Introduction to Algebraic Topology (Part - II)

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Lecture - 43
Jordan-Brouwer

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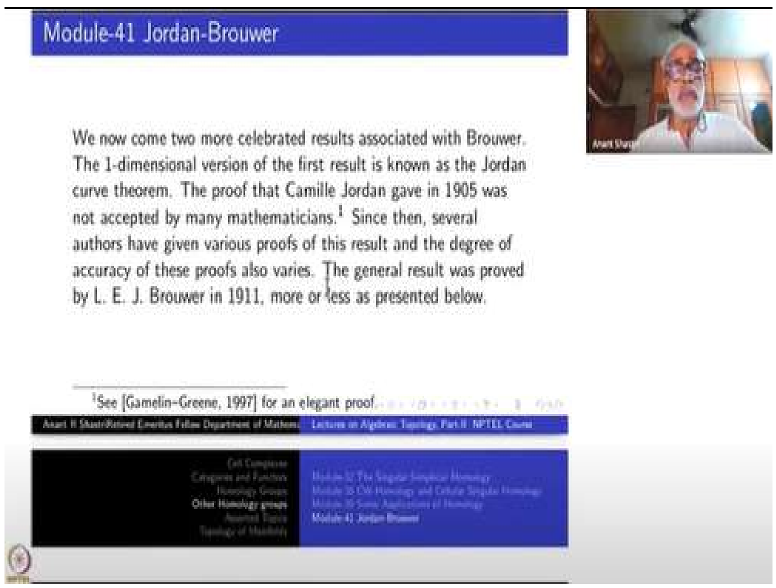

So, now we come to the celebrated results of Brouwer namely Jordan-Brouwer, separation theorem and then using that will prove the Brouwer invariance of domain also. I would like to tell you that in part one we had already proved the Brouwer invariance of domain directly using cellular simplicial approximation theorem and Sperner lemma. But we could not prove the Jordan Brouwer separation theorem there.

So, here we are proving the Jordan Brouwer separation theorem first then using that we will prove the Brouwer invariance of domain. So, this is completely a different approach and historically, Brouwer, more or less came with these ideas of developing homology just to prove this invariance of domain theorem one might say. So, the basic idea of the proof that we are going to present is more or less as in Brouwer's original paper of 1911.

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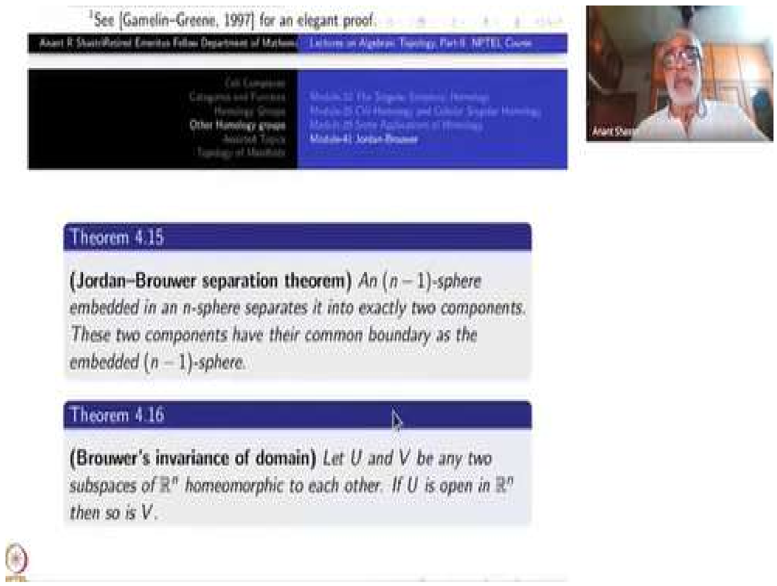

So, let me state them together because they go hand in hand in some sense, though the statements are quite different. So, Jordan Brouwer separation theorem is an extension of Jordan curve theorem which is for a topologically embedded circle in the complex plane, $\mathbb{R}^2$. Such a thing is called a Jordan curve, a one-to-one continuous mapping of $\mathbb{S}^1$ into $\mathbb{R}^2$.

So, this theorem was a sensation at that time because after the formal definition of a continuous function was accepted, lots of weird examples were constructed by many mathematicians. Examples like Anthony's necklace, Piano's space filling curves which were continuous functions on closed intervals but filling up $\mathbb{I}^2$ or $\mathbb{I}^3$ etc and even the countable product $\mathbb{I}^\infty$, were found.

And there are many other kinds of weird examples. Nowhere differentiable, continuous functions and so on. So, this positive result came as a pleasant surprise. But Jordan's proof was rejected by contemporaries. Later on, it was Veblen who gave a proof that was accepted. After that, several people have given several proofs and so-called proofs. Why is it `so called'? Most of them are wrong.

Here we are not proving that special case separately, we are going to prove directly the n-dimensional version of that, namely an $(n-1)$ sphere embedded in the n-sphere separates it into exactly two components. These two components have their common boundary as the embedded $(n-1)$-sphere, so this is the statement of Jordan Brouwer separation theorem.

The Brower invariance of domain says that if U and V are any two subspaces of $\mathbb{R}^n$, homeomorphic to each other and if U is open, then V is also open. So, that is the meaning of invariance of domain, homeomorphic invariance of domain. One is open then the other must be open. They must be subspaces of the same $\mathbb{R}^n$, to begin with.

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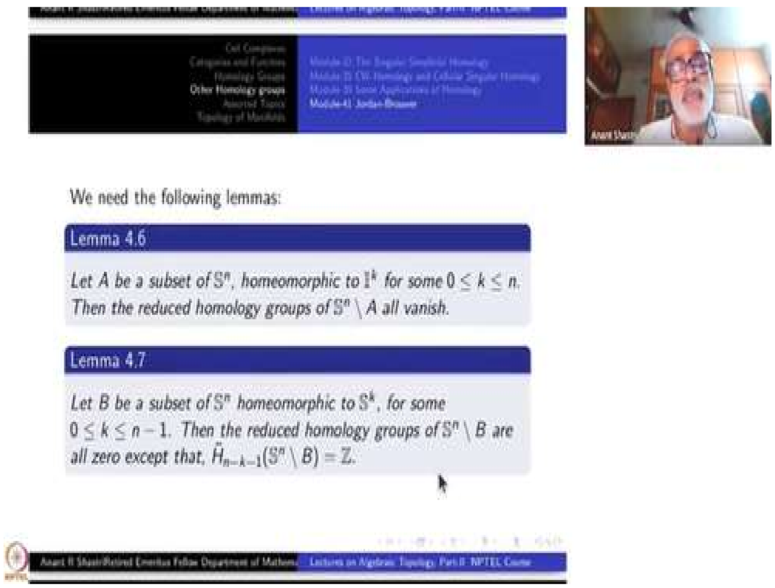

So, toward the proof of them we will state two lemmas which themselves give you a better picture of what is going on. Sometimes the lemmas can be quoted and used elsewhere, than the theorems. So, these are quite interesting lemmas.

So, the first lemma 4.6: Let A be a subset of $\mathbb{S}^n$, which is homeomorphic to $\mathbb{I}^k$, the closed interval cross itself k times. (Instead of the sphere it is a disc that we are studying first. I am taking the square model here, rather than the round model of the disc $\mathbb{D}^k$).

The integer k is between 0 and n, like you can take a point viz, $\mathbb{I}^0$ or you can take an arc when $k = 1$ or you can take a square when $k = 2$ and so on. The subset A need not be a rectangle even or polygon etc). Then the reduced homology groups of $\mathbb{S}^n \setminus A$ all vanish. For instance when $k = 0$, you know it; if you throw away one point from $\mathbb{S}^n$, the result is contractible.

So, reduced homology vanishes. So, this is a far reaching generalization of that phenomenon. For example, in the general case, we do not know whether $\mathbb{S}^n \setminus A$ is contractible or not. I can say that all the reduced homology groups are trivial, no matter A is provided it is homeomorphic to $\mathbb{I}^k$. So, I would say that this lemma itself is quite useful.

So, the next lemma 4.7: Suppose B is a subset of $\mathbb{S}^n$ homeomorphic to $\mathbb{S}^k$, for some k between 0 and n–1. Then the reduced homology groups of $\mathbb{S}^n \setminus B$ are all 0 except in one single dimension viz., $H_{n-k-1}(\mathbb{S}^n B) = \mathbb{Z}$. So, this lemma is also of importance in many theoretical problems. So, what we are going to do is first we assume these two lemmas and complete a proof of these two theorems. Then we will prove 4.7 first which is easier though the statement is a little more complicated. The crux of matter viz., lemma 4.6 will be taken at last. That is the plan. So, granting 4.6 and 4.7, we shall prove theorems 4.15 and 4.16. So 4.15 first.

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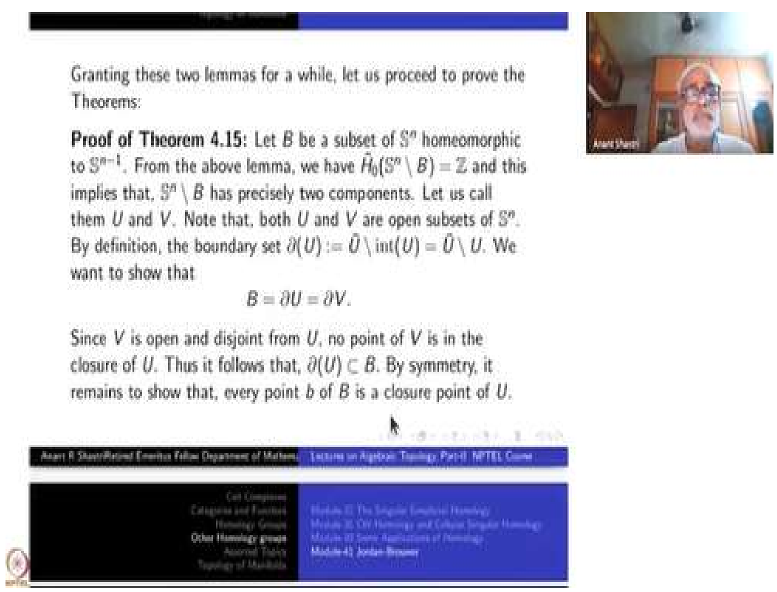

Namely, Jordan Brouwer separation theorem. Let B be a subset of $\mathbb{S}^n$ and homeomorphic to$\mathbb{S}^{n-1}$. We have to show that $\mathbb{S}^n \setminus B$ has exactly two connected components to begin with. Being open subsets of $\mathbb{S}^n$, they will be also path connected. Call them U and V. We have to show that boundary of U must be equal to boundary of V must be equal to B. These are two things that we have to show.

Now look at the above lemma. What is this lemma? Lemma is B is homeomorphic to $\mathbb{S}^k$, where $k = n–1$. So, it follows that $\tilde{H}_{n-1}$, i.e., $\tilde{H}(\mathbb{S}^n \setminus B)$ is isomorphic to $\mathbb{Z}$, which is the same thing as saying that the space has exactly has two connected components. That is very easy.

Let us call these two components U and V. Now use the fact that both U and V are open subsets. By definition, boundary of U is $\bar{U} \setminus \text{interior } (U)$, but interior of U is U itself and hence boundary of U is $\bar{U} \setminus U$. Similarly, boundary of V is $\bar{V} \setminus V$. We have to show that boundary of U is equal to boundary of V equal to B. This is what we have to show. Note that V is open and disjoint from U , $U \cup V \cup B$ is equal to the whole space $\mathbb{S}^n$. That is very clear. So, no point of V is in the closure U because U is disjoint from V and V is open. Take any point of V, V itself is a neighbourhood that and it does not intersect U. Therefore, it is not a boundary point. Therefore, boundary of U is contained inside B, it is not inside U because by definition the boundary of U is $\bar{U} \setminus U$. So, there is a chance that this point may be inside V, if it is not there also the rest of the thing is just B. This is purely set theoretic. So, the boundary of U is inside B, one way out. So, now we have to show that B is inside the boundary of U, i.e., every point of B is a closure point of U, then we are done. By symmetry the same will hold for V as well.

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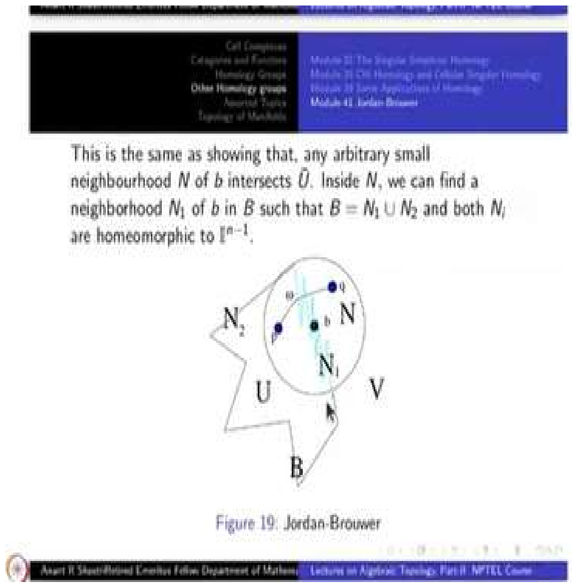

This is the same as showing that any arbitrary small neighbourhood N of $b \in B$ take a point intersects $\bar{U}$. So, inside N, we can find a neighbourhood N_1 of $b \in B$ such that B is actually a union of N_1 and N_2 and both N_i are homeomorphic to $\mathbb{I}^{n-1}$. Here we are using the fact that B is homeomorphic to $\mathbb{S}^{n-1}$. In other words, what I am doing is that I am taking N_1 to be a small nice disc around this point, so that its complement in B is also a disc. Inside a sphere, you can take a small disc, so that the complement will also be a homeomorphic disc.

So, B is a union of N_1 and N_2 and both the N_i are homoeomorphic to a disc of the same dimension equal to $n-1$. I am using the square model here $\mathbb{I}^{n-1}$. Here is a picture. In this picture, after all, $n = 2$ and B is a copy of a circle, and we are working inside a 2-sphere. This picture is for curve inside $\mathbb{R}^2$. This green part is N_1 and I have taken a point here b on the curve, B, this is B. I want to show that every neighbourhood of N of b will intersects $\bar{U}$. Because in the picture it is obvious. That can be the problem, you should not use a picture to conclude something. Use only the hypotheses so far and the results you have proved. Keep track of that. That is all.

So, there is a neighbourhood of a point $b \in B$, I would show that this neighbourhood intersects $\bar{U}$. So, what I have chosen is, this N_1 is homeomorphic to $\mathbb{I}^{k-1}$, its complement in B is also homoeomorphic to $\mathbb{I}^{n-1}$.

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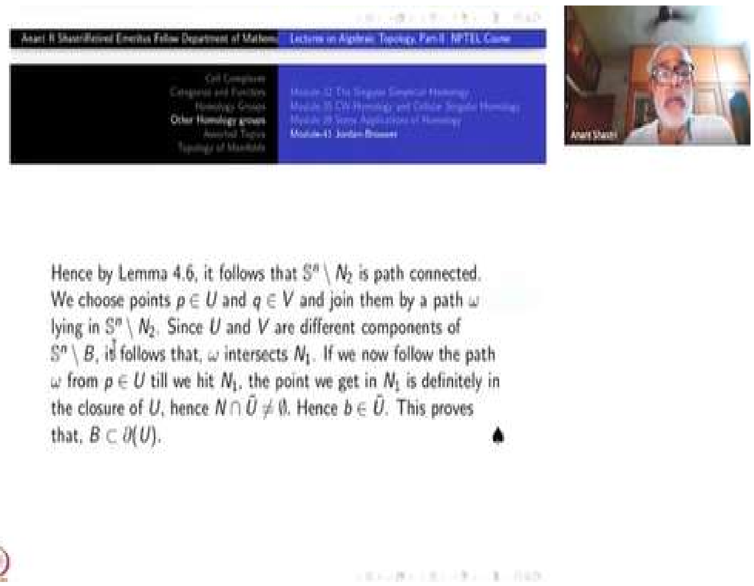

Put $k = n - 1$ in the previous lemma 4.6. It says that $\mathbb{S}^n \setminus N_2$ has its reduced homology 0. In particular, H_0 of that $\mathbb{Z}$ and so $\mathbb{S}^n \setminus N_2$ is path connected. That is from the first lemma. So, we take two points, p belonging to U and q belonging to V.

One is in here and other is there both are out of N_2. So, join them by a path in $\mathbb{S}^n \setminus N_2$. But U and V are different components of $\mathbb{S}^n \setminus B$. Therefore, you cannot join them in $\mathbb{S}^n \setminus B$. Therefore the path must intersect B and hence it must intersect N_1. Let us say that we have parameterized the path from p to q, by ω defined on $[0, 1]$ so that $\omega(0) = p$ and $\omega(1) = q$. There will be a first t or the smallest t for which $\omega(t)$ belongs to N_1, this green part. That point $\omega(t)$, whatever the point is, is obviously in the closure of U, because the entire $\omega[0, t)$ is in U.

What does that mean? We have chosen N_1 inside N. Therefore, $\omega(t)$ belongs to $\bar{U} \cap N$. Every neighbourhood N of every point in B intersects $\bar{U}$. That is what we have shown. So, that completes a proof of Jordan Brouwer separation theorem.

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Now we will come to Brouwer's invariance of domain theorem. So, U and V are subspaces of $\mathbb{R}^n$. By taking one-point compactification, we can work inside $\mathbb{S}^n$. If U(or V) is open in $\mathbb{R}^n$ iff U (resp. V) is open in $\mathbb{S}^n$. The assumption is that U and V are homeomorphic. So, let us fix a homeomorphism phi from U to V.

For definiteness, assume that U is open. I have to show that V is open. So take a point inside V which I can always write as $y = \phi(x)$, where x itself is inside U. What do I have to do? I must produce an open subset in $\mathbb{S}^n$ actually, which is contained inside V and contains the point $y = \phi(x)$. If this is true for every x, then I have completed the proof that V is open. So, we should produce an open subset of $\mathbb{S}^n$ containing y and contained in V.

So, now Jordan Brouwer separation theorem comes to help here in a miraculous way, in proving this great theorem. So, let A be a neighbourhood of x which is homeomorphic to the closed disc $\mathbb{D}^n$ and contained inside U. That is possible because U is open in $\mathbb{S}^n$.

Then the boundary of A is homeomorphic to $\mathbb{S}^{n-1}$. If we apply ϕ which is a homeomorphism from all of U into V, ϕ(boundary of A)= B will be an embedding of $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$. Of course it is contained inside V also. By Jordan-Brouwer separation theorem $\mathbb{S}^n \setminus B$ has precisely two components and both components are obviously open subsets of $\mathbb{S}^n$.

Look at A minus boundary of A, that is homeomorphic to the open disc. (I have taken A to be homeomorphic to the closed disc.) It is connected and its image under ϕ is also connected. Therefore, ϕ(A - boundary of A)=$\phi(A) \setminus B$ is connected. By lemma 4.6, $\mathbb{S}^n \setminus \phi(A)$ is connected because $\phi(A)$ is a homeomorphic a disc. Hope you have got the picture so far.

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On the other hand, $\mathbb{S}^n \setminus B = \mathbb{S}^n \setminus \phi(A)$ disjoint union with $\phi(A) \setminus B$. Because $\phi(A)$ is closed, $\mathbb{S}^n \setminus \phi(A)$ is open. B is closed, of course so, $\phi(A) \setminus B$ is open in $\phi(A)$. Just from this, we cannot conclude that $\phi(A) \setminus B$ is open in $\mathbb{S}^n$.

However, these two sets are connected subsets. Therefore each must be contained in one of the two components of $\mathbb{S}^n \setminus B$. There are only two components here. Since the whole space is the union of these two connected subsets, they must be the two components. This is an elementary topological result that I am using. In particular, $\phi(A) \setminus B$ being a connected component of $\mathbb{S}^n \setminus B$ is open. (So, this would have been obvious if you think Jordan Brouwer theorem is obvious.)

So, $\phi(A) \setminus B$ is an open subset of $\mathbb{S}^n$ and also it contains the point $y = \phi(x)$ and it is a subset of V, because the entire map ϕ is from U to V. Thus, we have succeeded in producing an open subset around $\phi(x)$, contained in V. So, this proves that V is open.

Thus, using the two lemmas, we have completed the proof of two big theorems. So, the two lemmas, we will prove next time. Thank you.