Introduction to Algebraic Topology (Part-II)
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Lecture-42
Applications of LFT

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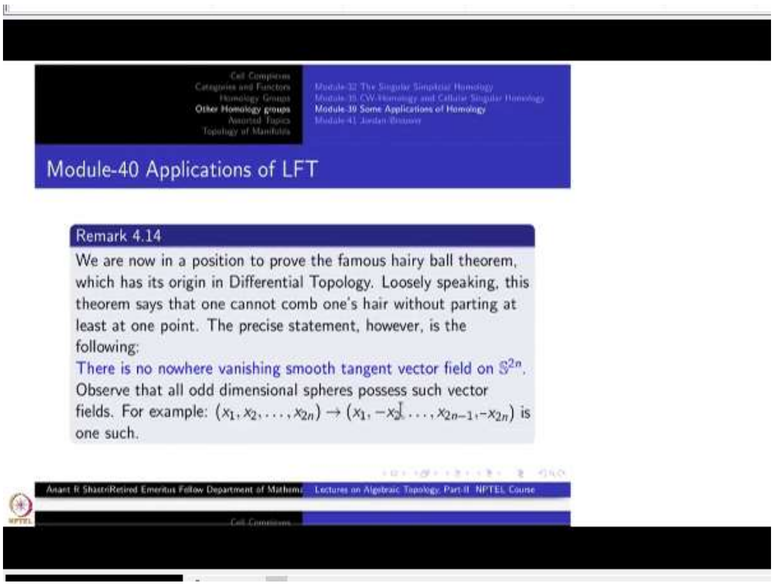

We are discussing the applications of homology. And in particular last time we saw Lefschetz's Fixed point theorem theorem today we will give you some applications of it. Recall that we have proved the hairy ball theorem by just using our computation of the degree of antipodal maps right? Here we will do it in a slightly different way using Lefschetz's fixed point theorem.

In any case, recall that the famous hairy ball theorem, which has its origin in different topology, says that you cannot comb your hair without parting at least at one point. The precise mathematical statement is that there is no nowhere vanishing smooth tangent vector field on $\mathbb{S}^{2n}$, where $\mathbb{S}^{2n}$ denotes the even dimensional standard unit sphere in $\mathbb{R}^{2n+1}$.

However, you should also see that all odd dimensional spheres have plenty of such tangent vector fields. For example, you can directly write down a formula, namely, $x = (x_1, x_2, \ldots, x_{2n})$, coordinates of an odd dimensional sphere, mapping to $f(x) = (x_1, -x_2, \ldots, x_{2n-1}, -x_{2n})$. okay? Note that $f(x)$ is a unit vector and is orthogonal to x. And hence defines a vector tangent to the sphere at the point x. So, there are many other possibilities also. But for even dimesion, you cannot do that. That is the statement of this Hairy- Ball- Theorem. Okay?

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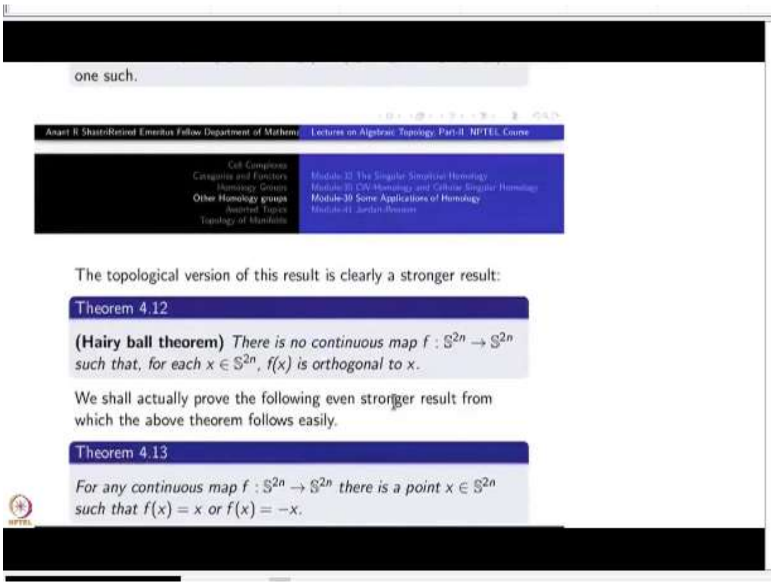

The topological version removes the smoothness part. It says that there is no continuous map f from $\mathbb{S}^{2n}$ to $\mathbb{S}^{2n}$ such that for each point in $\mathbb{S}^{2n}$, $f(x)$ is orthogonal to x. Okay? So, we need not worry about tangent fields etc. here because this is not a smooth version. This is just a continuous version, which makes sense and is a stronger than the smooth version above. So, here is theorem that I am going to prove, namely, for any continuous function from an even dimnsional sphere to itself, there is a point x in $\mathbb{S}^{2n}$ such that either $f(x)$ is x or $f(x)$ is $-x$, which is clearly, stronger than the Hairy Ball Theorem. So, you see Brouwer fixed point theorem says every continuous function from $\mathbb{D}^n$ to $\mathbb{D}^n$ has a fixed point. And this is slightly away from that, namely, either f or $-f$ has a fixed point.

Unfortunately or otherwise, it is only for even dimensional spheres. For odd dimensional sphere we have seen that it is not true okay? Suppose the statement is not true. That means that there exists a continuous function f from $\mathbb{S}^{2n}$ to $\mathbb{S}^{2n}$ such that $f(x)$ is not equal to $\pm x$.

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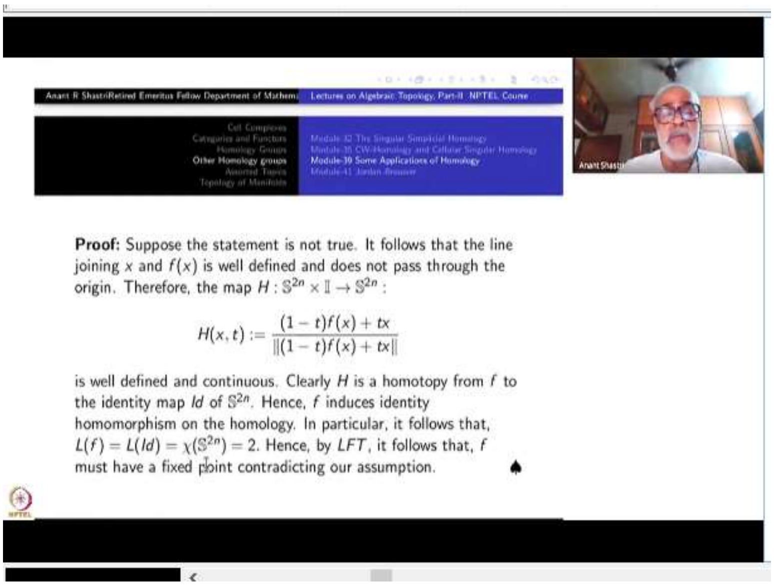

So, x and f(x) are always distinct and not antiphonal. Therefore, the line joining them will not pass through the origin. Is that clear? whenever you have distinct two points there is a unique line segment joining them. Because $f(x)$ is not equal to $-x$ also, okay, this segment will not pass through the origin. That means, our entire line segment consists of nonzero elements. If you write $(1-t)f(x)+tx$, where t lies between 0 and 1, this will give precisely all points of the line segment. They are all nonzero vectors in $\mathbb{R}^{2n+1}$. So, I can divided by the norm to get a map into $\mathbb{S}^{2n}$ itself. With this definitions, you get a continuous function H from $\mathbb{S}^{2n} \times \mathbb{I}$ to $\mathbb{S}^{2n}$. For $t=0$, what is this? Put $t=0$, it is $f(x)/\|f(x)\|$. But $\|f(x)\|$ is already 1. So, it is $f(x)$. Similarly when $t=1$, this would be x. So, $x/\|x\|$ is again x.

So, H is a homotopy of f with the identity map. So, every such map must be homotopic to the identity map. okay? That means that the Lefschetz number $L(f)$ is equal to the Lefschetz number of the identity map, which is the Euler characteristic of $\mathbb{S}^{2n}$. Euler characteristic of $\mathbb{S}^{2n}$ is very easy to compute. $H_0(\mathbb{S}^2 n) = \mathbb{Z}$ and $H_n(S^{2n}) = \mathbb{Z}$ and the rest of the H_i are zero. So, we get $\chi(\mathbb{S}^{2n}) = (-1)^0 + (-1)^2 n = 1 + 1 = 2$. Therefore, the fixed point theorem says that f must have a fixed point. But by our assumption, there is no fixed point. So, that is a contradiction.

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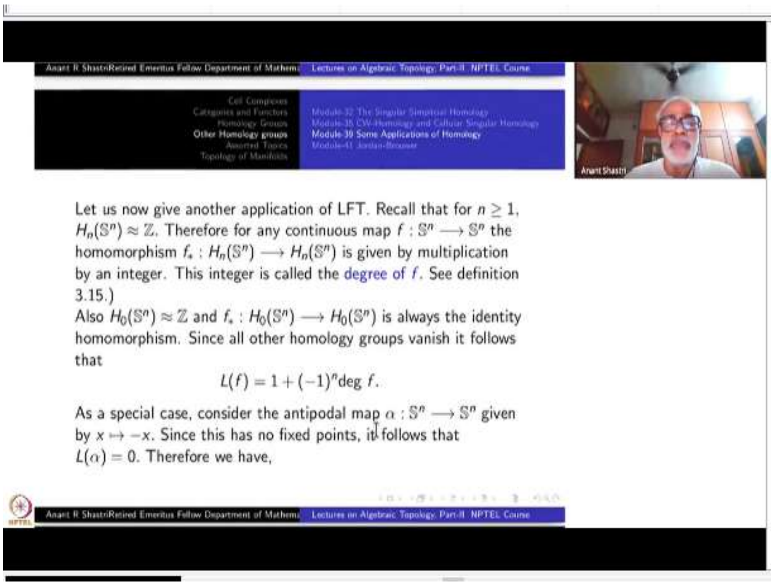

Next, recall that we have computed the degree of the antipodal map on the spheres right? We can do that in a slightly more general situation also here okay? Let us give another application of LFT. Fix $n \geq 1$. Recall that we have $H_n(\mathbb{S}^n) = \mathbb{Z}$. We have done this we have done it earlier. Therefore, for any continuous function from $\mathbb{S}^n$ itself, the induced homomorphism on n-th homology, a endomorphism of $\mathbb{Z}$, is given by multiplication by an integer. This integer is called the degree of f, right? This was the latest definition of degree. There are several definitions. So, some of them I have asked you to check whether they are equal or not okay.

Also, $H_0(\mathbb{S}^n) = \mathbb{Z}$. Indeed, for recall that any continuous function from a path connected space to itself, the induced homomorphism on H_0 is always identity map. Therefore, the trace there will be exactly 1. okay? Since all other homology groups vanish, it follows that $L(f) = 1 + (-1)^n \text{trace } f_*(H_n)$, which is nothing but the the degree of f. So, $L(f) = 1 + (-1)^n \text{degree } f$.

So, this formula can be used to compute $L(f)$ if you know the degree. Or you can compute the degree of f if you know $L(f)$. So you can use this in either direction. okay? For instance consider the case when f map from $\mathbb{S}^n$ to $\mathbb{S}^n$. This has no fixed points, okay? It follows that $L(f) = 0$. Okay? If you take antipodal map of of an odd degree n, okay, antipodal map of no fixed points in any case okay. So, $L(f)$ must be 0. So, that means for $L(\alpha)$ must be 0. Therefore, the degree will be equal to $(-1)^{n+1}$.

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So, this argument is applicable to any map without a fixed point, why only antipodal? So, this is the extra thing that we get. (There was no way to do this kind of things without the fixed point theorem).

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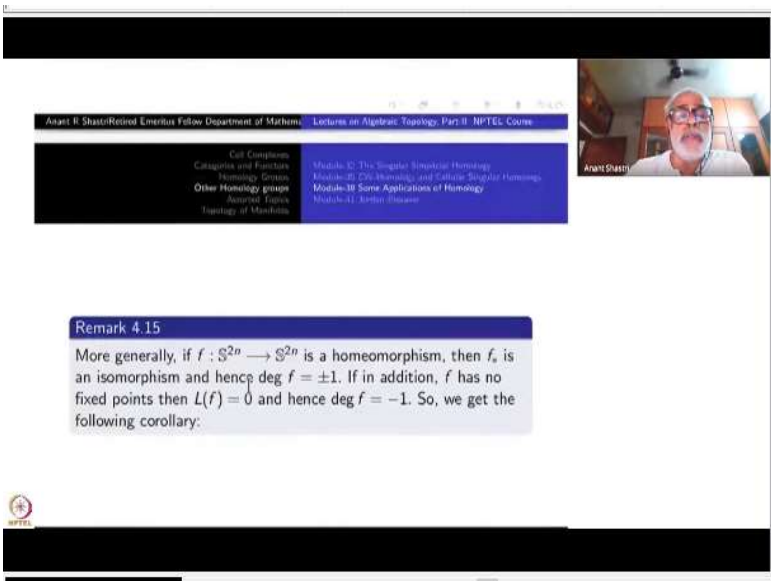

So, more generally, if you take a homeomorphism from $\mathbb{S}^{2n}$ to $\mathbb{S}^{2n}$, then f_* is an isomorphism on the homology. Therefore the degree must be ± 1, because under an isomorphism of $\mathbb{Z}$, the generator must go to plus or minus of the generator. So, degree with ± 1. If, in addition, f has no fixture points then $L(f)$ is 0 and hence degree must be -1. Therefore, we have another corollary here for you. See we have not done much deeper mathematics here, but we are just seeing the different sides of the same coin and deriving different results.

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Let G be a group of odd order acting on $\mathbb{S}^{2n}$, okay? For any (continuous) group action on a spaces X, automatically, the left translations L_g taking x to gx will define a homeomorphism of X, for each $g \in G$.

So, for each $g \in G$, there exists a v belong to $\mathbb{S}^{2n}$ such that $gv = v$. So, you cannot have a fixed-point-free action of an odd order group on an even dimensional sphere. This negative result is the starting point of a big theory anyway okay? We are just touching it and leave at that. We state it: G be a group of odd order acting continuously on $\mathbb{S}^{2n}$. Then it must have fixed point. Okay?

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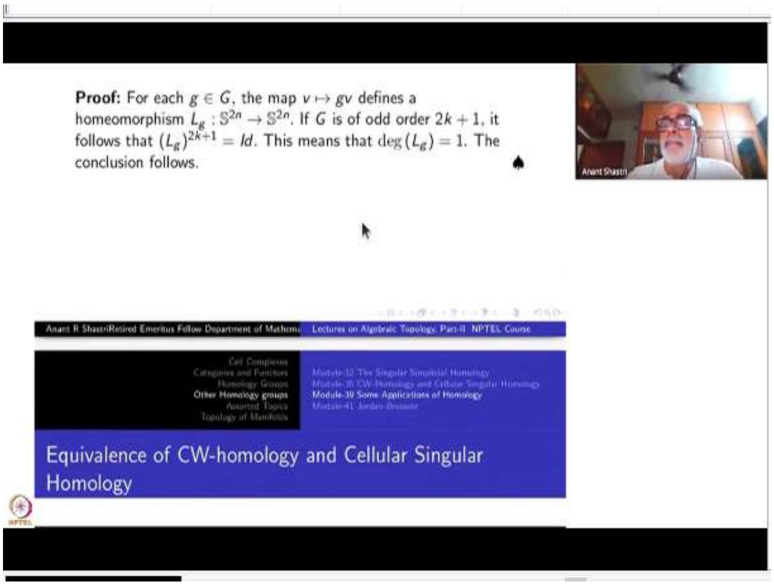

How do you prove that. Take g belongs G, v go going to gv defines a homeomorphism L_g from $\mathbb{S}^{2n}$ to $\mathbb{S}^{2n}$, okay? Note that $L_{g^{-1}}$ is the inverse of L_g. Now, the degree of a map has the property that $deg(f \circ g) = deg(f)\ deg(g)$. Therefore we have $1 = (deg(L_g))^{2k+1}$, the odd power. Therefore degree of L_g must be 1. Therefore, the Lefschetz number $L(L_g) = 2$. This means L_g has a fixed point.

In the remaining time, we will take up one of the postponed proofs which is very easy and would not take much time.

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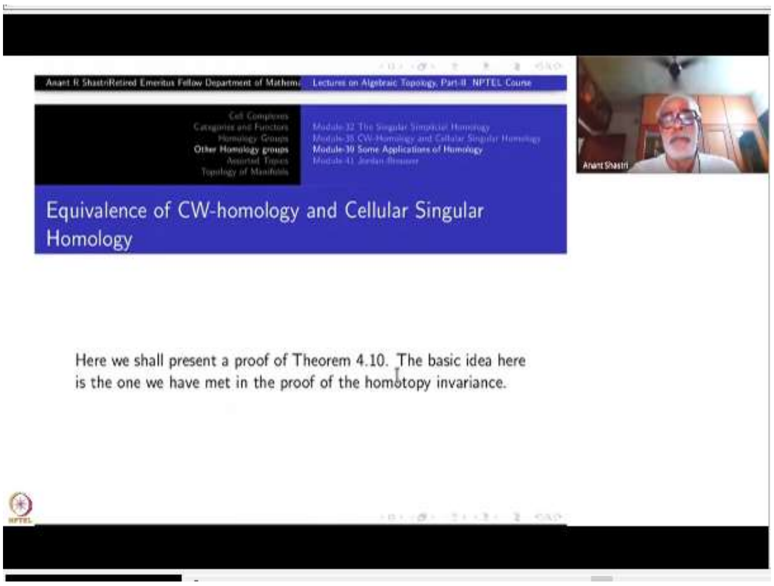

The equivalence of CW-homology and cellular homology, okay? Cellular singular homology okay. For a CW complex, the CW homology itself is equivalent to the singular homology, we shall see later. So, we have also introduced a cellular singular homology in between these two. So, I want to say that that is also equivalent to the singular homology. So, this is the statement of 4.10. I have just stated here what it is; the basic idea here will be used elsewhere also, for instance, in the proof of the homotopy invariance. Here it is much, much simpler. There we have more elaborate structure.

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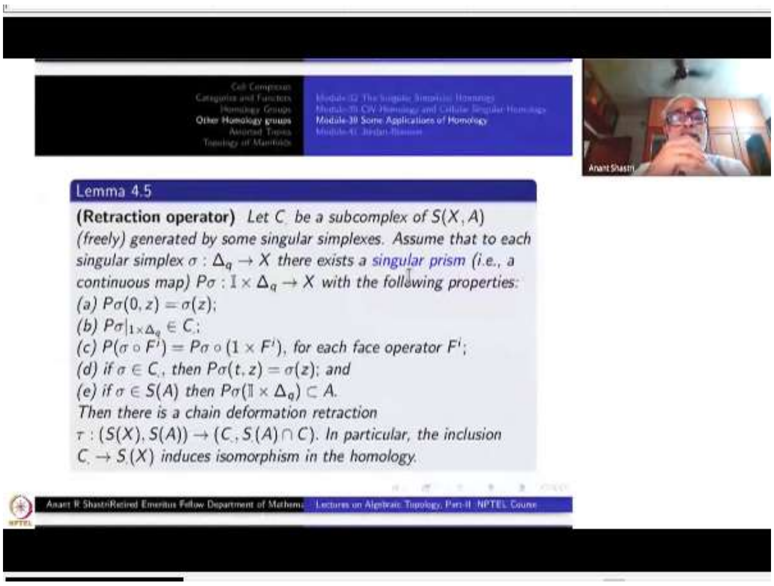

So, this is called a retraction operator, I have stated it this way, so that you can quote it elsewhere. Suppose you want to study something later on, this will be very useful, elsewhere in algebraic topology. So, I have stated this step separately as a lemma instead of just proving the theorem directly. So, this lemma plays the key role in proving this one and this can be used to do something else also.

So, let $C.$, (it is just a generic notation now, and not the simplicial chain complex of a simplicial complex as used earlier) be any sub chain complex of the singular chain complex $S.(X, A)$, freely generated by some singular simplexes. Note that every submodule of $S.(X, A)$ is free, but I am stating it very clearly, that it is really generated by some simplexes okay, the basis itself is a subset of the standard basis for $S.(X, A)$.

Assume that to each singular simplex σ from Δ_q to X, (I am stating the whole thing here in an unrelated way, but relative version is got easily from this, okay?) So, take a singular simplex sigma inside X, those are generators for $S.(X)$, remember that. There exists a singular prism. Okay? What is the meaning of a prism here? It is map P_σ from $\mathbb{I} \times \Delta_q$ to X, So the domain is the prism, closed interval cross Δ_q. With the following properties okay? So, what are these properties? At the 0-th level P_σ is just σ. At the 1-th level, it is inside C_q, okay? $C.$ is a subcomplex of $S.(X, A)$ okay, that is a hypothesis.

Next, $P_{\sigma \circ F^i} = P_\sigma \circ (Id \times F^i)$ from $I \times \Delta_{q-1}$ to X. That is composing with F^i, it is $P_\sigma \circ (1 \times F^i)$ Okay? So, at the top level for each face operate F^i, it should have this property.

Next if σ is already in $C.$ then this must be just the identity, $P_\sigma(t, z) = \sigma(z)$ for all $t \in \mathbb{I}$, and all $z \in \Delta_q$, okay? Finally, if P_σ is in $S.(A)$ then $P_\sigma(\mathbb{I} \times \Delta_q)$ should be completely inside A. This condition is for the relative part.

So, such a thing will be called a retraction operator. Suppose you have such a retraction operator P. Okay? For each σ, a generator for $S.$, P gives a homotopy to an element of $C.$ okay? P is a retraction why? Because if σ is already inside $C.$, the entire homotopy is identity. That is why the name retraction operator is justified. okay?

Conclusion is that then there is a chain deformation retraction tau from the pair $(S.(X), S.(A))$ to $(C., C. \cap S.(A))$. In particular, the inclusion of map $C.$ to $S.(X)$, is a chain deformation attraction. That means it is retraction and there is a chain homotopy which is identity on $C.$ In particular the inclusion map induces isomorphism in homology. So, this part, the last part is clear because deformation retraction has that property.

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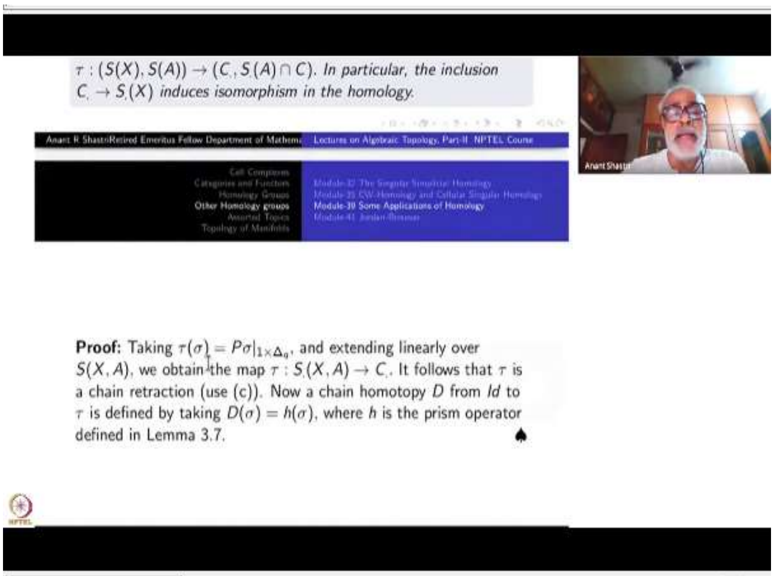

So, the proof is very easy. All you have to do: taking $\tau(\sigma)(z) = P_\sigma(1, z)$, restricted to $1 \times \Delta_q$, the 1-th level, you have a function defined on the generating set of $S.$ to a module $C.$.

Therefore, you can extend it linearly over all of $S.(X)$ okay, to get a chain map tau from $S.(X)$ to $C.$

It follows that τ is a chain retraction, okay? All that you do is to appeal to part(c). So, you have to verify that tau commutes with the boundary operator, which is absolutely trivial verification. These hypothesis on P have been chosen for this purpose.

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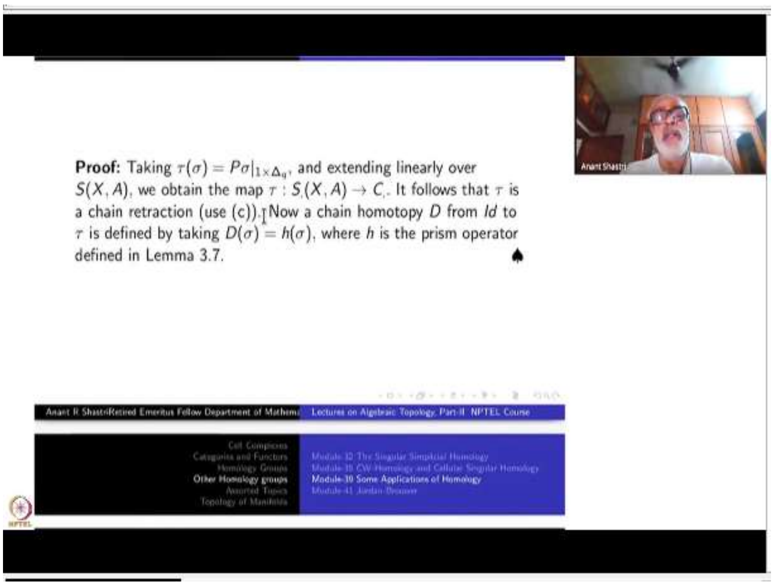

Now a chain homotopy D from identity map to τ is defined by $D(\sigma) = (P_\sigma).(h(\xi_q))$, where h is the prism operator which we have defined earlier okay. So, I will not go into this one now. So, when you come across with this one again we can explain that.

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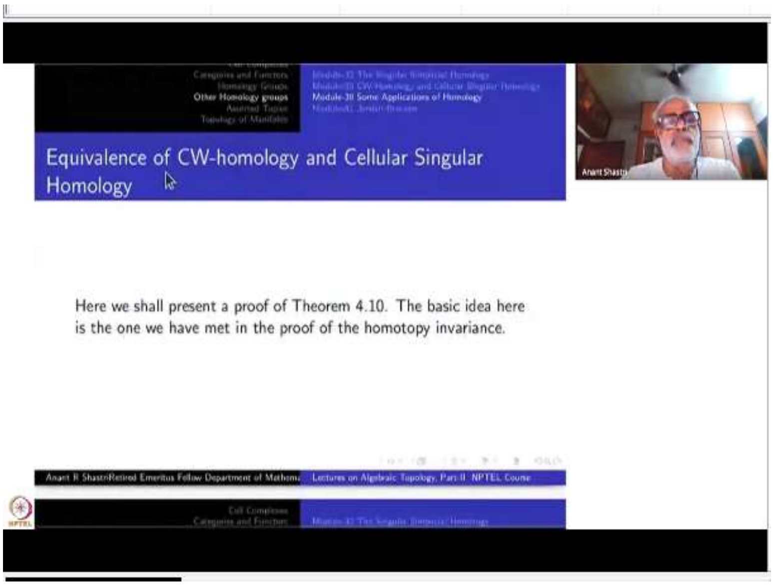

Now, what we have to prove is the equivalence of these two homologies. All that I need to do is to construct such a retraction operator. Okay? where this $C.$ will be now taken as $C_.^{CW}(X, A)$ inside $S.(X, A)$. Take this special case. So, that is what I will do. Okay.

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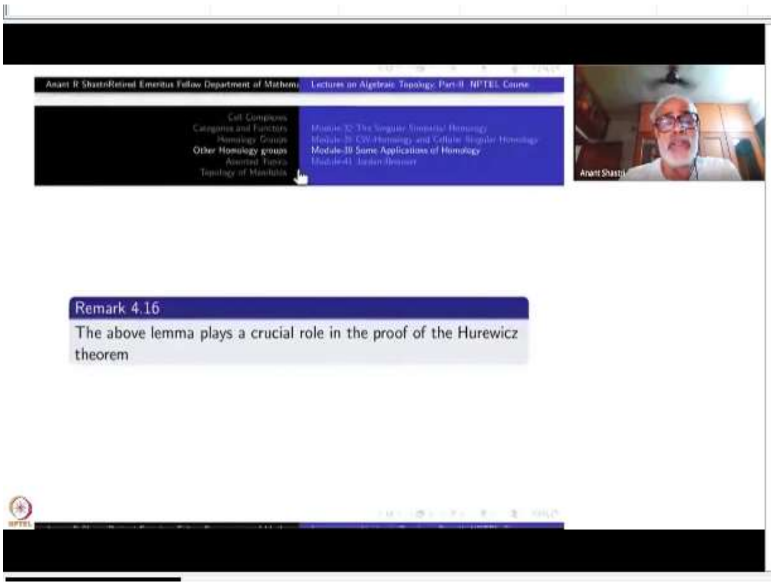

So, that is what I will do.

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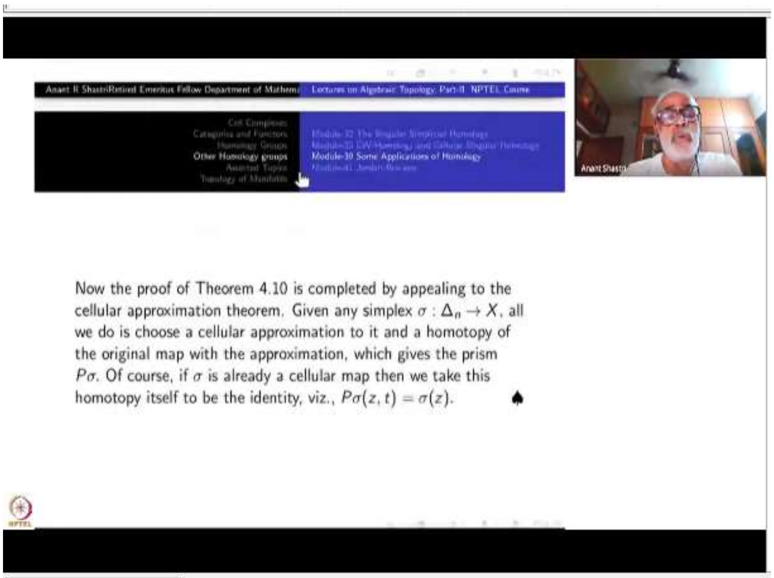

I complete this one by appealing to the cellular approximation theorem. Given any continuous function σ from Δ_q into X, all we do is to choose a cellular approximation to it and the homotopy of the original map with the approximation, which gives the prism P_σ. A cellular approximation comes with a homotopy; that homotopy is P_σ.

Start with a σ, take a cellular approximation. For that what you have to do? Nothing, you do not have to do anything, there is cellular approximation theorem okay? There is a homotopy also, just take that homotopy of σ to the cellular function which is in C. Okay? And if σ is already a cellular map, do not do anything you have to choose P_σ to be the identity homotopy. Each P_σ as to be defined independently. There is no continuity argument here at all, because for each σ which is a generator, I have to do separately for that. Okay? So, what I do, if it is already cellular, I keep P as identity cross σ, that is all. So, we can just take P that equal to σz in that. So, this is automatically satisfied.

Next time, we will continue with the applications of this homology, namely, the big promise, Jordan-Brouwer separation theorem, Jordan-Brouwer invariance of domain and so on, thank you.