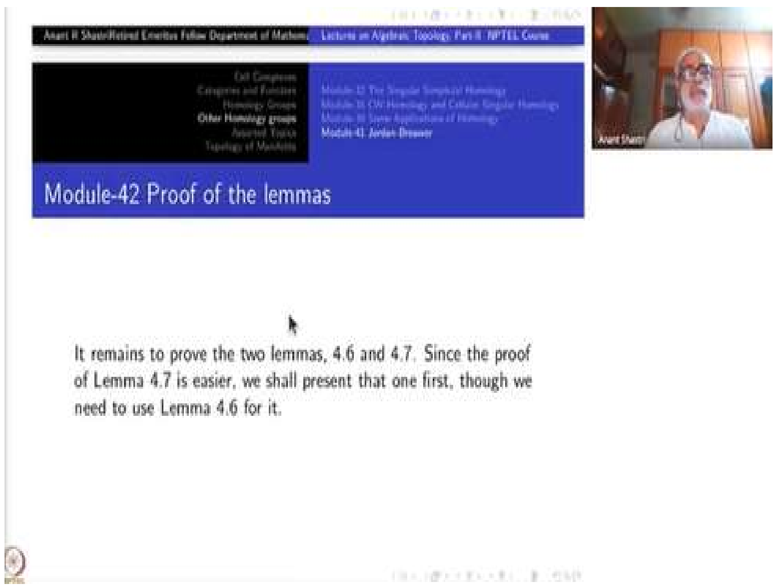
Introduction to Algebraic Topology (Part - II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay

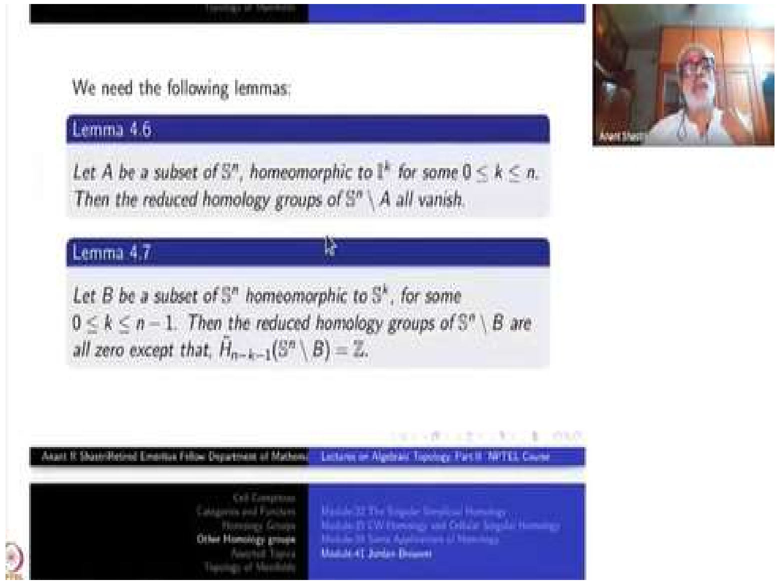
Lecture - 44
Proof of Lemmas

(Refer Slide Time: 00:13)



So, we are in the middle of proving Jordan Brouwer theorem and Brouwers' invariance of domain theorem. We proved these two statements modulo 2 important lemmas.

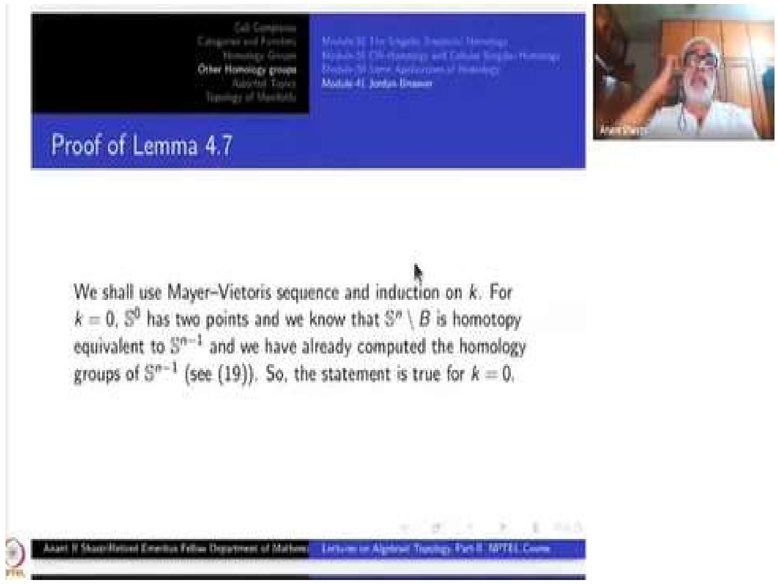
(Refer Slide Time: 00:37)



So, these two lemmas we need to prove now. The first lemma 4.6 says that for a subset of $\mathbb{S}^n$ homeomorphic to $\mathbb{I}^k$, where k is between 0 to n, the reduced homology groups of the complement all vanish. The second lemma 4.7 says that B is a subset $\mathbb{S}^n$, homeomorphic $\mathbb{S}^k$, for k between 0 and $n-1$, the reduced homology groups of the complement are zero except in the dimension $n-k-1$ and at that level it is an infinite cyclic group. So, we shall prove 4.7 first and then we will prove 4.6.

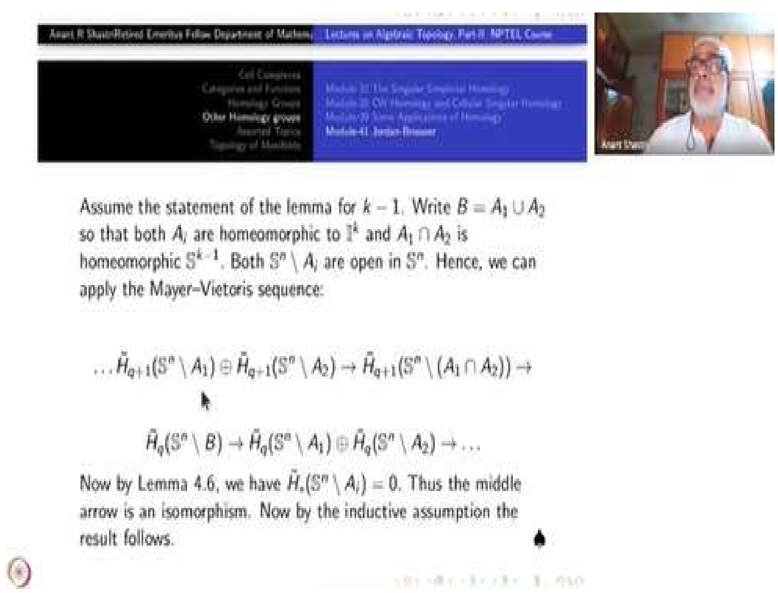
So, while proving 4.7, we use 4.6 and then once you prove 4.6 also, the two statements namely Jordan Brouwer separation theorem and Jordan Brouwer invariance of domain theorem will get proved. So, this is what we need to prove now. Now proof lemma 4.7.

(Refer Slide Time: 02:10)



So, we have to use Mayer-Vietoris sequence and apply induction on k. For $k = 0$, $\mathbb{S}^0$ has two points and $\mathbb{S}^n$ setminus two points is homotopy equivalent to $\mathbb{S}^{n-1}$. If you remove one point, you get a space homeomorphic to $\mathbb{R}^n$ or an open disc. You remove one more point then the space can be strongly deformed to a suitably chosen copy of $\mathbb{S}^{n-1}$. In particular $\mathbb{S}^n \setminus B$ is homotopy equivalent to $\mathbb{S}^{n-1}$. We have already computed the homology of $\mathbb{S}^{n-1}$ from which the statement of the lemma for $k = 0$ follows. Now assume that the statement is true for $k = 1$.

(Refer Slide Time: 03:28)



Now, take B which is homeomorphic to $\mathbb{S}^k$ in $\mathbb{S}^n$. It can be written as union of two closed sets A_1 and A_2, each homeomorphic to $\mathbb{I}^k$. In $\mathbb{S}^k$, I can take them as standard discs intersecting along the equator and then take their images as A_1 and A_2. So, $A_1 \cap A_2$ is homeomorphic to $\mathbb{S}^{k-1}$. Both $\mathbb{S}^n \setminus A_i$, (we throw away the closed subsets), are open sets in $\mathbb{S}^n$. Hence, we can apply Mayer-Vietoris sequence. So, you get an exact sequence of reduced homology groups: $\ldots H_{q+1}(\mathbb{S}^n \setminus A_1) \oplus H_{q_1}(\mathbb{S}^n \setminus A_2)$ to $H_{q+1}(\mathbb{S}^n \setminus (A_1 \cap A_2))$ to $H_q(\mathbb{S}^n \setminus B)$ to $H_q(\mathbb{S}^n \setminus A_1) \oplus H_q(\mathbb{S}^n \setminus A_2) \ldots$ This is the union of the two sets, the morphism here is the connecting homomorphism and the next one here is the intersection of the two sets $\mathbb{S}^n \setminus A_1$ and $\mathbb{S}^n \setminus A_2$. Put tilde everywhere.

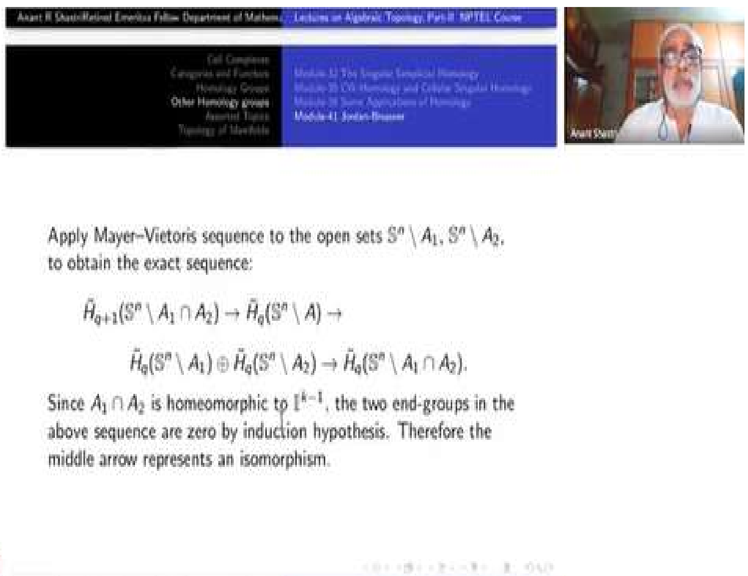
(Refer Slide Time: 06:15)

By the previous lemma 4.6, all of $\tilde{H}_*(\mathbb{S}^n \setminus A_i)$ are zero. Therefore the arrow representing the connecting homomorphism must be an isomorphism. So, the statement for k follows from the induction hypothesis. Now the proof of 4.6: Here also we induct on k. Now I have to prove that $\mathbb{S}^n$ setminus any disc has the reduced homology trivial. For $k = 0$, What is that? It is just $\mathbb{S}^n$ set minus a point which is actually homeomorphic to $\mathbb{R}^n$. So, it is conductible, hence the reduced homology groups vanish. Now assume the result for $k-1$.

Now, suppose $\tilde{H}_q(\mathbb{S}^n \setminus A)$ is not 0 for some q. (So, we are going to prove this one by contradiction.) Then you should have a q-cycle c, representing a nonzero element in $H_q(\mathbb{S}^n \setminus A)$. Remember that a q-cycle means a q-chain with its boundary being zero.

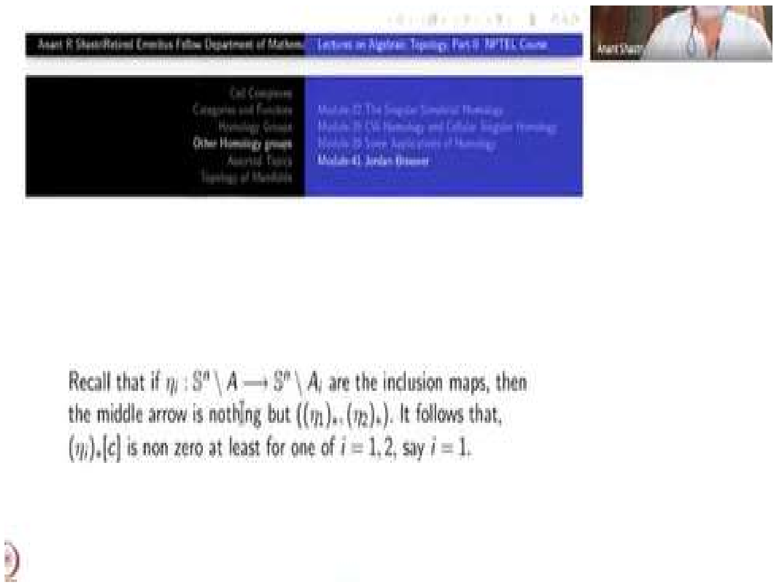
A q-chain being a finite sum of singular q-simplexes, its support is going to be a compact set. So, this is the topological fact that we are going to use now. So, let h be a homomorphism from $\mathbb{I}^k$ to A. (By hypothesis, A is homeomorphic to some $\mathbb{I}^k$). Write $\mathbb{I}^k = A_1 \cup A_2$. What are A_i's? We are going to split this disc A into two halves along $\mathbb{I}^{k-1}$ cross the last coordinate $1/2$. So, cut it in the middle along last coordinate. $A_1 = h(\mathbb{I}^{k-1} \times [0, 1/2])$ and $A_2 = h(\mathbb{I}^{k-1} \times [1/2, 1])$. Both of them are copies of again the disc itself. We can apply Mayer-Vietoris sequence and conclude something. (This is similar to what you did in the proof of Cauchy-Goursat theorem in complex analysis and easier).

(Refer Slide Time: 09:34)



Apply Mayer Vietoris sequence, to the excisive pair $\{\mathbb{S}^n A_1, \mathbb{S}^n \setminus A_2\}$ to obtain the exact sequence $\tilde{H}_{q+1}(\mathbb{S}^n \setminus (A_1 \cap A_2))$ to $\tilde{H}_q(\mathbb{S}^n \setminus A)$ to $\tilde{H}_q(\mathbb{S}^n \setminus A_1) \oplus \tilde{H}_q(\mathbb{S}^n \setminus A_2)$ to $\tilde{H}_q(\mathbb{S}^n \setminus (A_1 \cap A_2) \ldots$Now what we are assumed, we have assumed that there is some element in $H_q(\mathbb{S}^n \setminus A)$ which is not 0. Since $A_1 \cap A_2$ is homeomorphic to $\mathbb{I}^{k-1}$ by induction hypothesis that these two end groups are zero and hence this middle arrow is an isomorphism. So the non zero element goes to a non zero element in the direct sum.

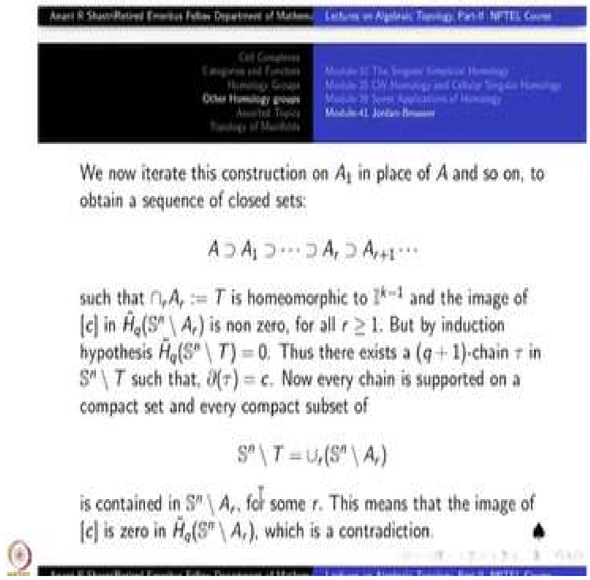
(Refer Slide Time: 11:16)



But what is this isomorphism? It is $[c]$ maps to $((\eta_1)_*(c), (\eta_2)_*(c))$, where, η_i from $\mathbb{S}^n \setminus A$ to $\mathbb{S}^n \setminus A_1$ are inclusion maps. It follows that at least one of the two $(\eta_i)_*(c)$ is non zero. By relabeling if necessary, we may assume that we have a half space A_1 of A homeomorphic to $\mathbb{I}^{k-1} \times B_1$ such that under the inclusion map η_1 from $\mathbb{S}^n \setminus A$ to $\mathbb{S}^n \setminus A_1$, we have $(\eta_1)_*(c)$ is non zero.

That is what you get using the Mayer Vietoris theorem. Now repeat this process, by replacing A by A_1, cut A_1 into two halves and so on to obtain a A_2 which is homeomorphic to $\mathbb{I}^{k-1} \times B_2$ and such that the image of c under the inclusion induced homomorphism is non zero.

(Refer Slide Time: 12:39)



So, we keep getting a nested sequence A_1 containing A_2 containing ..., where each $A_i = h(\mathbb{I}^{k-1} \times B_i)$ with B_i being a nested sequence of closed intervals of length $1/2^i$ and such that the image of c under the inclusion induced map η_i from $\mathbb{S}^n \setminus A$ to $\mathbb{S}^n \setminus A_i$ is non zero in $H_q(\mathbb{S}^n \setminus A_i)$. By Cantor's intersection theorem, intersection of all the B_i is equal to a single point $\{p\}$, in the last coordinate of $\mathbb{I}^k$.

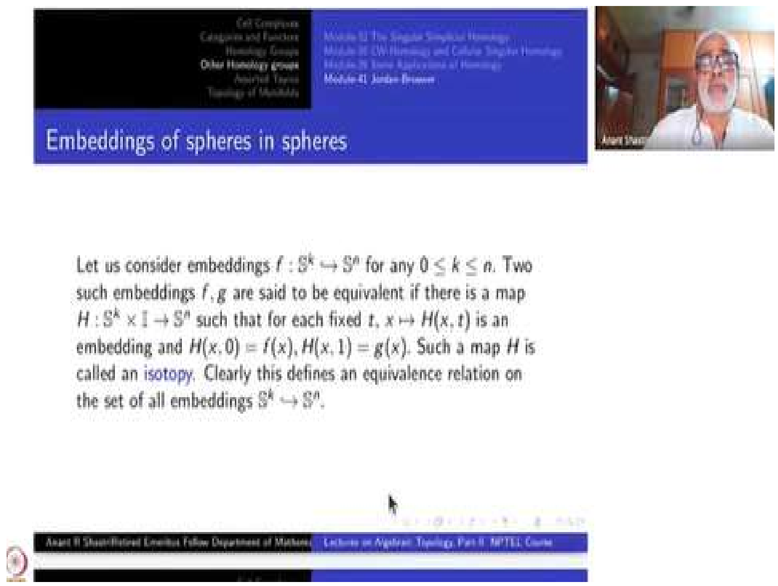
Put $T = h(\mathbb{I}^{k-1} \times \{p\})$. Then by induction hypothesis, $\tilde{H}_q(\mathbb{S}^n \setminus T) = 0$. Thus, $\eta_*(c) = 0$ where η from $\mathbb{S}^n \setminus A$ to $\mathbb{S}^n \setminus T$ is the inclusion map. Therefore there exists a $(q+1)$-chain τ in $\mathbb{S}^n \setminus T$ such that $\partial(\tau) = c$.

As before, the support of σ is compact in $\mathbb{S}^n \setminus T$ which covered by an increasing sequence of open sets $\mathbb{S}^n \setminus A_i$. But then this implies $(eta_i)_*(c) = 0$ which is a contradiction. So, that completes the proof of the lemma and thereby the proofs of all the theorems.

As I told you, these two lemmas, they give you much more information than the theorems and they will be quite useful elsewhere also. Now, I would like to make a few comments on Jordan Brouwer theorem itself. This theorem belongs to a class of problems or results, namely study of embedding of one sphere into another.

These two lemmas tell you a lot about them. There are many more things of interest which we cannot discuss here. So, Brouwer, while proving this famous theorem, he did not just prove one single result but he has done a lot of work. Now let us understand in this slide a little bit about the above problem in general.

(Refer Slide Time: 17:22)

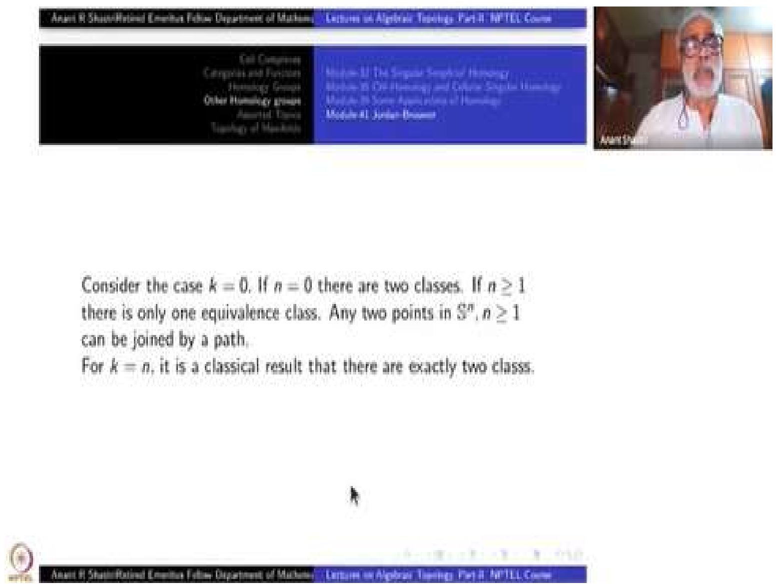


Let us consider embeddings of $\mathbb{S}^k$ in $\mathbb{S}^n$, embedding means just topological embeddings-- one-to-one continuous functions. Domain and codomain are compact Hausdorff spaces, automatically such functions are homeomorphisms onto their image. Two such embeddings are set to be equivalent if you can `move' one embedding to the other embedding. What is the meaning of `moving'? We have already defined homotopy.

But here homotopy will not work, homotopy with an additional condition, namely, at each stage it must be an embedding, i.e., homotopy through embeddings. (To begin with we have two embedded objects you can move it from one to the other through embeddings, do not collapse, or make overlap, cross itself and so on.) So, more specifically, you must have a continuous function H from $\mathbb{S}^k \times \mathbb{I}$ to $\mathbb{S}^n$ such that for each fix $t \in I$, x maps to $H(x, t)$ is an embedding, $H(x, 0)$ is $f(x)$ and $H(x, 1)$ is $g(x)$. Then we can say that f is isotopic to g. Clearly, f is always isotopic to f itself, f is isotopic with g implies g is isotopic to f, etc. Just like homotopy is a transitive relation you can even prove that isotopy is a transitive also. Therefore isotopy is an equivalence relation. So, one is interested in knowing equivalence classes of embeddings of $\mathbb{S}^k$ in $\mathbb{S}^n$. So this is a general problem.

Now quite a bit is known about this problem. I would like to give you just some information here, no proofs.

(Refer Slide Time: 19:46)



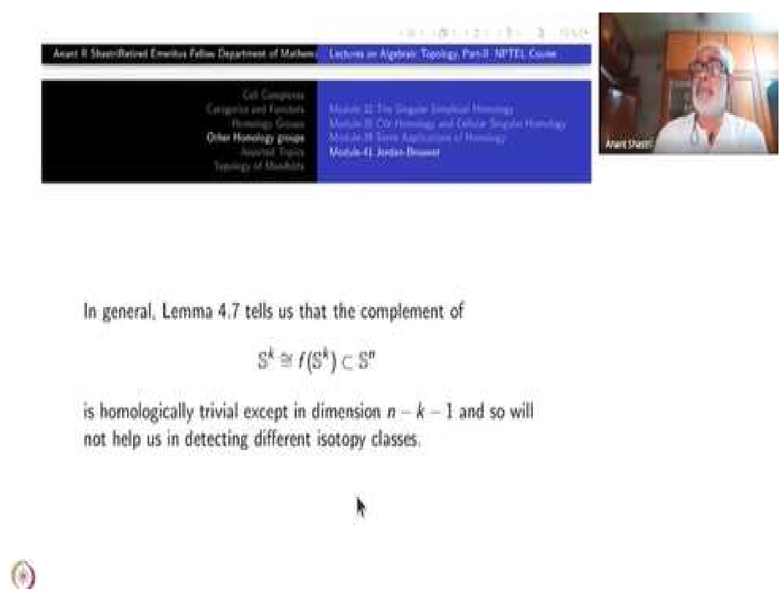
Consider case, $k = 0$. What is the meaning of $k = 0$? $\mathbb{S}^0$ has two points. Embedded in $\mathbb{S}^n$? If n is also 0, then there are only two possibilities either the identity map or the transposition and you cannot isotope one to the other. So, there are two classes. If $n > 0$ we consider two points in $\mathbb{S}^1$, or $\mathbb{S}^2$, etc. then since we know that all $\mathbb{S}^n$, $n > 0$ are connected, we can join any two points by a path. An isotropy of a point is nothing but just a homotopy, nothing more than that. Actually, we have to see a little more here.

Take any two pairs of points, $\{a_1, b_1\}$ and $\{a_2, b_2\}$ inside $\mathbb{S}^n$. You can join a_i to b_i by paths ω_i which do not intersect each other. That is not very difficult to see for $n > 1$. For $n = 1$, you must check that.

Next consider the case $k = n$. What is the meaning of an embedding now. It must be homeomorphism from $\mathbb{S}^n$ onto the whole of $\mathbb{S}^n$. By the way, this requires a proof. Can you have homeomorphic embedding of $\mathbb{S}^n$ inside $\mathbb{S}^n$, which is smaller than the whole of $\mathbb{S}^n$? No. I am not going to tell you the proof here. It actually follows whatever well have done so far. But I am not going to do that one now. You have to think about that.

So, now the question is what are the self-homomorphism isotopy-classes of $\mathbb{S}^n$? This was a classical problem at least in the differentiable case. It was solved quite long back by Smale and Hirsch, etc. The answer is that there are exactly two classes. When you are dealing with just homeomorphisms which may not be smooth, there are always more difficulties than in the smooth case. In the smooth case, namely diffeomorphisms, there are exactly two isotopy classes; this is the standard result. Namely, those which preserve orientation and those which reverse orientation. It is easy to see that these two are distinct classes. Much more difficult to prove that these are the only two classes.

(Refer Slide Time: 23:13)



Now, consider the general case, $0 < k < n$. Our lemma 4.7 is ineffective here. The complement of any embedded sphere has trivial homology accept in dimension $n - k - 1$. So, in some sense to distinguish between two embeddings it is useless. It is a very good theorem but it is useless as far as in distinguishing any two of them upto isotopy. The complements are homologically same. So, what to do, there is no simpler way. So, we consider further special cases. When $k = 1$ and $n = 3$ seems to be the most interesting of all the cases.

(Refer Slide Time: 24:07)

So, before that I will tell you a deep result in PL topology. It tells you that instead of topological embeddings, consider the so called PL embeddings of $\mathbb{S}^k$ in $\mathbb{S}^n$. If the codimension $n - k$ is bigger than equal to 3, then any two embeddings are PL-isotopic. It is a fantastic result. So, $\mathbb{S}^1$ embedded in $\mathbb{S}^4$, $\mathbb{S}^5$, or $\mathbb{S}^2$ embedded in $\mathbb{S}^5$, $\mathbb{S}^6$, there is only one isotopy class.

In other words, whatever embedding you choose, you can move it around and bring it back to the standard embedding. So, that is the meaning of saying that any two embeddings are isotopic to each other. In particular, the complements of any two embeddings of $\mathbb{S}^k$ in $\mathbb{S}^{k+r}$ for $r > 2$ are homeomorphic.

(Refer Slide Time: 25:13)

So, the only non trial cases are when $n - k = 1$ or 2. The case $n - k = 2$ is generally known as knot theory, which is quite deeply studied branch of algebraic topology, with a lot of applications in many other sciences like chemistry, string theory and so on. So, we shall not be able to discuss this problem in any more detail, except that I want to tell you a few things. The case when $n = 3$ and $k = 1$ is the most interesting one, that is where lots of problems are there. And classically, the only thing that was done was to look at the fundamental group of the complement.

Now none of these theorems and lemmas not tell you anything about the fundamental group of the complement. They tell you only about the homology. The fundamental group can be computed using Van-Kampen's theorem and what are called knot presentations. So, that will give you a lot of information.

Nevertheless, till 1980s, a lot of problems were unsolved till C. R. F. Jones came into picture and cracked the whole thing by opening up a new way of thinking about the knots. So, it is now quite a flourishing branch of algebraic topology.

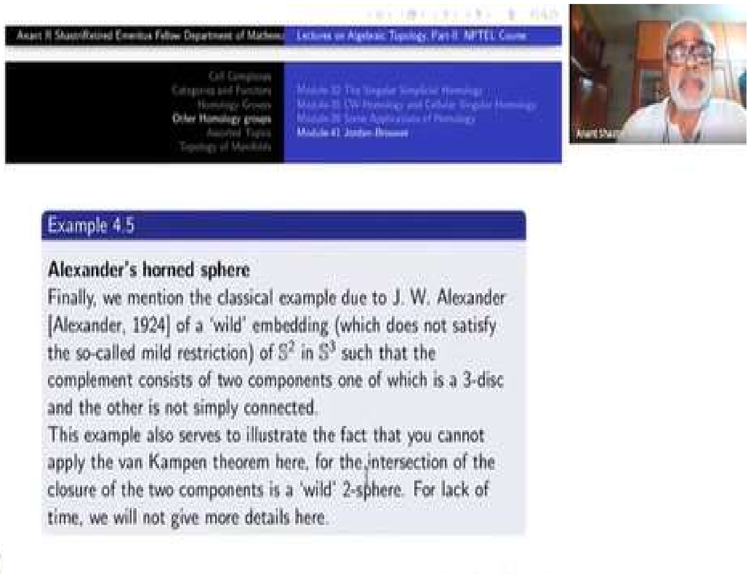
(Refer Slide Time: 27:53)

So, finally for the case, $k = n - 1$, let me give you a little more information on this one. For $n = 2$, this is Jordan Brouwer theorem. Here we have a very strong theorem. From complex analysis. Namely, Jordan-Schoenflies theorem (which is an extension of Riemann mapping theorem) says that any two embeddings of a circle inside $\mathbb{R}^2$ are equivalent. In other words, if you have any simple closed curve in the plane, you can bring it to the standard circle by an isotopy.

With a little more extra assumptions, the same thing can be done in all other dimensions. First it was Mazur, then by Morse and then further by Brown, each time weaker and weaker assumptions. But still there is some assumption I do not want to go into the technicalities of that. You can read this in Bredon's book or one of Brown's papers I have listed. Brown's paper is very readable. So, you can look into that one.

The last thing that I wanted to tell you is that suppose you remove the extra assumptions, namely, take any topological embedding of $\mathbb{S}^2$ inside $\mathbb{S}^3$. What can you say? That was cracked by Alexander long back in a negative way, constructing an example. It is now called Alexander's horned sphere.

(Refer Slide Time: 29:42)



Obviously, the embedding is somewhat wild. So, he is going to put `solid horns' to the $\mathbb{D}^3$ and what happens is it is actually an embedding of $\mathbb{D}^3$, and therefore, the boundary is obviously a 2-sphere. But the complement is a complicated space, whereas the complement of the standard disc in the sphere is also a disc. So, Alexander is are able to show that the complement of his embedded disc is not simply connected and therefore it cannot be isotopic to the standard embedding. So, for lack of time I will not be able to present that one.

(Refer Slide Time: 30:40)

But I will just give you the original picture of Alexander's horned sphere. So, this is supposed to be the disc, if you cut it away here it is just a twisted disc with a two horns. This is a big horn, this is another horn. Each of them grow two smaller horns again like stags horns. And then all the four of them will have two horns each. They all tend to get interlinked.

As they keep growing, they are coming closer and closer. In the limiting case, there will be cantor set of points wherein they meet finally. So, one can show that the limiting space is homeomorphic to disc, so the boundary is a 2-sphere.

But the complement of this embedded disc is complicated. One can show that its fundamental group is very complicated. So, that is all I can tell you at this stage. Let us at a stop here. Thank you.