy property also. So, let now summerize the properties of Sd, so that we can quote it in future use.

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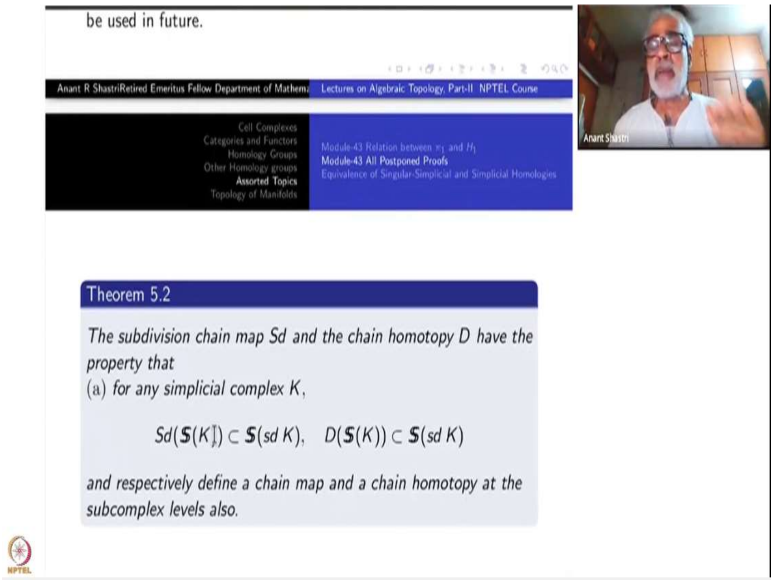

The subdivision chain map Sd from $S.(X)$ to $S.(X)$ and the chain homotopy D from $S.(X)$ to $S.(X)$ have the following properties: What are properties?

(a) First of all, for any simplicial complex K, Sd of the singular simplicial chain subcomplex double $S.(X)$ goes inside double $S.(sdK)$ and D of the singular simplicial chain subcomplex double $S.(K)$ goes inside double $S.(sdK)$ and hence define a chain map and a chain homotopy on the chain complex double $S.(K)$ into double $S.(sdK)$.

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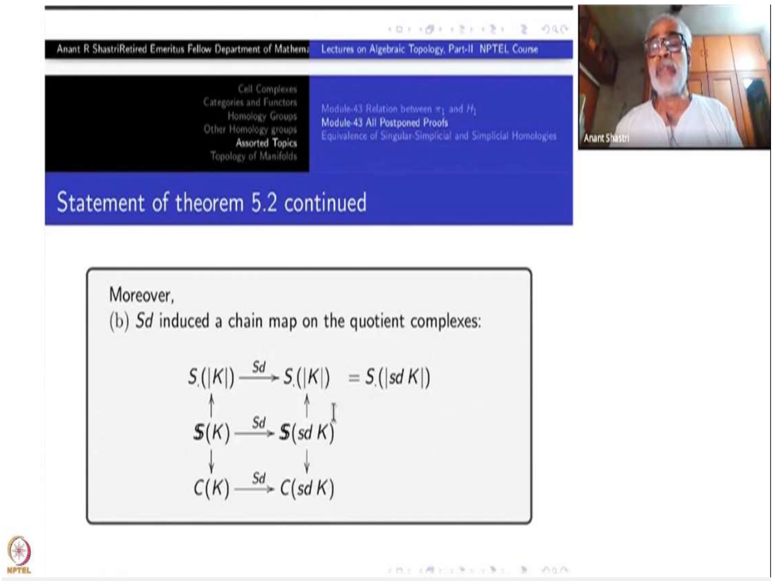

(b) The chain map defined in (a) further induces a chain map on the quotient complexes, $C.(K)$ to $C.(sdK)$. Verification of the above two statements is easy.

So, this generalizes the old construction D of $S(K)$ is contained this $S(Sd(K))$. So, our new chain complex, our new Sd the subdivision chain map has the property that it preserves the simplicial maps. Simplicial singular subcomplex are preserved. This is what defines a chain map and a chain homotopy at the subcomplex levels also.

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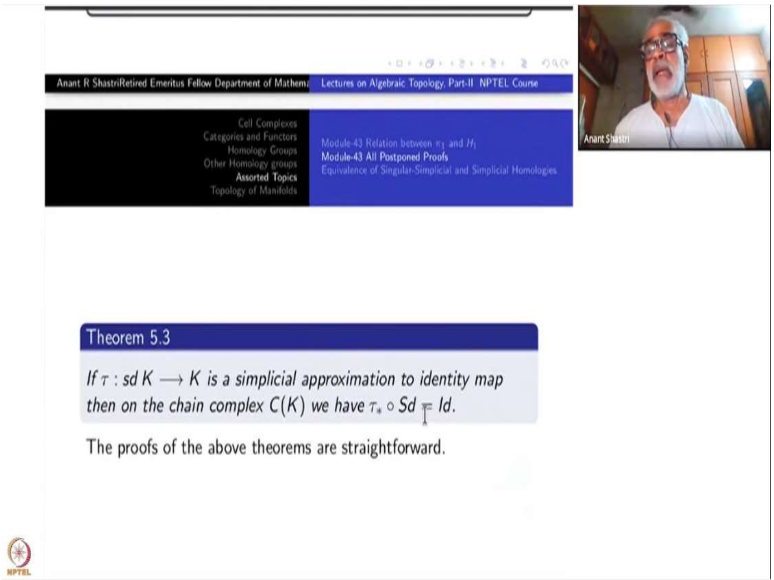

The next theorem states that: If τ from $sd(K)$ to K is a simplicial approximation to the identity map, then on the chain complex level, we have $\tau_* \circ Sd = identity$. We proved this one earlier working quite hard. Now this follows easily. Let us stop here and next time we should prove the excision, basically excision and a few other things which are left out. Thank you.