Introduction to Algebraic Topology (Part - II)
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Lecture - 45
Relation Between Pi 1 and H 1

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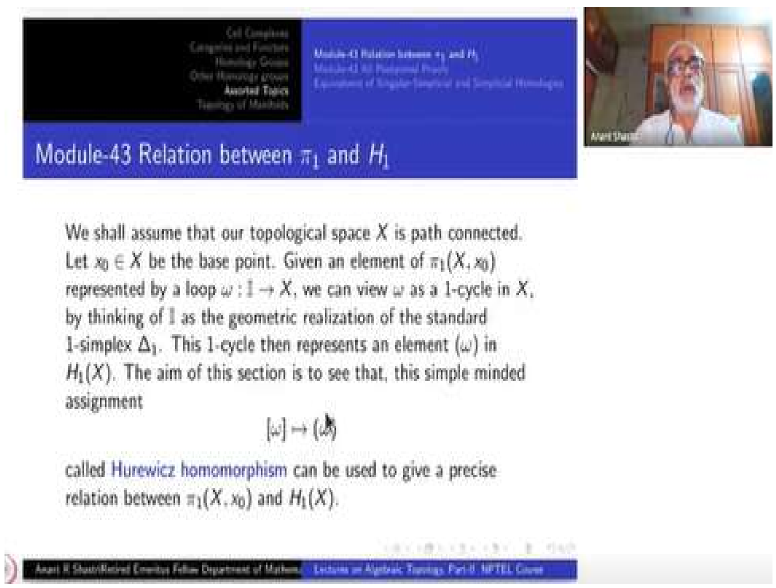

So, now we shall start an entirely new topic which is again the starting point of a big topic namely relation between homotopy groups and homology groups. So, we shall only deal here with relation between the fundamental group and H_1. To begin with we can and will assume that X is path connected because, if it is not path connected, it will be the study of each path connected components separately.

So, you can pick up a base point because we are discussing π_1 the fundamental group. So, fundamental group always should be discussed with a base point x_0 in the space X, though we do not always mention it explicitly.

Now, take an element in $\pi_1(X)$, namely, represented by loop in X at the point x_0. A loop is, first of all a continuous function from a closed interval to X. The closed interval can be thought of as the underlying space of the standard 1-simplex Δ_1. Therefore, a closed loop can be thought of as a one cycle. It is a singular 1-simplex in X, but since both the endpoints are the same, the boundary of that 1-simplex will be 0. Therefore, it is a 1-cycle. So, if we have a loop ω, I will denote the (ω) to be the element represented by this ω in $H_1(X)$. The $[\omega]$ will denote the element represented by ω in $\pi_1(X)$.

So, starting with an element in $\pi_1(X)$ namely $[\omega]$ to (ω), I get a function from $\pi_1(X)$ to $H1(X)$. I want to claim, first of all that this is well defined. In other words, if ω_1 and ω_2 are path homotopic loops, i.e., they represent the same element in π_1 then as 1-cycles, they should represent the same element in H_1, that is what we have to prove. Otherwise, the function will not be well defined.

After that we want to prove that this function is actually a homomorphism and then of course we will prove that this homomorphism is surjective and so on. So, this homomorphism is called Hurewicz homomorphism. So, let us do all these things now elaborately.

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The first lemma is that suppose, $\omega_i, i = 1, 2$ are paths in X such that their composition is defined. That means that endpoint of ω_1 is the starting point of ω_2. Then $\omega_1 * \omega_2$ is defined. As a singular 1

-chain you can take $\omega_1 * \omega_2 - \omega_1 - \omega_2$. This 1-chain is null homologous. So, this can proved by 2 -chain in X such that its boundary is this given 1-chain.

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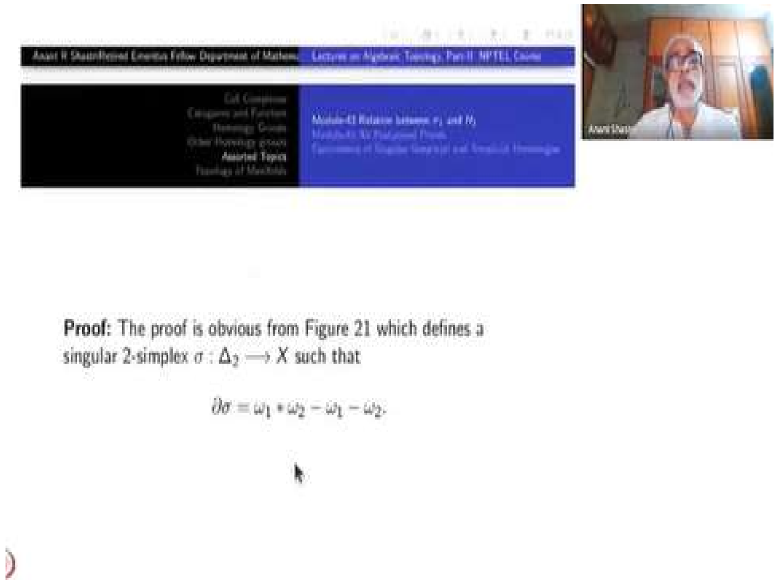

So, this is the starting lemma. The proof is completely obvious if you look at this picture.

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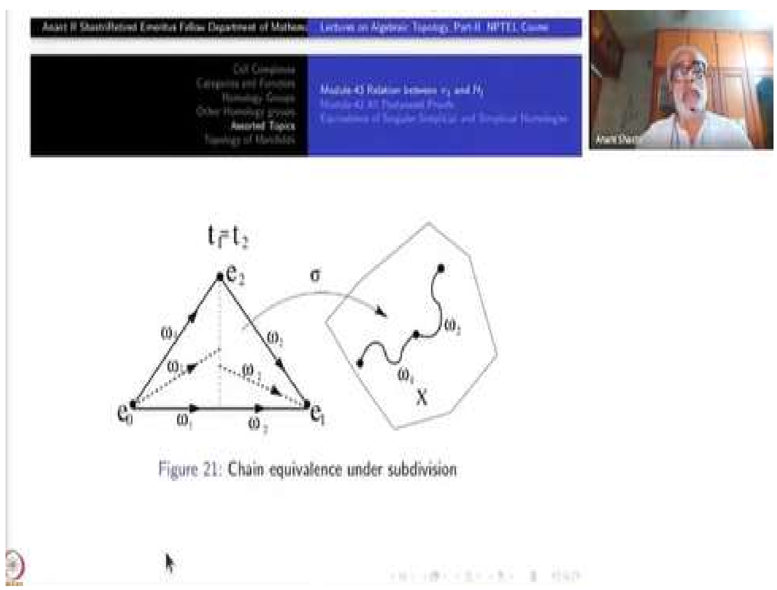

In this picture, we have this triangle. I have cut it into 2 parts. Its vertices are e_0, e_1 and e_2. From e_2, you draw a perpendicular to the base edge $[e_0, e_1]$. The base edge is divided into 2 parts, the first part will take ω_1, second part will take ω_2 so that the original edge $[e_0, e_1]$ is the domain of $\omega_1 * \omega_2$. Also put ω_1 from e_0 to e_2 here in this way and ω_2 from e_2 to e_1 here this way. So, if you trace the boundary of this 2-simplex e_0 to e_1 to e_2 back to e_0, what you get is the 1-chain $\omega_1 * \omega_2 - \omega_2 - \omega_1$. So, point is that I can fill up this whole triangle there. viz, extend continuosly, the function defined on the boundary to a function into X. How do I do this? Draw the perpendicular divided L from the vertex e_2. Then the triangle is divided into two parts. In the first part, every point lies on a line segment joining e_0 to a point p on L and define the function to be ω_1 as shown by the arrow. In the second part, every point line on a segment joining p to e_1, define the function to be ω_2 as shown by the arrow. The proof of the lemma is over.

However, if you are not convinced with this kind of geometric argument, you can better. However, the geometric argument given above is actually necessary at least to get the idea how to fill up the triangle.

But finally, what you have to do is to explicitly write down the formula for the continuous function. I have been training you in this respect, so that you must be able to write down this formula on your own. Here is the formula for the 2-simplex σ.

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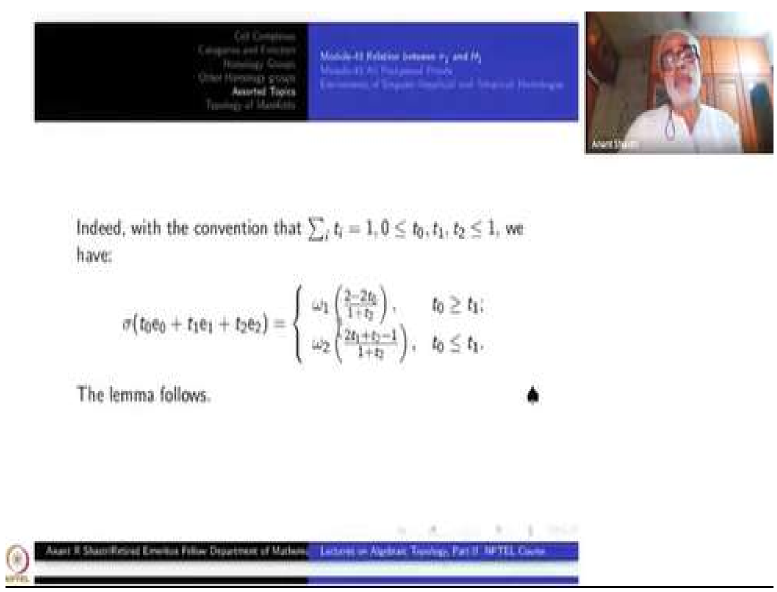

Every point in this triangle is uniquely expressible as $z = t_0e_0 + t_1e_1 + t_2e_2$, where $\sum t_i = 1$ and t_0, t_1, t_2 are all between 0 and 1 (the triangle is the convex hull of the three vertices.) This is, by definition the geometric realization of a standard 2-simplex Δ_2. In that expression, z will be in the first half of the triangle iff $t_0 \geq t_1$. z lies on $[e_0, e_1]$ if $t_2 = 0$, on $[e_0, e_2]$, if $t_1 = 0$ and on $[e_2, e_1]$ if $t_0 = 0$. Put $\sigma(t_0 e_0 + t_1 e_1 + t_2 e_2)$ equal to ω_1 or ω_2 according as z is in the first half of the triangle or in the second half. But you have to reparametrize. So, check whether this is done correctly or not. There may be some typos and so on. So, you have to check that this is the correct definition.

Of course, you can immediately realize that this is not the only unique way to do it. You may have different formulae. This is one neat way. So, what we have proved is that subdivision of path gives a homologous element. Take a path and subdivide it. Then think of this as a conctanation of the parts. As a chain you get a sum which is homologous to the chain represented by the original path.

This idea descends from the experience we have in line integrals. The integral on the entire path is equal to sum of the integral on the subdivisioned paths. So, that is the kind of thing that is happening here. So, in homology, this is what subdivision gives you. We will do more sophisticated things soon.

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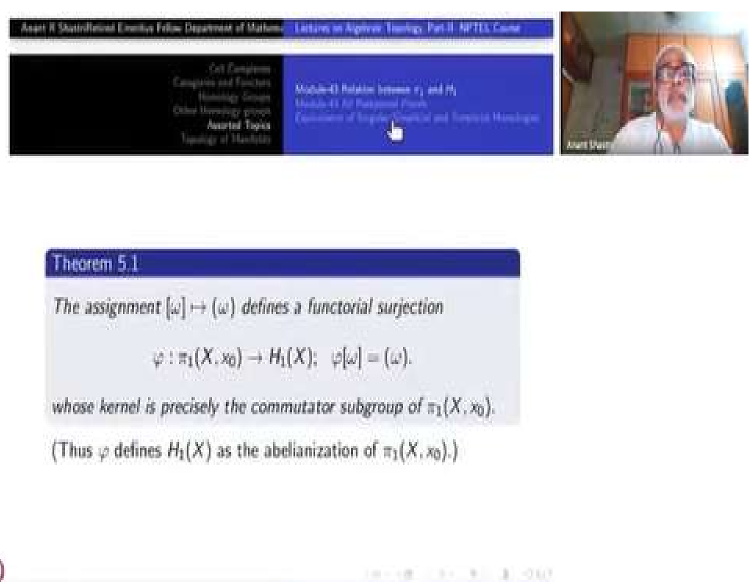

So, next theorem is that the assignment $[\omega]$ to (ω) defines a functorial surjection from $\pi_1(X, x_0)$ to $H_1(X)$, (it is denoted by a $\phi([\omega])$,) whose kernel is precisely the commutator subgroup of $\pi_1(X, x_0)$.

The functoriality is easy to verify. Thus, ϕ defines a surjective homomorphism with its kernel precisely the commutator subgroup means what? π_1 modulo its commutator subgroup is nothing but the abelianization of π_1. It is isomorphic to H_1, by the first isomorphic theorem. So, H_1 is abelianization of π_1. That is very easy to remember. H_1 is by definition an abelian group. It is actually the abelianization of π_1 is the final result here. Let us go ahead and prove this one.

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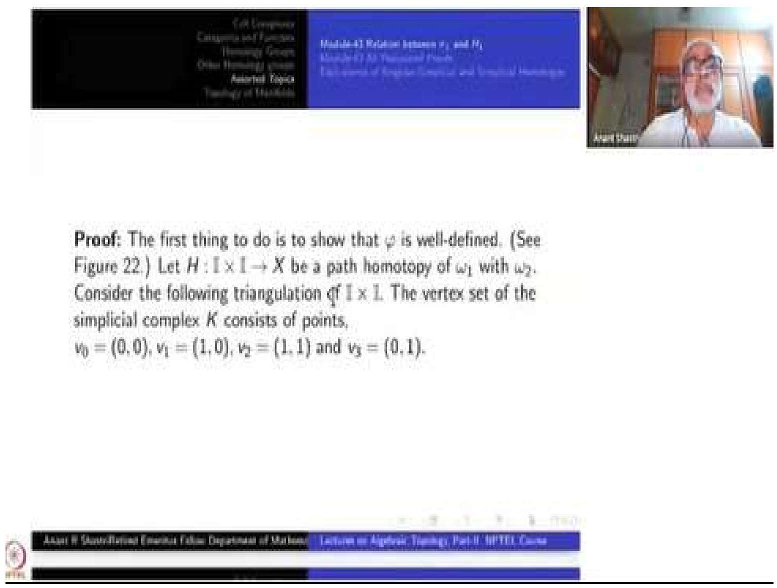

The first thing to do is to verify that ϕ is well defined, without which we have not yet defined the function. So, let H from $\mathbb{I} \times \mathbb{I}$ to X be a path homotopy from ω_1 to ω_2. Consider following triangulation of $\mathbb{I} \times \mathbb{I}$; the vertex of the simplicial complex K consists of these four vertices, $v_0 = (0,0), v_1 = (0,1), v_2 = (1,1)$ and $v_3 = (0,1)$ and five edges $\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2\}$ and $\{v_2, v_3\}$ and the two obvious triangles. $\{v_0, v_1, v_2\}$ and $\{v_0, v_2, v_3\}$. $\mathbb{I} \times \mathbb{I}$ have to be triangulate and then look at the homotopy H and think of it as a sum of singular 2-simplexes. That is what I am trying to do.

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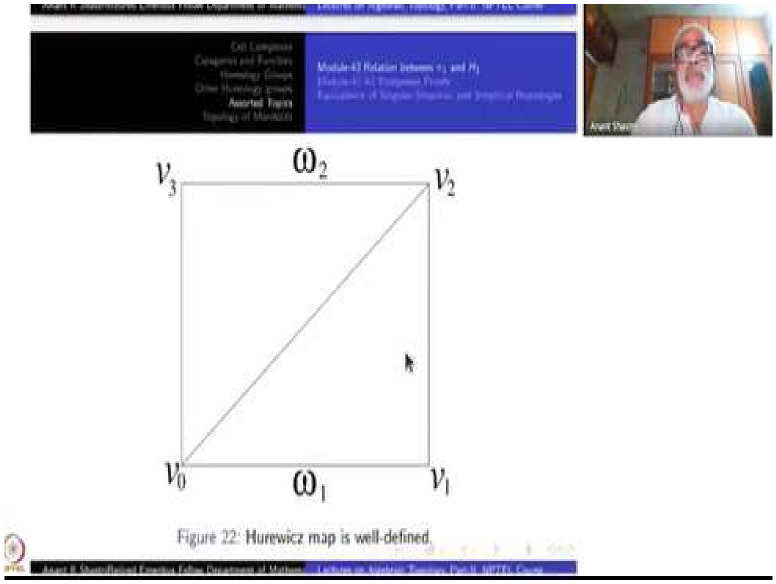

So, put $\alpha = H \circ [v_0, v_1, v_2]$ and $\beta = H \circ [v_0, v_2, v_3]$. We claim that $\partial(\alpha + \beta) = \omega_1 - \omega_2$. That means that $\omega_1 - \omega_2$ is null homologous, or equivalently, ω_1 is homologous to ω_2.

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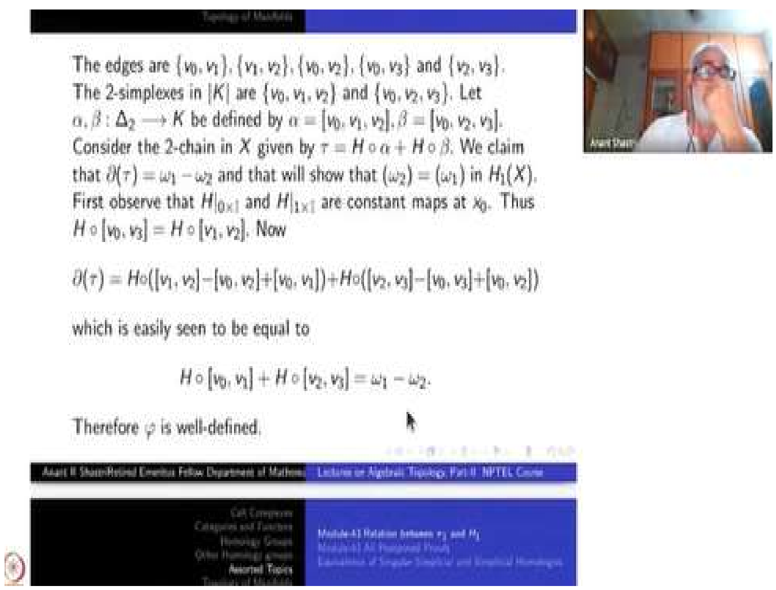

By the very definition a homotopy of two paths has a property that on the end points the homotopy is a constant. This just means that $H \circ [v_1, v_2]$ and $h \circ [v_1, v_2]$ are singletons and hence they are degenerate 1-simplexes.

What is $\partial(\alpha)$? It is $H \circ \partial([v_0, v_1, v_2])$. Now $\partial[v_0, v_1, v_2]$ is $[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. Similarly, $\partial[v_0, v_2, v_3] = [v_2, v_3] - [v_0, v_3] + [v_0, v_2]$. So, $[v_0, v_2]$ occurs with opposite signs and cancels out. It follows that $\partial(\alpha + \beta) = H \circ [v_0, v_1] + H \circ [v_2, v_3] = \omega_1 - \omega_2$.

So, we have verified that ϕ is well defined. Now, the first lemma will tell you that phi is a homomorphism. Functoriality of ϕ is something which is totally obvious here. But I will leave it to you to think about that. So, you think about it and write down a proof.

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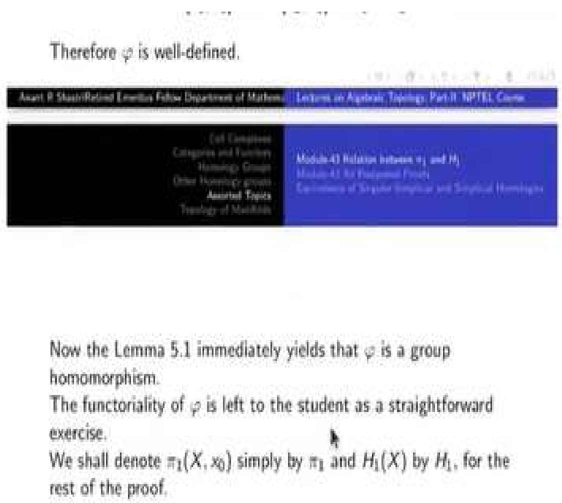

For the rest of the proof we shall use a lazy notation π_1 for $\pi_1(X, x_0)$ and H_1 for $H_1(X)$. Also, in the diagrams if I write the full notation these, it will be difficult to adjust within a slide.

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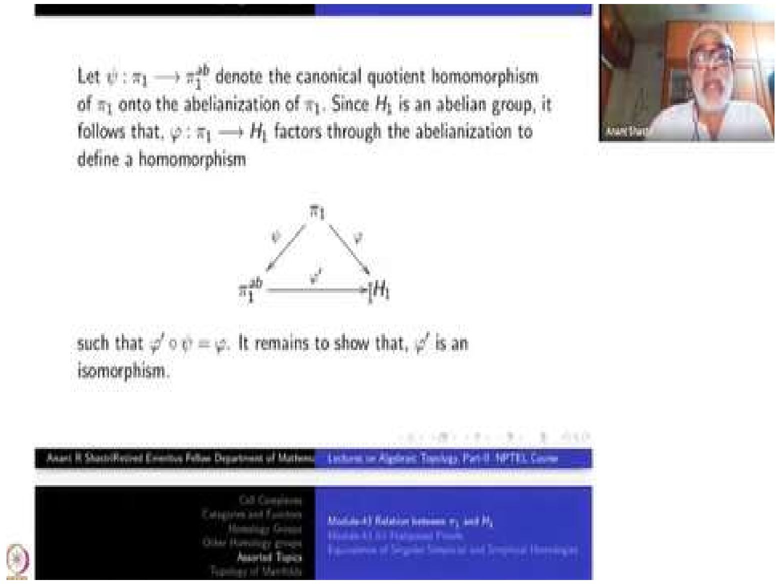

Let ψ from π_1 to π_1^{ab} denote the canonical quotient homomorphism obtained by taking the quotient by the commutator subgroup $[\pi_1, \pi_1]$. So, $\hat{\pi_1}^{ab}$ denotes the fundamental group abelianised, namely, the quotient of π_1 by the commutator subgroup.

Now H_1 is an abelian group. Since any homomorphism from any group to an abelian group always factors down on the abelianization, it follows that ϕ factor down to give a homomorphism ϕ' from π_1^{ab} to H_1 such that $\phi = \phi' \circ \psi$. Our claim is that this ϕ' an isomorphism. Instead of proving that ψ' is a bijection, what I am going to do is to produce an explicit inverse for ϕ'.

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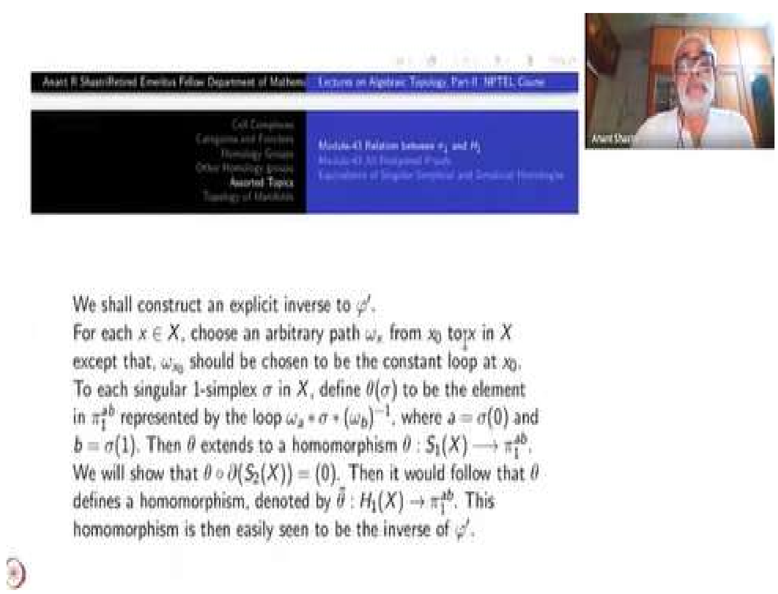

So, how does one do this. We use the typical calculus experience here especially the one in constructing primitives. We have fixed a point x_0 in X. Use path connectivity of X. For each x in X, take a path from x_0 to x and call it ω_x. There are many paths. You take anyone. However, Let us resolve not make life complicated. So, when you joining x_0 to x_0, just take the constant path at x_0, that is the only specification. You could have taken any other loop there is nothing wrong but do not do that just to keep things simpler.

Now let σ be a singular 1-simplex in X. What is the singular 1-simplex? This is just a path in X. It has a starting point a and an end point b, say. There are paths from x_0 to both of them. So, I am going to define $\theta(\sigma)$ to be the element in π_1^{ab} represented by the loop $\omega_a * \sigma * \omega_b^{-1}$.

Start from x_0, trace the path ω_a then trace σ and come back to x_0 via $\omega_b i^{-1}$. That loop represents an element in π_1. Take its image under ψ in the abelianization. That is $\theta(\sigma)$. Thus we have a set theoretic function from the basis of $S_1(X)$ into the abelian group π_1^{ab}. So, θ extends to a homomorphism, I am writing it as θ itself, from the free abelian group $S_1(X)$ to π_1^{ab}.

Note that if we had stopped at π_1 level, since π_1 may not be an abelian group the set theoretic function may not extend to the hole of S_1. We claim that θ vanishes on $\partial(S_2(X))$ and hence defines a homomorphism $\tilde{\theta}$ from π_1^{ab} to H_1 and that $\tilde{\theta}$from here to here is the inverse of ϕ'. So, let us verify that one by one.

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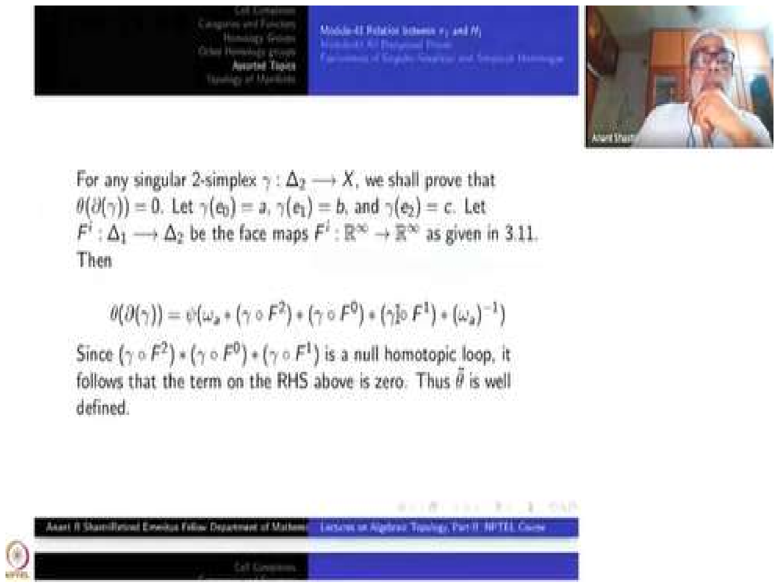

Take any singular 2-simplex gamma from Δ_2 to X. If I show that $\theta(\partial(\sigma))$ is 0 then, since S_2 is generated by all these γ's, i.e., every element of S_2 is a finite sum of such γ's that will prove that theta vanishes on $\partial(S_2(X))$.

Let F^i be the standard face operators that we have been using all the time in the definition of ∂. Then $\partial(\gamma) = \gamma \circ (F^0 - F^1 + F^2)$. To define θ of that, you have to join the end points of these 1-simplexes to the base point x_0. Because the boundary of any 2-simplex is a cycle, you can treat it as a loop at some point a. Therefore, $\theta(\partial(\gamma)) = \psi(\omega_a * (\gamma \circ F^0) * (\gamma \circ F^1)^{-1}) * (\gamma \circ F_2) * \omega_a^{-1})$.

But the loop $(\gamma \circ F^0) * (\gamma \circ F^1)^{-1}) * (\gamma \circ F_2)$ in X is the boundary of gamma defined on Δ_2 and hence is null homotopic in X. So, its conjugate is also null homotopic. Therefore the term on RHS is actually 0. I said that this element in $\pi_1(X)$ itself is trivial. So, when you go to π_1^{ab} this will go to 0 element. That proves that $\tilde{\theta}$ is well defined.

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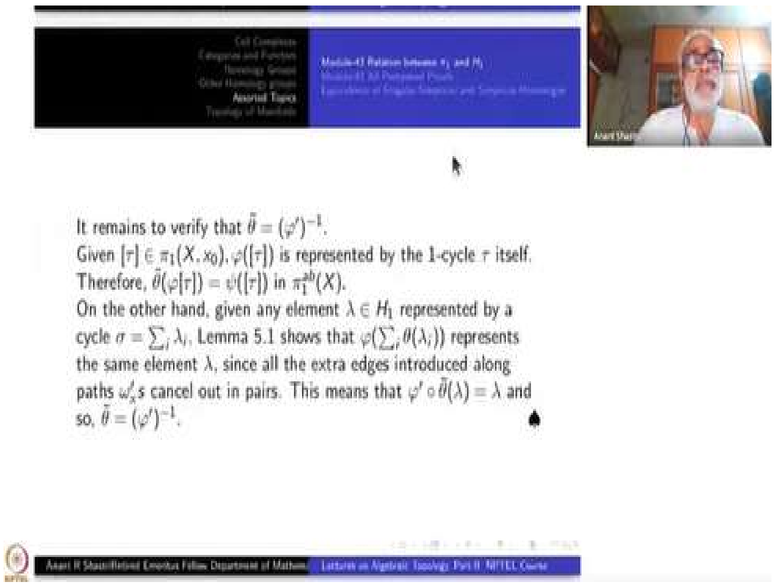

It remains to verify that $\tilde{\theta}$ is $(\phi')^{-1}$. Given $[\tau]$ in $\pi_1(X, x_0)$, $\phi(\tau)$ is represented by the 1-cycle τ itself. You see what is definition of ϕ? ϕ is just the be round bracket of a square bracket. Therefore, I do not have to search for some ω_x etc here, to define θ, τ being already a loop at x_0. $\theta(\phi[\tau])$ is (τ) itself. That shows that $\tilde{\theta} \circ \phi'$ is the identity of π_1^{ab}.

Finally to show that $\phi' \circ \tilde{\theta} = Id_{H_1}$, start with a 1-cycle in X, $c = \sum n_i \lambda_i$. For each λ_i, I am introducing paths from x_0 to the starting point as well as the end point of λ_i, and converting them into loops based at x_0. As a chain itself, all these extra 1-simplexes will cancel out in pairs, since c itself is a cycle. So, without loss of generality, I can assume that all the λ_i are loops based at x_0. Therefore, we get a loop omega based at x_0 viz., composition of $(\lambda)^{n_i}$ in whichever order you please. it follows easily that $\psi[\omega] = \tilde{\theta}(c)$ and $\phi'\psi[\omega] = c$. Recall that introducing extra edges suitably is a technique we learnt in complex analysis.

This result about the relation between π_1 and H_1 is very useful in algebraic topology. Let me take a few more minutes and give you one small application of product of CW-complexes. Again, this result is already familiar to you and not a new one, though arrived in a different way.

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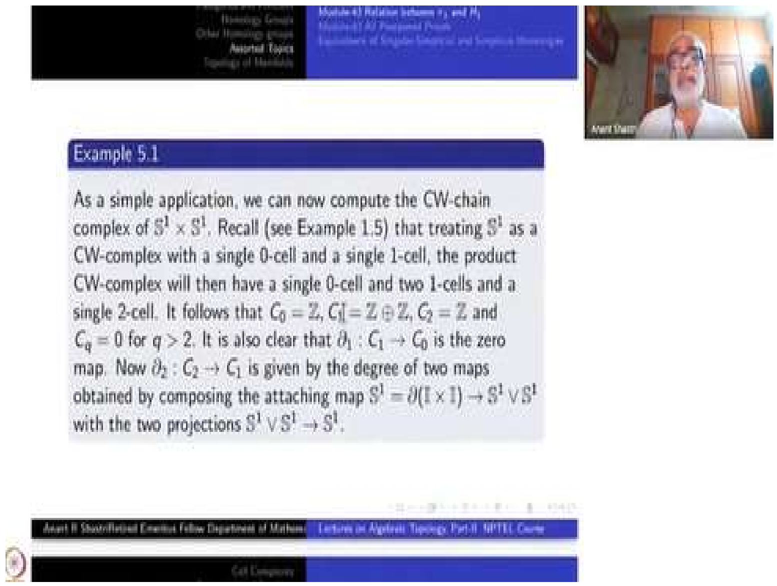

So, let us compute the CW-chain complex of of the 2-dimensional torus, $\mathbb{S}^1 \times \mathbb{S}^1$. Recall that we have given $\mathbb{S}^1$ a CW-structure with one 0-cell and one 1-cell. The product CW-complex will then have a single 0-cell, two 1-cells and a 2-cell. That is the product CW structure on $\mathbb{S}^1 \times \mathbb{S}^1$.

So, it follows that the associated CW-chain complex should have C_0, C_1 and C_2 respectively equal to $\mathbb{Z}, \mathbb{Z}^2$ and $\mathbb{Z}$. (I should actually write $C_i^{CW}(\mathbb{S}^1 \times \mathbb{S}^1)$ but I am just writing a short notation here.) And C_q is 0 for $q > 2$. It is also clear that the boundary operators d_1 from C_1 to C_0 is the zero map, because the vertex set is a singleton and the attaching maps for 1-cells have to the same constant function. So, when you take the boundary, it becomes point minus point, and cancels out. So, on both the generators of C_1, the boundary map will be 0. So, the entire boundary d_1 from C_1 to C_0 is the zero map.

What is the boundary map from d_2 from C_2 to C_1? There is only one 2-cell. The characteristic map for this is the product $\mathbb{I} \times \mathbb{I}$ to $\mathbb{S}^1 \times \mathbb{S}^1$, of the two characteristic maps of the 1-cells, $\mathbb{I}$ to $\mathbb{S}^1 \times \mathbb{S}^1$.

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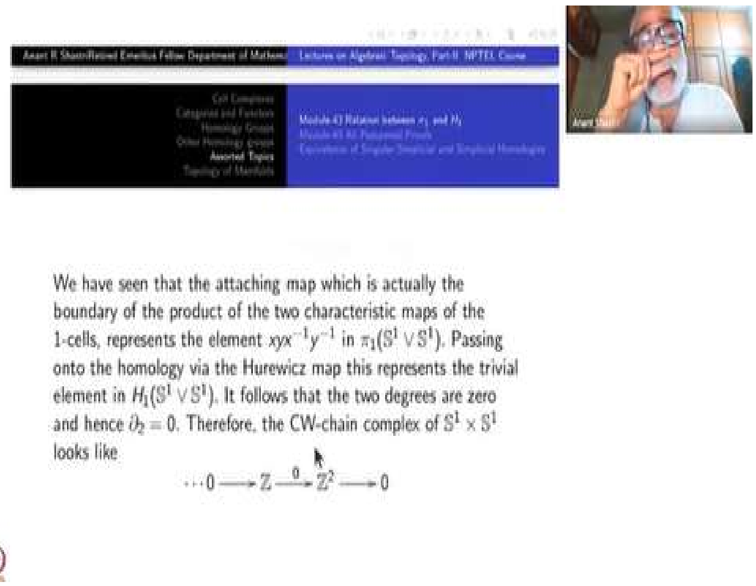

We have also seen that the attaching map of this 2-cell is the map $\mathbb{S}^1$ to $\mathbb{S}^1 \vee \mathbb{S}^1$ representing the loop $xyxy^{-1}$ in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$, where x, y represent the elements given by the two inclusions $\mathbb{S}^1$ to $\mathbb{S}^1 \vee \mathbb{S}^1$. I have also elaborately explained how to compute the boundary map in the chain complex C_*^{CW}.

So, look at the attaching map there and take projections to each of the two factors $\mathbb{S}^1$ and compute the degree of each of them. Then you can determine the function d_2. This boundary loop will have to trace the first factor once in a particular direction, then the second factor followed again by the first fact in the opposite direction and finally the second fact in the opposite direction. x in the opposite direction and y in the opposite direction. This is easily understood the process of obtaining the torus as a quotient of a rectangular piece of paper.

So from $H_1(X^{(1)}, X^{(0)}) = H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ iff you project onto $H_1(\mathbb{S}^1)$, the result will be zero under both projections, x and $-x$ (and similarly, y and $-y$) cancel out. Therefore, the d_2 from C_2 to C_1 is the zero map. The the entire chain complex looks like $\ldots 0 \to \mathbb{Z} \to 0 \to \mathbb{Z}^2 \to 0 \to \mathbb{Z}$. Therefore, kernel of each d_i is the entire domain and the image of each d_i is zero. Thus the homology groups of $\mathbb{S}^1 \times \mathbb{S}^1$ are $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$ and zero after that.

So, let us stop here. Next time we will be doing several of the proofs that we have postponed while doing homology theory. So, we will go through some of the proofs. Thank you very much.