

Introduction to Algebraic Topology (Part-II)
Prof. Anant R. Shastri
Department of Mathematics
Indian Institute of Technology-Bombay

Lecture-42
Applications of LFT

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Module-40 Applications of LFT

Remark 4.14
 We are now in a position to prove the famous hairy ball theorem, which has its origin in Differential Topology. Loosely speaking, this theorem says that one cannot comb one's hair without parting at least at one point. The precise statement, however, is the following:
 There is no nowhere vanishing smooth tangent vector field on S^{2n} .
 Observe that all odd dimensional spheres possess such vector fields. For example: $(x_1, x_2, \dots, x_{2n}) \rightarrow (x_1, -x_2, \dots, x_{2n-1}, -x_{2n})$ is one such.

We are discussing the applications of homology. And in particular last time we saw Lefschetz's Fixed point theorem today we will give you some applications of it. Recall that we have proved the hairy ball theorem by just using our computation of the degree of antipodal maps right? Here we will do it in a slightly different way using Lefschetz's fixed point theorem.

In any case, recall that the famous hairy ball theorem, which has its origin in different topology, says that you cannot comb your hair without parting at least at one point. The precise mathematical statement is that there is no nowhere vanishing smooth tangent vector field on S^{2n} , where S^{2n} denotes the even dimensional standard unit sphere in \mathbb{R}^{2n+1} .

However, you should also see that all odd dimensional spheres have plenty of such tangent vector fields. For example, you can directly write down a formula, namely, $x = (x_1, x_2, \dots, x_{2n})$, coordinates of an odd dimensional sphere, mapping to $f(x) = (x_1, -x_2, \dots, x_{2n-1}, -x_{2n})$. okay? Note that $f(x)$ is a unit vector and is orthogonal to x . And hence defines a vector tangent to the sphere at the point x . So, there are many other

possibilities also. But for even dimension, you cannot do that. That is the statement of this Hairy- Ball- Theorem. Okay?

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one such.

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The topological version of this result is clearly a stronger result:

Theorem 4.12

(Hairy ball theorem) *There is no continuous map $f : \mathbb{S}^{2n} \rightarrow \mathbb{S}^{2n}$ such that, for each $x \in \mathbb{S}^{2n}$, $f(x)$ is orthogonal to x .*

We shall actually prove the following even stronger result from which the above theorem follows easily.

Theorem 4.13

For any continuous map $f : \mathbb{S}^{2n} \rightarrow \mathbb{S}^{2n}$ there is a point $x \in \mathbb{S}^{2n}$ such that $f(x) = x$ or $f(x) = -x$.

The topological version removes the smoothness part. It says that there is no continuous map f from \mathbb{S}^{2n} to \mathbb{S}^{2n} such that for each point in \mathbb{S}^{2n} , $f(x)$ is orthogonal to x . Okay? So, we need not worry about tangent fields etc. here because this is not a smooth version. This is just a continuous version, which makes sense and is a stronger than the smooth version above. So, here is theorem that I am going to prove, namely, for any continuous function from an even dimensional sphere to itself, there is a point x in \mathbb{S}^{2n} such that either $f(x)$ is x or $f(x)$ is $-x$, which is clearly, stronger than the Hairy Ball Theorem. So, you see Brouwer fixed point theorem says every continuous function from \mathbb{D}^n to \mathbb{D}^n has a fixed point. And this is slightly away from that, namely, either f or $-f$ has a fixed point.

Unfortunately or otherwise, it is only for even dimensional spheres. For odd dimensional sphere we have seen that it is not true okay? Suppose the statement is not true. That means that there exists a continuous function f from \mathbb{S}^{2n} to \mathbb{S}^{2n} such that $f(x)$ is not equal to $\pm x$.

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Asant R Shastri, Retired Emeritus Fellow, Department of Mathematics, Lectures on Algebraic Topology, Part-II: NPTEL Course

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Other Homology groups
Associated Topics
Topology of Manifolds

Module-32: The Singular Homotopy Homology
Module-33: CW-Homology and Cellular Singular Homology
Module-39: Some Applications of Homology
Module-41: Jordan-Brouwer

Proof: Suppose the statement is not true. It follows that the line joining x and $f(x)$ is well defined and does not pass through the origin. Therefore, the map $H : \mathbb{S}^{2n} \times \mathbb{I} \rightarrow \mathbb{S}^{2n}$:

$$H(x, t) := \frac{(1-t)f(x) + tx}{\|(1-t)f(x) + tx\|}$$

is well defined and continuous. Clearly H is a homotopy from f to the identity map Id of \mathbb{S}^{2n} . Hence, f induces identity homomorphism on the homology. In particular, it follows that, $L(f) = L(Id) = \chi(\mathbb{S}^{2n}) = 2$. Hence, by *LFT*, it follows that, f must have a fixed point contradicting our assumption. ♣

So, x and $f(x)$ are always distinct and not antipodal. Therefore, the line joining them will not pass through the origin. Is that clear? whenever you have distinct two points there is a unique line segment joining them. Because $f(x)$ is not equal to $-x$ also, okay, this segment will not pass through the origin. That means, our entire line segment consists of nonzero elements. If you write $(1-t)f(x) + tx$, where t lies between 0 and 1, this will give precisely all points of the line segment. They are all nonzero vectors in \mathbb{R}^{2n+1} . So, I can divide by the norm to get a map into \mathbb{S}^{2n} itself. With this definition, you get a continuous function H from $\mathbb{S}^{2n} \times \mathbb{I}$ to \mathbb{S}^{2n} . For $t = 0$, what is this? Put $t = 0$, it is $f(x)/\|f(x)\|$. But $\|f(x)\|$ is already 1. So, it is $f(x)$. Similarly when $t = 1$, this would be x . So, $x/\|x\|$ is again x .

So, H is a homotopy of f with the identity map. So, every such map must be homotopic to the identity map. okay? That means that the Lefschetz number $L(f)$ is equal to the Lefschetz number of the identity map, which is the Euler characteristic of \mathbb{S}^{2n} . Euler characteristic of \mathbb{S}^{2n} is very easy to compute. $H_0(\mathbb{S}^{2n}) = \mathbb{Z}$ and $H_n(\mathbb{S}^{2n}) = \mathbb{Z}$ and the rest of the H_i are zero. So, we get $\chi(\mathbb{S}^{2n}) = (-1)^0 + (-1)^2 n = 1 + 1 = 2$. Therefore, the fixed point theorem says that f must have a fixed point. But by our assumption, there is no fixed point. So, that is a contradiction.

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Let us now give another application of LFT. Recall that for $n \geq 1$, $H_n(\mathbb{S}^n) \approx \mathbb{Z}$. Therefore for any continuous map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ the homomorphism $f_* : H_n(\mathbb{S}^n) \rightarrow H_n(\mathbb{S}^n)$ is given by multiplication by an integer. This integer is called the **degree** of f . See definition 3.15.) Also $H_0(\mathbb{S}^n) \approx \mathbb{Z}$ and $f_* : H_0(\mathbb{S}^n) \rightarrow H_0(\mathbb{S}^n)$ is always the identity homomorphism. Since all other homology groups vanish it follows that

$$L(f) = 1 + (-1)^n \deg f.$$

As a special case, consider the antipodal map $\alpha : \mathbb{S}^n \rightarrow \mathbb{S}^n$ given by $x \mapsto -x$. Since this has no fixed points, it follows that $L(\alpha) = 0$. Therefore we have,

Next, recall that we have computed the degree of the antipodal map on the spheres right? We can do that in a slightly more general situation also here okay? Let us give another application of LFT. Fix $n \geq 1$. Recall that we have $H_n(\mathbb{S}^n) = \mathbb{Z}$. We have done this we have done it earlier. Therefore, for any continuous function from \mathbb{S}^n itself, the induced homomorphism on n -th homology, a endomorphism of \mathbb{Z} , is given by multiplication by an integer. This integer is called the degree of f , right? This was the latest definition of degree. There are several definitions. So, some of them I have asked you to check whether they are equal or not okay.

Also, $H_0(\mathbb{S}^n) = \mathbb{Z}$. Indeed, for recall that any continuous function from a path connected space to itself, the induced homomorphism on H_0 is always identity map. Therefore, the trace there will be exactly 1. okay? Since all other homology groups vanish, it follows that $L(f) = 1 + (-1)^n \text{trace } f_*(H_n)$, which is nothing but the the degree of f . So, $L(f) = 1 + (-1)^n \text{degree } f$.

So, this formula can be used to compute $L(f)$ if you know the degree. Or you can compute the degree of f if you know $L(f)$. So you can use this in either direction. okay? For instance consider the case when f map from \mathbb{S}^n to \mathbb{S}^n . This has no fixed points, okay? It follows that $L(f) = 0$. Okay? If you take antipodal map of of an odd degree n , okay, antipodal map of no fixed points in any case okay. So, $L(f)$ must be 0. So, that means for $L(\alpha)$ must be 0. Therefore, the degree will be equal to $(-1)^{n+1}$.

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Theorem 4.14
 Let $f : S^n \rightarrow S^n$ be any map with no fixed point. Then $\deg f = (-1)^{n+1}$. In particular, the degree of the antipodal map on S^n is equal to $(-1)^{n+1}$ for all $n \geq 1$.

So, this argument is applicable to any map without a fixed point, why only antipodal? So, this is the extra thing that we get. (There was no way to do this kind of things without the fixed point theorem).

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Remark 4.15
 More generally, if $f : S^{2n} \rightarrow S^{2n}$ is a homeomorphism, then f_* is an isomorphism and hence $\deg f = \pm 1$. If in addition, f has no fixed points then $L(f) = 0$ and hence $\deg f = -1$. So, we get the following corollary:

So, more generally, if you take a homeomorphism from S^{2n} to S^{2n} , then f_* is an isomorphism on the homology. Therefore the degree must be ± 1 , because under an isomorphism of \mathbb{Z} , the generator must go to plus or minus of the generator. So, degree with ± 1 . If, in addition, f has no fixture points then $L(f)$ is 0 and hence degree must be -1 . Therefore, we have another corollary here for you. See we have not done much deeper mathematics here, but we are just seeing the different sides of the same coin and deriving different results.

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Corollary 4.3

Let G be a group of odd order acting on S^{2n} through homeomorphisms. Then for each $g \in G$, there exists $v \in S^{2n}$ such that $gv = v$.

Let G be a group of odd order acting on S^{2n} , okay? For any (continuous) group action on a spaces X , automatically, the left translations L_g taking x to gx will define a homeomorphism of X , for each $g \in G$.

So, for each $g \in G$, there exists a v belong to S^{2n} such that $gv = v$. So, you cannot have a fixed-point-free action of an odd order group on an even dimensional sphere. This negative result is the starting point of a big theory anyway okay? We are just touching it and leave at that. We state it: G be a group of odd order acting continuously on S^{2n} . Then it must have fixed point. Okay?

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Proof: For each $g \in G$, the map $v \mapsto gv$ defines a homeomorphism $L_g : S^{2n} \rightarrow S^{2n}$. If G is of odd order $2k+1$, it follows that $(L_g)^{2k+1} = \text{Id}$. This means that $\deg(L_g) = 1$. The conclusion follows.

How do you prove that. Take g belongs G , v go going to gv defines a homeomorphism L_g from S^{2n} to S^{2n} , okay? Note that $L_{g^{-1}}$ is the inverse of L_g . Now, the degree of a map has the

property that $\deg(f \circ g) = \deg(f) \deg(g)$. Therefore we have $1 = (\deg(L_g))^{2k+1}$, the odd power. Therefore degree of L_g must be 1. Therefore, the Lefschetz number $L(L_g) = 2$. This means L_g has a fixed point.

In the remaining time, we will take up one of the postponed proofs which is very easy and would not take much time.

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The equivalence of CW-homology and cellular homology, okay? Cellular singular homology okay. For a CW complex, the CW homology itself is equivalent to the singular homology, we shall see later. So, we have also introduced a cellular singular homology in between these two. So, I want to say that that is also equivalent to the singular homology. So, this is the statement of 4.10. I have just stated here what it is; the basic idea here will be used elsewhere also, for instance, in the proof of the homotopy invariance. Here it is much, much simpler. There we have more elaborate structure.

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Lemma 4.5

(Retraction operator) Let C_* be a subcomplex of $S(X, A)$ (freely) generated by some singular simplexes. Assume that to each singular simplex $\sigma : \Delta_q \rightarrow X$ there exists a singular prism (i.e., a continuous map) $P\sigma : \mathbb{I} \times \Delta_q \rightarrow X$ with the following properties:

- (a) $P\sigma(0, z) = \sigma(z)$;
- (b) $P\sigma|_{\mathbb{I} \times \Delta_q} \in C_*$;
- (c) $P(\sigma \circ F^i) = P\sigma \circ (1 \times F^i)$, for each face operator F^i ;
- (d) if $\sigma \in C_*$, then $P\sigma(t, z) = \sigma(z)$; and
- (e) if $\sigma \in S(A)$ then $P\sigma(\mathbb{I} \times \Delta_q) \subset A$.

Then there is a chain deformation retraction $\tau : (S(X), S(A)) \rightarrow (C_*, S(A) \cap C_*)$. In particular, the inclusion $C_* \rightarrow S(X)$ induces isomorphism in the homology.

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So, this is called a retraction operator, I have stated it this way, so that you can quote it elsewhere. Suppose you want to study something later on, this will be very useful, elsewhere in algebraic topology. So, I have stated this step separately as a lemma instead of just proving the theorem directly. So, this lemma plays the key role in proving this one and this can be used to do something else also.

So, let C_* , (it is just a generic notation now, and not the simplicial chain complex of a simplicial complex as used earlier) be any sub chain complex of the singular chain complex $S_*(X, A)$, freely generated by some singular simplexes. Note that every submodule of $S_*(X, A)$ is free, but I am stating it very clearly, that it is really generated by some simplexes okay, the basis itself is a subset of the standard basis for $S_*(X, A)$.

Assume that to each singular simplex σ from Δ_q to X , (I am stating the whole thing here in an unrelated way, but relative version is got easily from this, okay?) So, take a singular simplex σ inside X , those are generators for $S_*(X)$, remember that. There exists a singular prism. Okay? What is the meaning of a prism here? It is map P_σ from $\mathbb{I} \times \Delta_q$ to X , So the domain is the prism, closed interval cross Δ_q . With the following properties okay? So, what are these properties? At the 0-th level P_σ is just σ . At the 1-th level, it is inside C_q , okay? C_* is a subcomplex of $S_*(X, A)$ okay, that is a hypothesis.

Next, $P_{\sigma \circ F^i} = P_\sigma \circ (Id \times F^i)$ from $I \times \Delta_{q-1}$ to X . That is composing with F^i , it is $P_\sigma \circ (1 \times F^i)$ Okay? So, at the top level for each face operate F^i , it should have this property.

Next if σ is already in C . then this must be just the identity, $P_\sigma(t, z) = \sigma(z)$ for all $t \in \mathbb{I}$, and all $z \in \Delta_q$, okay? Finally, if P_σ is in $S.(A)$ then $P_\sigma(\mathbb{I} \times \Delta_q)$ should be completely inside A . This condition is for the relative part.

So, such a thing will be called a retraction operator. Suppose you have such a retraction operator P . Okay? For each σ , a generator for S , P gives a homotopy to an element of C . okay? P is a retraction why? Because if σ is already inside C , the entire homotopy is identity. That is why the name retraction operator is justified. okay?

Conclusion is that then there is a chain deformation retraction τ from the pair $(S.(X), S.(A))$ to $(C, C \cap S.(A))$. In particular, the inclusion of map C to $S.(X)$, is a chain deformation attraction. That means it is retraction and there is a chain homotopy which is identity on C . In particular the inclusion map induces isomorphism in homology. So, this part, the last part is clear because deformation retraction has that property.

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$\tau : (S(X), S(A)) \rightarrow (C, S(A) \cap C)$. In particular, the inclusion $C \rightarrow S(X)$ induces isomorphism in the homology.

Asmit R Ghosh-Retired Emeritus Fellow Department of Mathematics Lectures on Algebraic Topology, Part II, NPTEL Course

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Asmit Ghosh

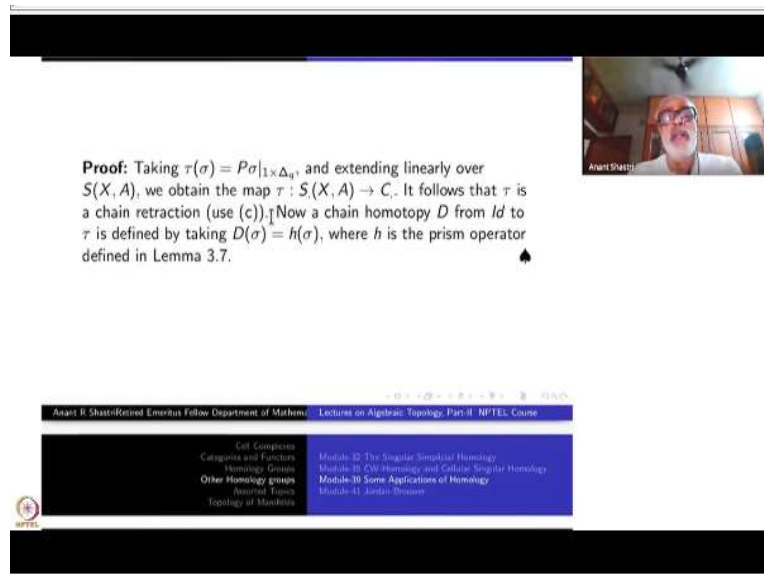
Proof: Taking $\tau(\sigma) = P\sigma|_{1 \times \Delta_q}$, and extending linearly over $S(X, A)$, we obtain the map $\tau : S(X, A) \rightarrow C$. It follows that τ is a chain retraction (use (c)). Now a chain homotopy D from Id to τ is defined by taking $D(\sigma) = h(\sigma)$, where h is the prism operator defined in Lemma 3.7.

So, the proof is very easy. All you have to do: taking $\tau(\sigma)(z) = P_\sigma(1, z)$, restricted to $1 \times \Delta_q$, the 1-th level, you have a function defined on the generating set of S . to a module C .

Therefore, you can extend it linearly over all of $S.(X)$ okay, to get a chain map τ from $S.(X)$ to C .

It follows that τ is a chain retraction, okay? All that you do is to appeal to part(c). So, you have to verify that τ commutes with the boundary operator, which is absolutely trivial verification. These hypothesis on P have been chosen for this purpose.

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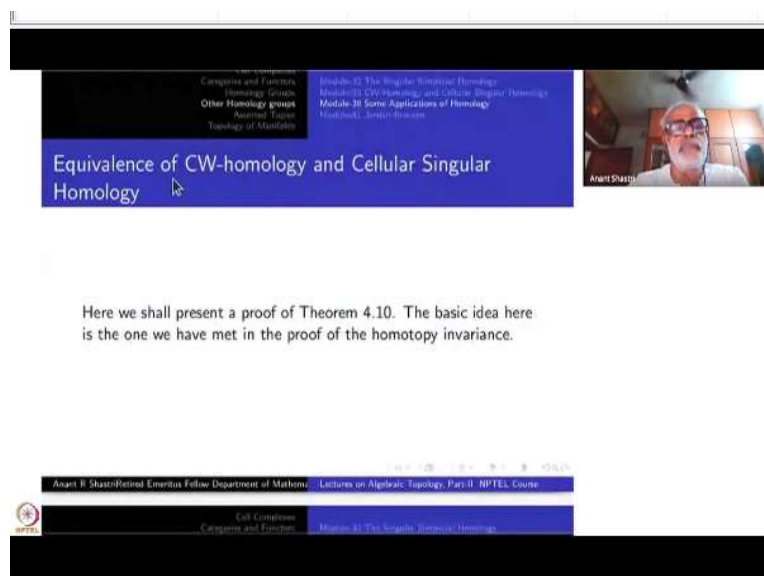
Proof: Taking $\tau(\sigma) = P\sigma|_{1 \times \Delta_q}$ and extending linearly over $S(X, A)$, we obtain the map $\tau : S(X, A) \rightarrow C$. It follows that τ is a chain retraction (use (c)). Now a chain homotopy D from Id to τ is defined by taking $D(\sigma) = h(\sigma)$, where h is the prism operator defined in Lemma 3.7.

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Now a chain homotopy D from identity map to τ is defined by $D(\sigma) = (P_\sigma).(h(\xi_q))$, where h is the prism operator which we have defined earlier okay. So, I will not go into this one now. So, when you come across with this one again we can explain that.

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Equivalence of CW-homology and Cellular Singular Homology

Here we shall present a proof of Theorem 4.10. The basic idea here is the one we have met in the proof of the homotopy invariance.

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Now, what we have to prove is the equivalence of these two homologies. All that I need to do is to construct such a retraction operator. Okay? where this C will be now taken as $C^{CW}(X, A)$ inside $S(X, A)$. Take this special case. So, that is what I will do. Okay.

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The screenshot shows a video lecture interface. At the top, there is a header bar with the text 'Anant B. Shrivastava Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course'. Below this is a navigation menu with items: 'Cell Complexes', 'Categories and Functors', 'Homology Groups', 'Other Homology groups', 'Algebraic Topology', and 'Topology of Manifolds'. The main content area displays 'Remark 4.16' in a blue box, followed by the text: 'The above lemma plays a crucial role in the proof of the Hurewicz theorem'. On the right side, there is a small video feed of the lecturer, Anant Shrivastava.

So, that is what I will do.

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The screenshot shows a video lecture interface. At the top, there is a header bar with the text 'Anant B. Shrivastava Retired Emeritus Fellow Department of Mathematics, Lectures on Algebraic Topology, Part-II, NPTEL Course'. Below this is a navigation menu with items: 'Cell Complexes', 'Categories and Functors', 'Homology Groups', 'Other Homology groups', 'Algebraic Topology', and 'Topology of Manifolds'. The main content area displays the text: 'Now the proof of Theorem 4.10 is completed by appealing to the cellular approximation theorem. Given any simplex $\sigma : \Delta_n \rightarrow X$, all we do is choose a cellular approximation to it and a homotopy of the original map with the approximation, which gives the prism $P\sigma$. Of course, if σ is already a cellular map then we take this homotopy itself to be the identity, viz., $P\sigma(z, t) = \sigma(z)$ '. On the right side, there is a small video feed of the lecturer, Anant Shrivastava.

I complete this one by appealing to the cellular approximation theorem. Given any continuous function σ from Δ_q into X , all we do is to choose a cellular approximation to it and the homotopy of the original map with the approximation, which gives the prism P_σ . A cellular approximation comes with a homotopy; that homotopy is P_σ .

Start with a σ , take a cellular approximation. For that what you have to do? Nothing, you do not have to do anything, there is cellular approximation theorem okay? There is a homotopy also, just take that homotopy of σ to the cellular function which is in C . Okay? And if σ is already a cellular map, do not do anything you have to choose P_σ to be the identity homotopy. Each P_σ as to be defined independently. There is no continuity argument here at all, because for each σ which is a generator, I have to do separately for that. Okay? So, what I do, if it is already cellular, I keep P as identity cross σ , that is all. So, we can just take P that equal to σz in that. So, this is automatically satisfied.

Next time, we will continue with the applications of this homology, namely, the big promise, Jordan-Brouwer separation theorem, Jordan-Brouwer invariance of domain and so on, thank you.