Introduction to Algebraic Topology (Part-II)
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Lecture-41
Some Applications of Homology

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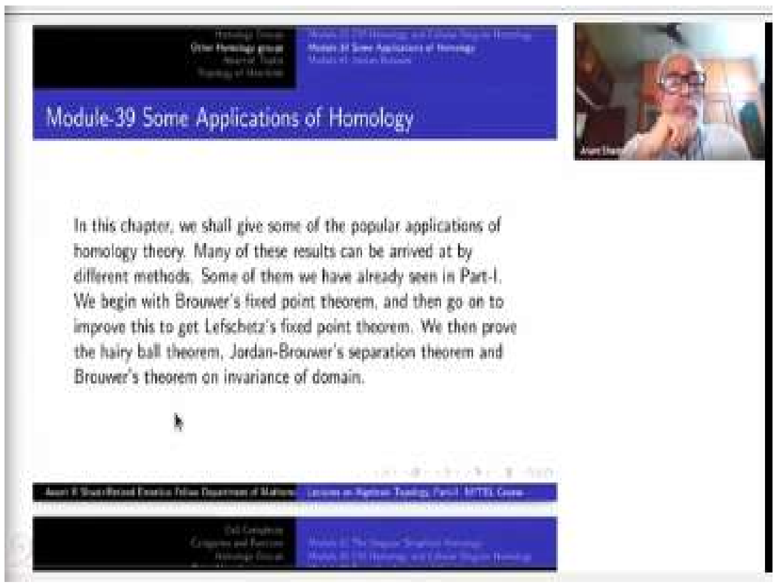

So, today we will start a new chapter namely applications of homology. Here, we will do a few popular applications of homology. Some of these results one can prove in different methods also. And some of them you have already seen in part I, without using the homology theory. Ok? So, having said that, the idea of proving these things is to emphasize how homology theory actually helps in many problem solving in topology especially in algebraic topology.

So, we begin with the famous Brouwer's fixed point theorem, then go on to prove the Leftschetz's fixed point theorem which is an improvement on Brouwer's theorem. And then we prove some popular results such as hairy ball theorem, Jordan-Brouwer separation theorem, Jordan-Brouwer invariance of domain etc. So, these are all very, very popular results. When these things were proved, algebraic topology itself became extremely popular because of such results. Ok?

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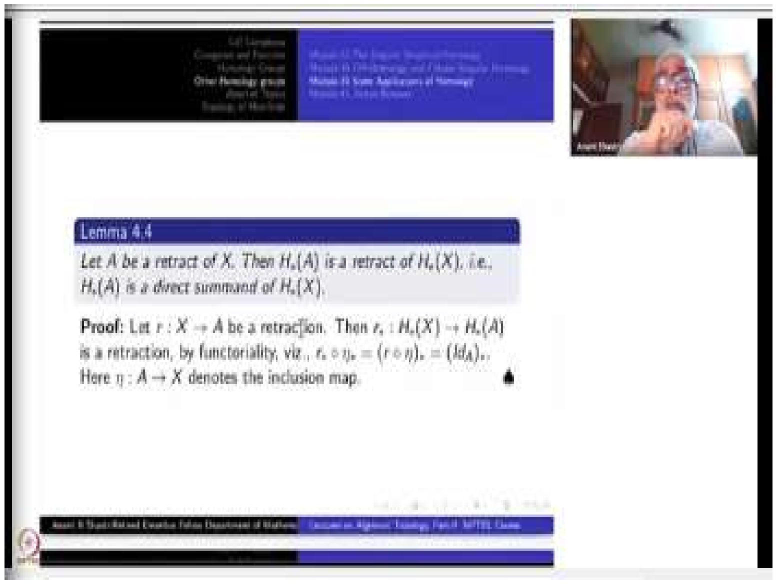

So, we begin with a lemma which has reminiscent of certain things that we have done in part I using the fundamental group. Instead, we now use the homology groups, ok? Let A be a retract of X, then $H_*(A)$ will be a retract of $H_*(X)$, that is, $H_*(A)$ is a direct summand of $H_*(X)$, which is somewhat stronger, Ok? For topological spaces A and X, A is a subspace of X, A is a retract of X does not yield such decomposition. All that you have is a continuous function from X to A which is identity on A, ok? r is the continuous function from X to A, i is inclusion map of A into X, start with i and then follow by r that is the identity of A. This is the meaning of saying that A is a retract of X.

When you pass to the homology, it will tell you that $H_*(A)$ is a retract of $H_*(X)$. That means there is a homomorphism from $H_*(X)$ to $H_*(A)$, etc., where to look for it? It is nothing but r_*. r itself induces a homomorphism and that will have this property $r_* \circ i_* = Id_{H_*(A)}$. Ok? So, r from X to A is a retraction, then r_* from $H_*(X)$ to $H_*(A)$ is a retraction. $r_* \circ i_* = Id_{H_*(A)}$. So, that is the meaning of saying that r_* is the retraction. And whenever you have retraction of abelian groups, the subgroup $H_*(A)$ becomes a direct summand, ok? This is a general theory for abelian groups. Ok?

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As a corollary we have: For any $n \geq 0$, $\mathbb{S}^n$ is not a retract of $\mathbb{D}^{n+1}$. So, this sounds like a negative result but this is a very positive result. ok? Remember, in part I, this was a stepping stone in the proof of Brouwer-fixed point theorem using simplicial approximation and Sperner lemma. So, here the same thing comes very easily. No need for a deeep result like Sperner lemma and so on. But of course we have to use whatever homology theory that we have done. Namely, the simplest thing that we have done was the computation of $H_n(\mathbb{S}^n)$ and $H_n(\mathbb{D}^{n+1})$. We know that $H_n(\mathbb{D}^{n+1}) = 0$ because the disc is contractible, and $H_n(\mathbb{S}^n)$ is infinite cyclic. Ok? If $\mathbb{S}^n$ were a retract of $\mathbb{D}^{n+1}$, then $H_n(\mathbb{S}^n)$ would have been a subgroup of $H_n(\mathbb{D}^{n+1})$, which is 0. But the subgroup is not 0. That contradiction proves the corollary.

So, observe that the functorality of the homology is strongly used here in this theorem. See from the hypothesis that $r \circ i$ is identity, in topology we have got the corresponding statement that $r_* \circ i_*$ is identity in the group theory, in the homology, Ok?

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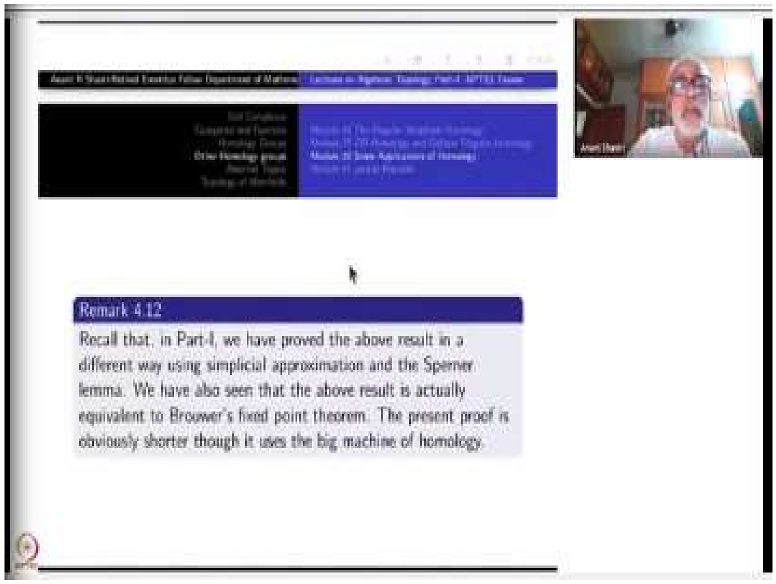

I have already told you, but I will repeat it. In part I we have proved the above result in a different way using simplicial approximation and Sperner lemma. The present proof is obviously shorter though it uses big machine of homology. Ok?

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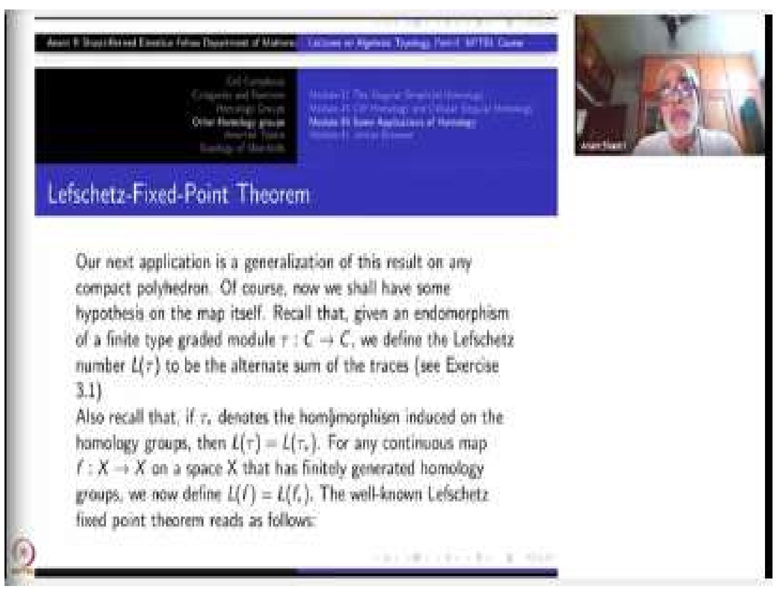

We have also seen that the above corollary is actually equivalent to Brouwer's fixed point theorem. So, we shall not bother about proving Brouwer's fixed point theorem again, once you have got this corollary, ok? You have got then of Brouwer's fixed point theorem. Namely, any continuous map from $\mathbb{D}^{n+1}$ to $\mathbb{D}^{n+1}$ has a fixed point. Ok? So, I am not going to discuss this one anymore, because it was done very thoroughly in part I.

So, let us directly go to the Lefschetz fixed point theorem now. As a consequence you will get another proof of Brouwer's fixed point theorem also. Therefore, we do not need spend any time on Brouwer's fixed point theorem separately.

So, our next application is a generalization of this BFT is for any any compact polyhedron. Since $\mathbb{D}^n$ is a compact polyhedron, it will be applicable to $\mathbb{D}^n$ also. Now we shall have some hypothesis on the map itself. Ok? On the polyhedron, only compactness is the assumption. On the map we have some additional assumption.

Recall that given an endomorphism τ of a finitely generated graded module C, here I take the liberty to shorten the notation $C.$ to just C, ok? We define the Lefschetz number $L(\tau)$ as the alternate sum of the traces of each τ_n. $L(\tau) = \sum(-1)^n trace(\tau_n)$. Ok?

So, I am just recalling a result. Recall that if τ_* denote the homomorphism induced on the homology groups of the chain complex C, then $L(\tau) = L(\tau_*)$, ok? So, this Lefschetz number will be used now in the statement of Lefschetz fixed point theorem. For any continuous map f from X to X on a space X that has finitely generate homology groups, we now define $L(f) = L(f_*)$. The well known Lefschetz fixed point theorem will be the following. Ok?

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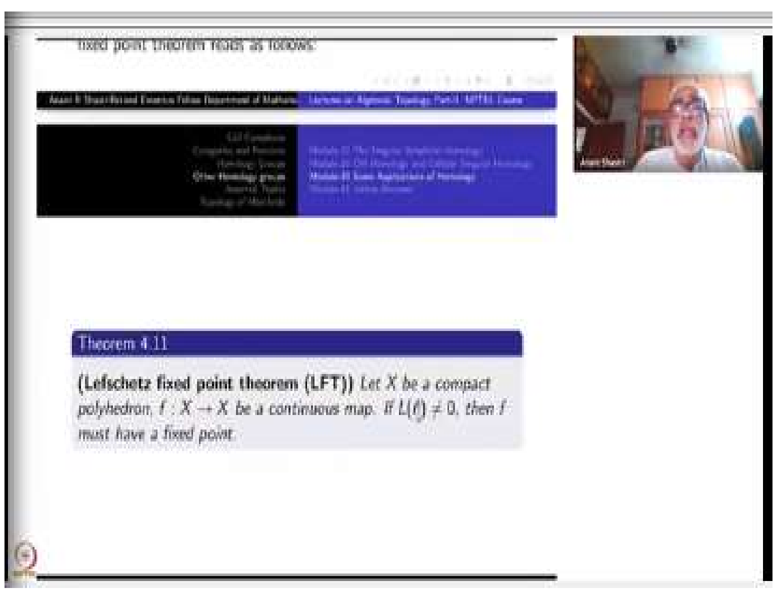

Let X be a compact polyhedron, which means what? There is a finite simplicial complex K, $|K|$ is homeomorphic to X. ok? (So, presently, do not worry about the simplicial structure here, just take the underlying topological space X. Let f from X to X be a continuous function. If $L(f)$ is not 0, then f must have a fixed point.

In Brouwer's fixed point theorem, there was no assumption on f except that f is continuous, whereas X itself was very special viz., just the closed the unit disc. Any thing homeomorphic to a disc will also do ok? But here this X is any the underlying $|K|$ where K is arbitrary finite simplicial complex. The extra assumption was on the continous function. So, how do you derive Brouwer's fixed point theorem from Lefschetz's?

So, if X is $\mathbb{D}^n$, ok, any map f from X to X is homotopic to the identity map, this $L(f)$ is a homotopy invariant, because it depends only on f_* on homology groups. Therefore, $L(f)$ will be equal to $L(Id_X)$. Now L of the identity map is easy to compute. In fact, on a path connected space X, we have proved that for any continuous map f from X to X, f_* from $H_0(X)$ to $H_0(X)$ is the identity map. Since $H_0(\mathbb{D}^n)$ is infinite cyclic, it follows that trace of $f_* = 1 = L(f)$. So, the hypothesis for LFT viz., that $L(f)$ not zero, is satisfied. That gives an easy proof of Brouwer's fixed point theorem from this theorem. Ok?

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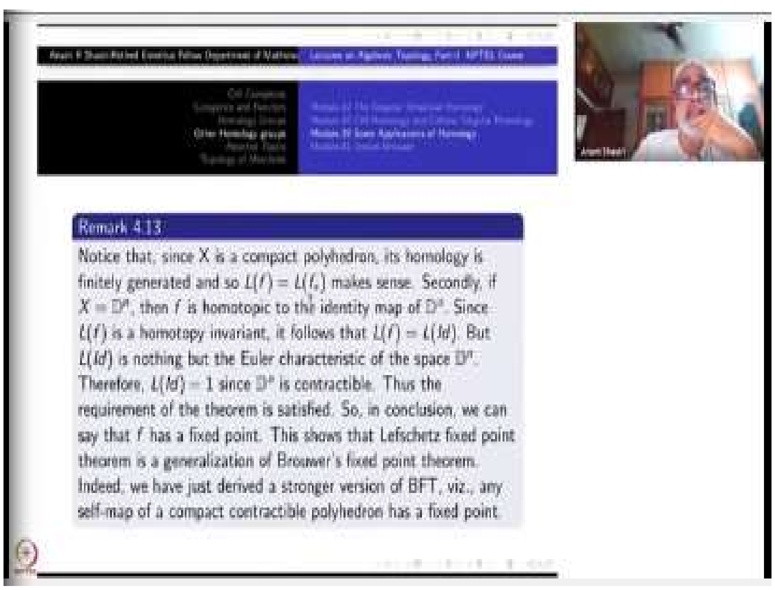

So, let us go to the proof of this itself now. So, this is what I have said here, I will repeat it. If X is $\mathbb{D}^n$ then f is homotopic to identity map of $\mathbb{D}^n$, because $\mathbb{D}^n$ is contractible. Any two functions into $\mathbb{D}^n$ are homotopic. Since $L(f)$ is a homotopy invariant, it follows that $L(f)$ is $L(Id)$. But L of identity map is nothing but Euler characteristic $\mathbb{D}^n$ which is equal to 1, because $\mathbb{D}^n$ is contractible. Ok? Thus the requirement of the theorem is satisfied. So, in conclusion, we can say that f has a fixed point. This shows that Lefschetz fixed point theorem is a generalization of Brouwer's fixed point theorem.

Indeed, we have just derived a stronger version of Brouwer's fixed point theorem, namely any self map of a compact contractible polyhedron has a fixed point. We never used that X is actually homeomorphic to $\mathbb{D}^n$, only contractibility was used. Ok So, this is another intermediary result, you may note down which follows from Lefschetz fixed point theorem. So, let us see some more results also. But before that let us try to complete a proof of this.

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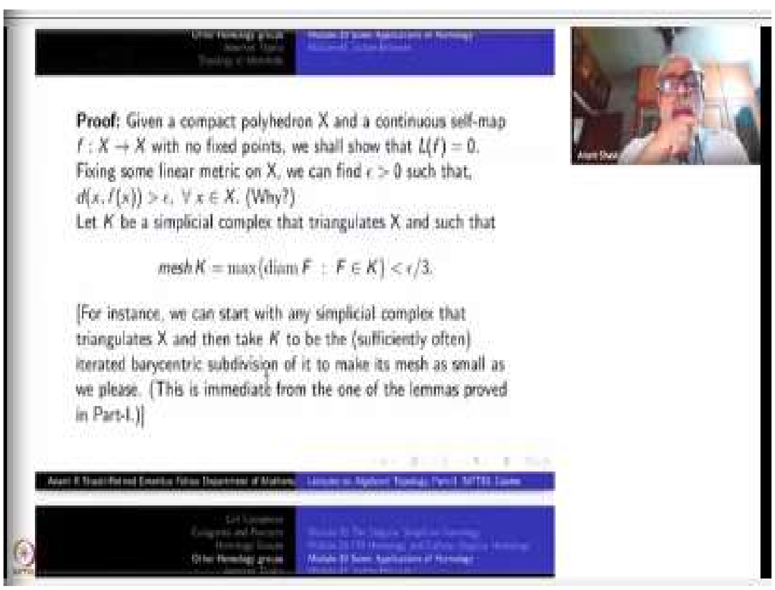

Given a compact polyhedron and a continuous self map, suppose it has no fixed points. Then we want to show that $L(f)$ is 0. Now this being a compact polyhedron, viz., we have a finite simplicial complex K such that$|K| = X$. We have seen that there is always a linear metric, ok, you can choose any linear metric and find an ϵ such that distance between x and $f(x)$ is bigger than this ϵ for every x belonging to X. Alright, why this is true?

You have to do some topology here. On a compact metric space, the distance between x and $f(x)$ is a continuous function, ok? A real valued continuous function, positive. So, it attains its minimum and that means that minimum will be strictly positive. Take that positive number to be epsilon. Then everything else will be bigger than that. ok?

Now, I will fix a finite simplicial structure on X, I do not care what it actually is but that it is a finite triangulation on X. Now, X is nothing but $|K|$, Ok? Next I can choose a subdivion K' of K, (by repeating barycetric subdivsion as many times as required, for instance) so that the mesh of K' which is the by definition, maximum of the diameters of all the simplexes, is less than epsilon by 3. Ok. We have done this in part I. All that you have to do is to keep taking barycentric subdivisions. Each time the diameter becomes r times original one where $r < 1$. So, if you repeat it several times then it will be less than given number. Ok

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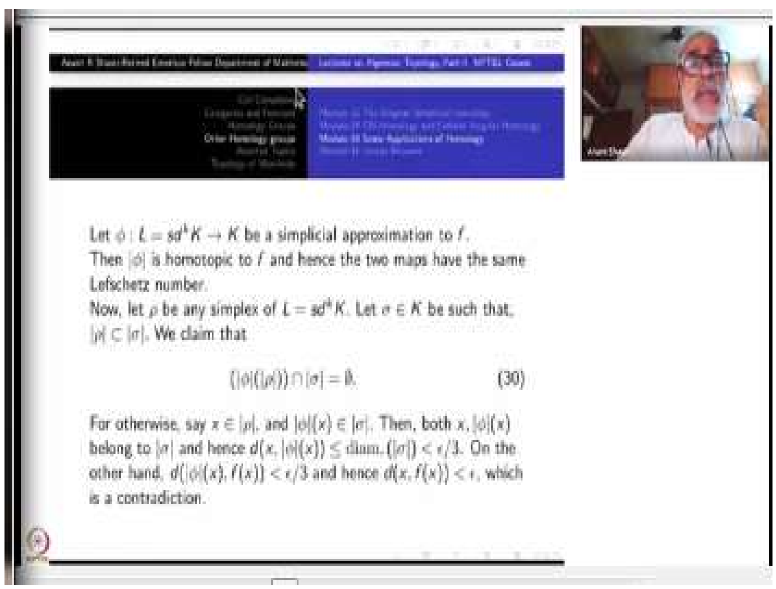

Replacing K' by K, we can as well assume that mesk of K itself is less than $\epsilon/3$. Now by simplicial approximation theorem applied to the continuous function f from $|K|$ to $|K|$, we get a integer k and a simplicial map ϕ from $sd^k(K)$ to K to the function f. That means, in particular that $|\phi|$ is homotopic to f. Therefore, $L(|\phi|)$ is the same thing as $L(f)$. You want to show that a $L(f)$ is 0. We can do that by showing that $L(|\phi|)$ is 0. Ok? So, that is what we are trying to do now. How do we how do we use this information all this information? Why all this was done?

So, the first thing we claim is the following. Take ρ to be any simplex of $L := sd^k(K)$. Then there is a σ belonging to K such that $|\rho|$ is contained inside $|\sigma|$. That is a property of subdivisions. The claim is that $|\phi|(|\rho|) \cap |\sigma|$ is empty. For this we have to use the facts that ϕ is a simplicial approximation to f, f has no fixed point, ϵ is chosen in a particular way and then the simplicial complex is chosen in a particular way viz, mesh of K is smaller than $\epsilon/3$, etc. Since ρ was inside sigma, where as points of ρ will be carried away from σ. This is a strong claim that we want to prove now. ok?

Suppose this is not true, that could mean that there is a point x in the intersection. That means x in $|\rho|$ and such that $|\phi|(x)$ belongs to $|\sigma|$. Ok? Then both x and $|\phi|(x)$ will be inside $|\sigma|$. Now the diameter of $|\sigma|$ is less than $\epsilon/3$. Therefore distance between x and $|\phi|(x)$ will be less than $\epsilon/3$. On the other hand ϕ is a simplicial approximation to f. So, $f(x)$ is $|\phi|(x)$ will be the same same simplex of K. Since the mesk of K is less than $\epsilon/3$, it follows that distance between $f(x)$ and $|\phi|(x)$ is less than $\epsilon/3$. By triangle inequality it follows that the distance between x and $f(x)$ will be less than $2\epsilon/3$ which is a contradiction to the choice of ϵ, being the minimum of such distances. In particular, it follows that the simplicial map ϕ has the property that ρ and $\phi(\rho)$ are disjoint for every simplex ρ in $sd^k(K)$.

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Now the only problem is that when you take the simplicial approximation, you have taken a subdivision of K and then the domain compelx and codomain are different. Thus the chain complexes of the domain and the codomain are different. So, there is no concept of trace of a morphism. Trace is defined only for an endomorphism of the module to itself and not for an arbitrary homomorphism. So, that is the difficulty here and it has to be overcome somehow and there are different methods of doing that. So, I will give you twp methods here which I like, some other methods I do not like and I even doubt them.

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So, in the first method I use the CW-structure on X associated with the simplicial complex K on both domain and codomain. Then it follows that $|\phi|$ from X to X is a cellular map. Any k-cell will be mapped into the union of k-cells by $|\phi|$ since it is simplicial from $sd^k(K)$ to K. Therefore we can now pass of the morphism induced by $|\phi|$ on $C_n^{CW}(X) = H_n(K^{(n)}, K^{(n-1)})$.

Let this map be denotes by α_n from $H_n(K^{(n), K^{(n-1)}})$ to itself. So, I have the domain and codomain are same here. If an n-simplex τ is a generator, (this is the meaning of oriented n-simplex, ok), it follows from the above property of $|\phi|$ that $\tau \cap |\phi|(\tau)$ is empty, right? That $\alpha_n(\tau)$ is a finite sum of certain oriented n-simplexes the support of none of which will intersect $|\tau|$. But this will means that coefficient of τ in the $\sum \alpha_n(\tau) = 0$.

It follows that if you write down the matrix for α_n using the set of n-simplexes in K as the basis, the diagonal entries of this matrix will be all 0. That means the trace of α_n is 0. Therefore trace of

α_n is 0. This is true for all n right? Therefore, the alternating sums will be also 0. That means $L(|\phi|) = 0$. That is what we wanted to prove.

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The second method is slightly more elaborate, but it teaches you something about the subdivision chain map also. So, how to use subdivision map? Ok. The property of sibdivision chaoin map, whatever, we are going to use here, will be useful elsewhere also. So, here take the subdivision chain map Sd, which you have defined from $C.(K)$ to $C.(sd(K))$. Little sd is the barycentric subdivision, whereas Sd is the subdivision chain map. Ok.

We have seen that this Sd induces identity homomorphism in the homology. That is the beauty. You can forget about this $C.$ and go to $H_*(|K|)$ to $H_*(|K|)$, it is identity morphism there. By repeated application of this, take compositions of Sd with itself k times, it follows that Sd^k from $C.(K)$ to $C.(sd^k(K))$ induces the identity map on the homology.

So, I can compose it with $|\phi|_*$ from $H_*(|sd^k(K)|)$ to $H_*(K)$ to get an endomorphism of $C.(Sd^k(K))$ to $C.(Sd^k(K))$, viz., $Sd^k \circ \phi$.. When you pass on to homology, it follows that the induced morphism is the same as $|\phi|_*$. Therefore, $L(|\phi|) = L(Sd^k \circ \phi)$ which can be computed at the chain complex level.

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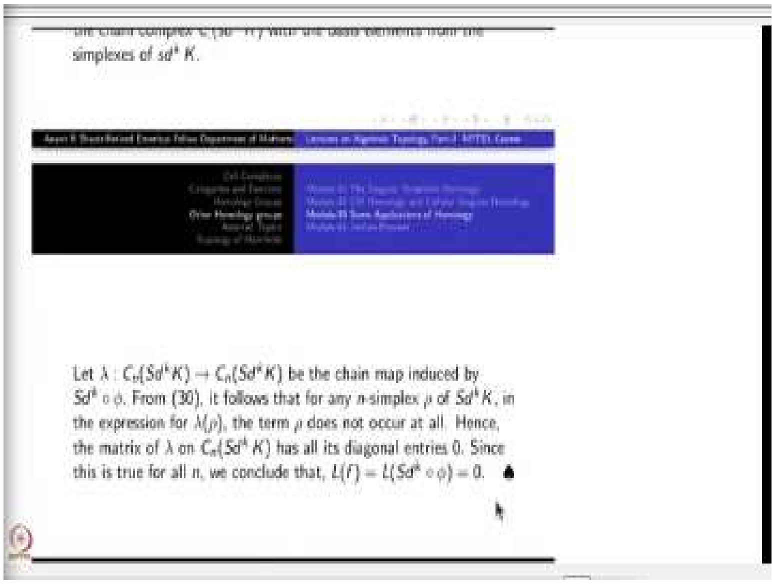

Taking the n-simplexes of $Sd^k(K)$ as a basis, consider the matrix associated to $\lambda = (Sd^k \circ \phi)_n$ at the n-chain group. Property (35) here will tell you that if you write the expression for $\lambda(\rho)$ for any simplex ρ in $Sd^k(K)$, it will not consist of ρ at all. That argument is the same as in the earlier case, ok? Hence the matrix of λ will have all diagonal entries 0, so that its trace is 0. Alternate sum of these traces will be also 0.

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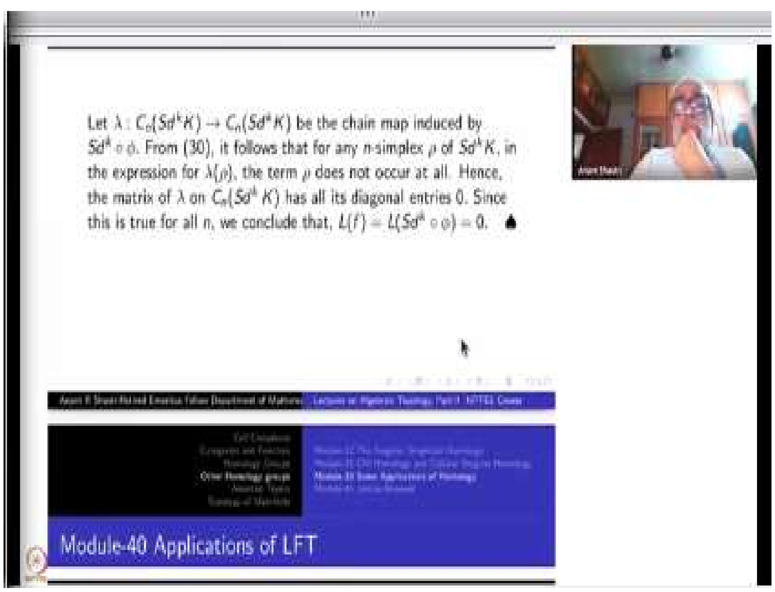

So I think I will have stop here. We shall see some interesting applications of the Lefschetz fixed point theorem itself next time. Thank you.