Introduction to Algebraic Topology (Part-II)
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Lecture-40
Assorted Topics

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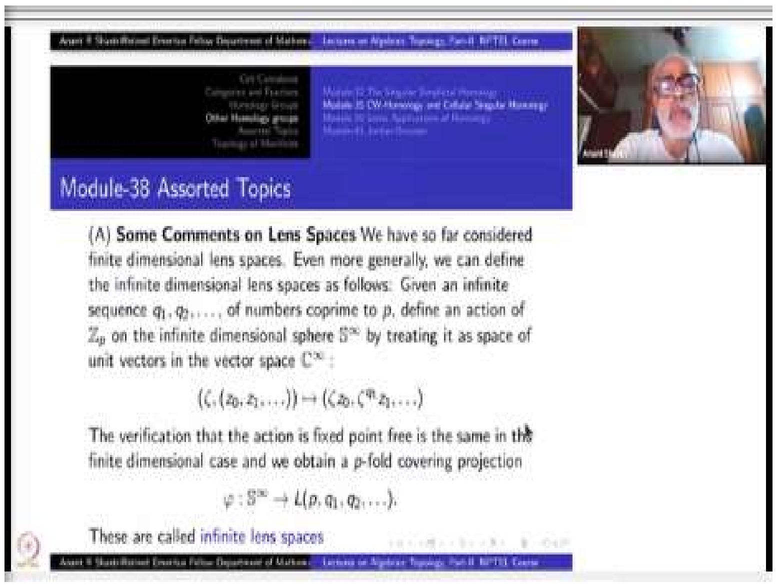

So, today we shall take care of few side remarks and certain left out things and so on. So, this is not exactly a topic module, so I have called it assorted topics. To carry on with whatever we were doing, namely, the study of lens spaces. Let me first give you a few more information on lens spaces. So far, we considered only finite dimensional lens spaces, namely, finite quotients of odd dimensional sphere. We made a group action and then take the quotient. Right?

We can do this in a little more general fashion, namely, Each time the action that you take on a lower dimensional sphere extended to a higher dimensional sphere using the containment $\mathbb{S}^3$ contained in $\mathbb{S}^5$ containend in $\mathbb{S}^7$ and so on. We have been extending the action. Therefore you can do this all the way to infinite dimensional sphere as well. Ok?

So, what the starting data? Instead of a finite sequence $q_1, q_2, \ldots, q_n$ of numbers which are coprime to a given number p, we start with an infinite sequence of numbers which are all coprime to p. Ok? Then all that you have to do is take this infinite sequence of complex numbers such that after a finite stage they are all 0, that is the meaning of this one $z_0, z_1, \dots, z_n$ etc, sitting inside $\mathbb{S}^\infty$ thought of as the unit sphere in $\mathbb{C}^\infty$ which is nothing but the direct sum of countably infinite copies of $\mathbb{C}$. Take ζ to be a primitive p-th root of unity and send $(\zeta, (z_0, z_1, \dots, z_n, \dots)$ to $(\zeta z_0, \zeta^{q_1} z_1, \zeta^{q_2} z_2, \dots)$ i.e., the 0-th coordinate by ζ and multiply the i-th coordinate by ζ^{q_i}. Verification that, this defines a fixed point free action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{S}^\infty$ is actually done earlier, because though the sequences are infinite, each time you have to handle only finitely many non zero entries. What you have do? At each time you have to verify it at the finite level.

It follows that the quotient map is a p-fold covering from $\mathbb{S}^\infty$ to the orbit space, which we denote by $L := L_{p,q_1,q_2,\dots}$. So, that is called an infinite dimensional lens space.

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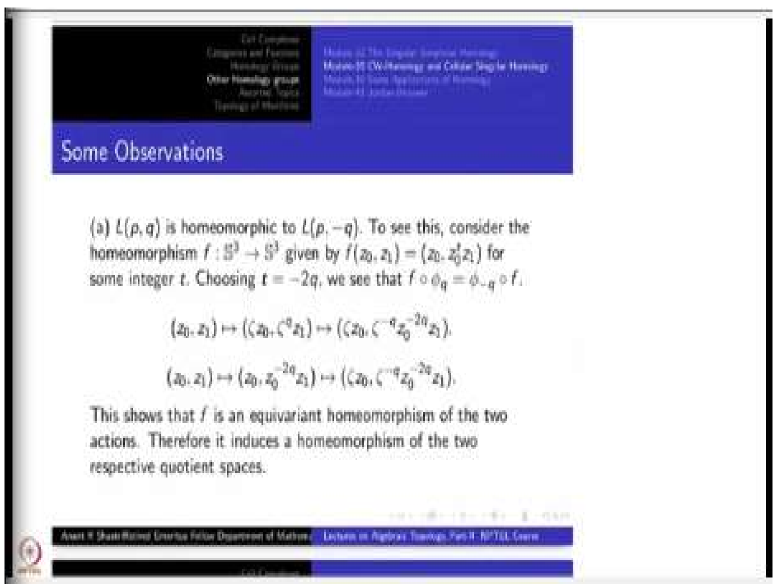

It has some theoretical importance. I will let you know about it a little later. Right now, coming back to the finite dimensional lens spaces, there are some easy observations you can make. $L_{p,q}$, remember, as such depends upon both p and q, right? Indeed, q is congruent to q' modulo p then $L_{p,q}$ and $L_{p,q'}$ are the same spaces. This allows us to take q to be negative integers also.

But there are some more interesting relations, namely, $L_{p,q}$ is diffeomorphic $L_{p,-q}$. How to see this? There are many different ways of seeing. I am telling you one way. Consider the map f from $\mathbb{S}^3$ to $\mathbb{S}^3$ given by (z_0, z_1) going to $(z_0, z_0^t z_1)$, where t is some integer. Clearly, it is a smooth map and its inverse is given by (z_0, z_1) going to $(z_0, z_0^{-t} z_1)$, ok? Anyway it is a homeomorphism if you do not know what a diffeomorphism is. Now for a special t, namely. $t = -2q$, you can verify that f actually becomes an equivariant map from $\mathbb{S}^3$ to $\mathbb{S}^3$ with respect to the two given actions of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{S}^3$, viz., $f \circ \phi_q$ is equal to $\phi_{-q} \circ f$. it is sensing a ϕ_{-q} of of f. Therefore f factors down to define a map $\hat{f}$ from $L_{p,q}$ to $Lp, -q$, ok?

(z_0, z_1) under ϕ_q would go to $(\zeta z_0, \zeta^q z_1)$. Applying f now gives you $(\zeta z_0, \zeta^{-2q} \zeta^q z_1)$. There is some typo here in the slide, you better change it. On the other hand (z_0, z_1), under f will first go to $(z_0, \zeta^{-2q} z_1)$ and then under ϕ_q, it goes to $(\zeta z_0, \zeta^q \zeta^{-2q} z_1)$. The end results are the same.

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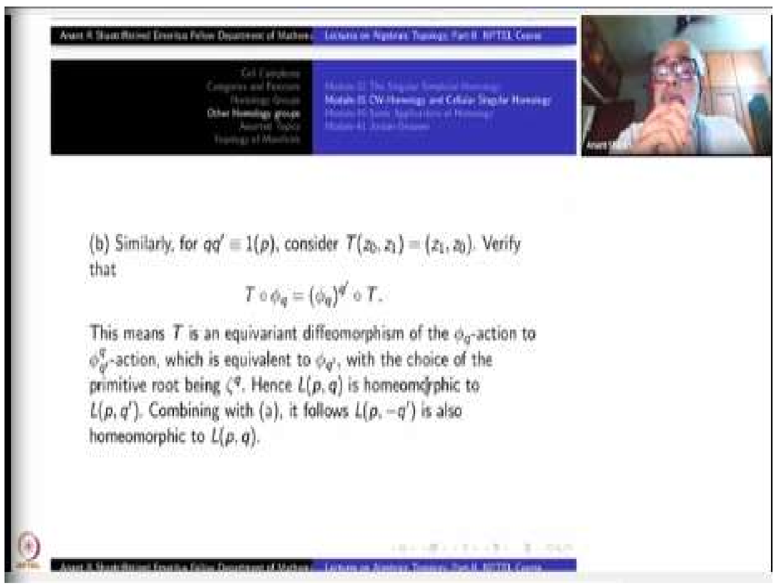

The second thing is that if q and q' are such that one is the inverse of the other modulo p, i.e, in the ring $\mathbb{Z}/p\mathbb{Z}$, these two numbers must be inverses of each other, viz, qq' is congruent to $1\ mod(p)$, ok? then again $L_{p,q}$ is homeomorphic (diffeomorphic) to $L_{p,q'}$. So, to see this you look at the map which interchanges the two factors viz., $T(z_0, z_1) = (z_1 z_0)$. That is a diffeomorphism of $\mathbb{S}^3$ to $\mathbb{S}^3$.

We can easily verify that $T \circ \phi_q$ is the same thing as $(\phi_q)^{q'} \circ T$ i.e., ϕ_q operated q' times repeatedly. But that is the same as if we choose the primitive root $\zeta^{q'}$ for our ϕ_q action. Note that the action is independent of what primitive root of unity you choose, because if ζ and ζ' are two primitive roots then multiplication by $\zeta' \zeta^{-1}$ defines an equivariant map from one ζ-action to ζ'-

action. Ok? The assumption that both q and q' are coprime to p is important here. Hence $L_{p,q}$ is homeomorphic to $L_{p,q''}$.

If you combine these two observations, you will get $L_{p,-q'}$ is also homeomorphic to L_{pq}, ok? The first number is the same, the second number q can be replaced by its additive inverse, or multiplicative inverse or its negative, the diffeomorphic type of $L_{p,q}$ does not change. Ok?

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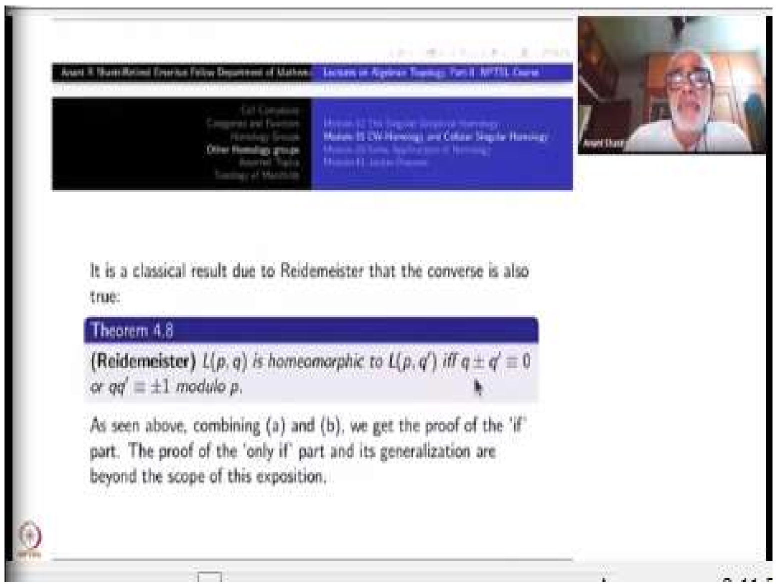

What Reidemeister has done is to prove the converse and thus giving complete classification of 3 -dimensional lens spaces, very classical result, viz., $L_{p,q}$ is homeomorphic to $L_{p,q'}$, if and only iff $q \pm q'$ is 0 $mod p$, or qq' is ± 1 $mod p$. Ok?

The `only if' part viz., the converse part is beyond the scope of this course, which requires us to introduce new concepts such as Reidemeister torsions and so on, a deeper theory. Ok.

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Similarly there is a theorem of Whitehead which gives you a homotopy classification. So let me state that one. In any case all these things available in a nice article by Cohen written in 1073. So, if you are interested in you can read that article.

So, Whitehead's theorem: $L_{p,q}$ is homotopy equivalent to $L_{p,q'}$ if and only if qq' is a square modulo p, ok? So, homotopy equivalence automatically takes care of homeomorphisms also, but it is a much larger class. So this is Whitehead's result. Ok.

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So, now, I will let you know something about infinite dimensional lens spaces. The strangeness with infinite lens space is that its homotopy type depends only on the first integer p. All of them have their fundamental group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Thus $\mathbb{S}^{\infty}$ is a p-fold cover of the Lens space as in the finite case. But what is important here is that $\mathbb{S}^{\infty}$ is contractible, unlike finite dimensional sphere. From this fact one can deduce that if s, s' are any two infinite sequences of numbers all of them coprime to p, then $L_{p,s}$ and $L_{p,s'}$ have the same homotpy type. This follows from a general fact that Eilenberg-Maclane spaces of type a given type are unique up to homotopy type. Now let us go to another topic here namely, Euler characteristic revisited.

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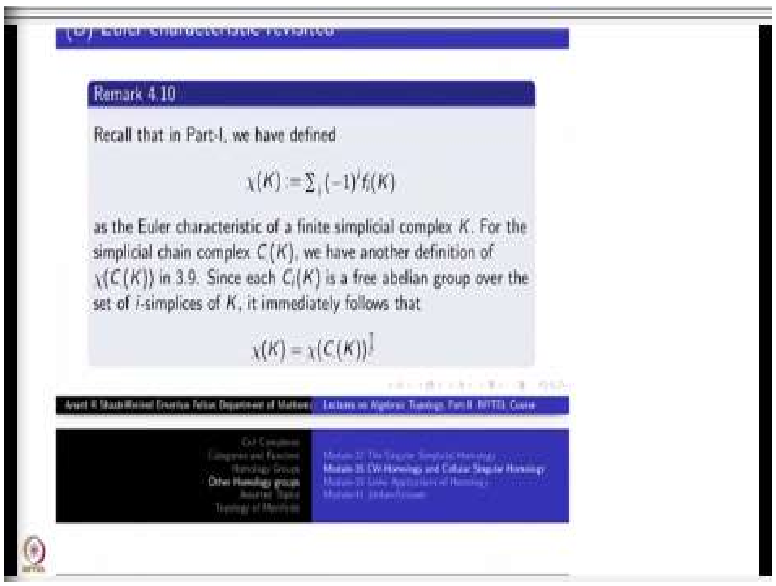

Remember, in part I, wedefined Euler characteristic of a finite simplicial complex. A finite simplicial complex K has a finite sequence $(f_0, f_1, f_2, \ldots, f_n)$, where f_i is the number of i-dimensional simplexes in K. This ordered $(n+1)$-tuple is called the face vector in combinatorics. What we are interested in here is that the Euler characteristic of K is defined as the alternate sum of f_i's for a finite simplicial complex K.

For the simplicial chain complex $C.(K)$ which is finitely generated, we have another definition of the Euler Characteristic, $\chi(C(K))$, namely the alternate sum of the ranks of these $C_n(K)$. We have seen that this is also equal to the alternate sum of the ranks of the corresponding homology groups. Ok?

From our study of the simplicial chain complex of a simplicial complex K, we know what the groups $C_i(K)$. They are the free abelian groups with basis consisting of i-dimensional simplexes and therefore, they are of rank f_i. Therefore, these two definitions are the same, whether you use the simplicial chain complex to define the Euler characteristic or directly do the way we have done it in part I. Ok? So, that is an important observation.

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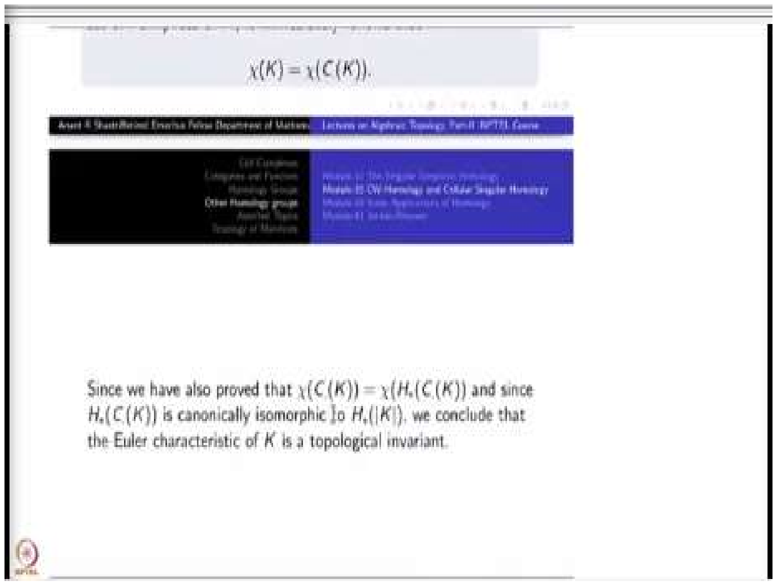

But then we have also stated that the simplicial homology is canonically isomorphic to the singular homology of the underlying topological space. From this it follows that the Euler characteristic is independent of what triangulation you choose on the given topological space, provided it is defined.

To define the Euler characteristic for topological spaces directly, of course, there will be some restrictions. Ok? For instance, anytime X has finite simplicial complex structure this will valid. For example, if you start with a compact space which can be triangulated, then all simplicial structures on X will be automatically finite and they all yield the same Euler-characteristic.

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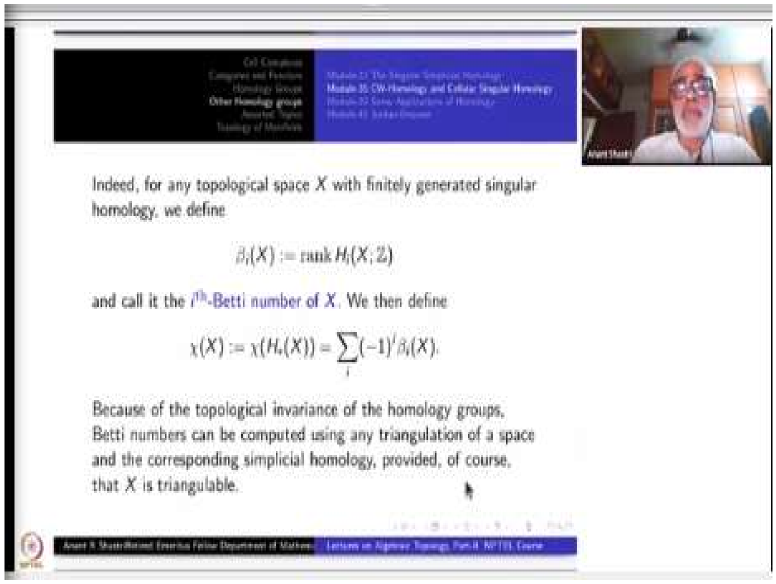

By the way, Euler characteristic at least for surfaces, were studied even before Poincare. The rank of $H_i(X)$ is itself an important homotipy invariant of a space. It is classically denoted by $\beta_i(X)$, whether it is finite or infinite, and it is called i-th Betti number of X. The alternate sum of i-th Betti numbers, if it is defined (i.e. all of them finite and most of them 0) is called Euler characteristic in the most general case. If these things are all finite, then you call it as $\chi(X)$. To compute this one usually used a suitable triangulation.

We can go one more step ahead because now we know what is a CW complex instead of a simplicial complex, and what is the CW-homology.

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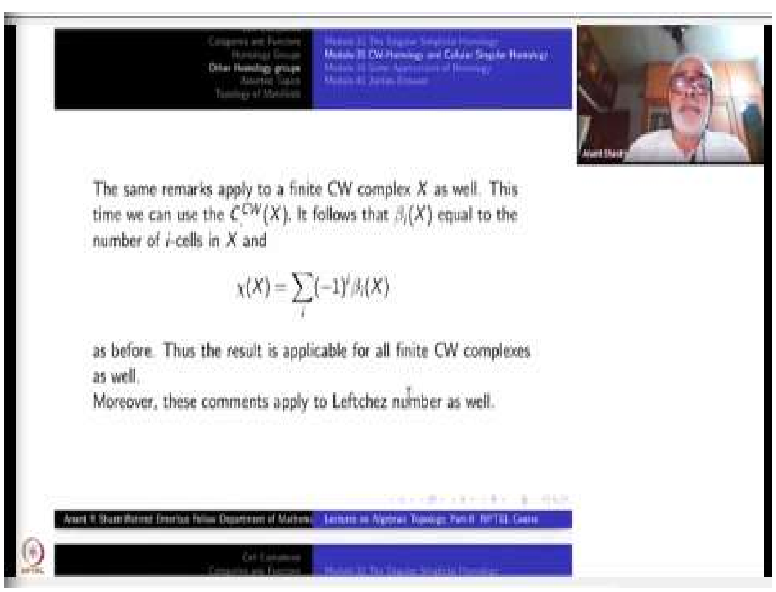

Exactly as in the case of simplcial complexes we can define a face vector and compute the Euler-characteristic directly. So, this is valid for all finite CW complexes which is a slightly larger category than finite simplicial complexes. All these comments apply to the Lefschetz number also. Recall what is the Lefschetz number of a continuous function f from one topological space X into itself.

Now suppose X is a space such that its homology $H_*(X)$ is finitely generated. Then take the induced homomorphisms f_* from $H_i(X)$ to $H_i(X)$. Look at the trace of each of them.(These are $\mathbb{Z}$-linear maps you can talk about the trace.) Take the alternate sum. That is the definition of Lefschetz number $L(f)$. ok? So, this can be computed either using the CW structure or using simplicial structure for X, provided that the map f is cellular or simplicial respectively.

But final result says that it is independent of all that. It can defined for any map from any X to X provided X has finitely degenerated homology. Ok?

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Here is another kind of homology: cellular singular homology. We now come to this version of homology which is between CW homology and singular homology of the underlying space. Ok? This is similar to the case of simplicial homology and singular simplicial homology of a simplicial complex. Ok?

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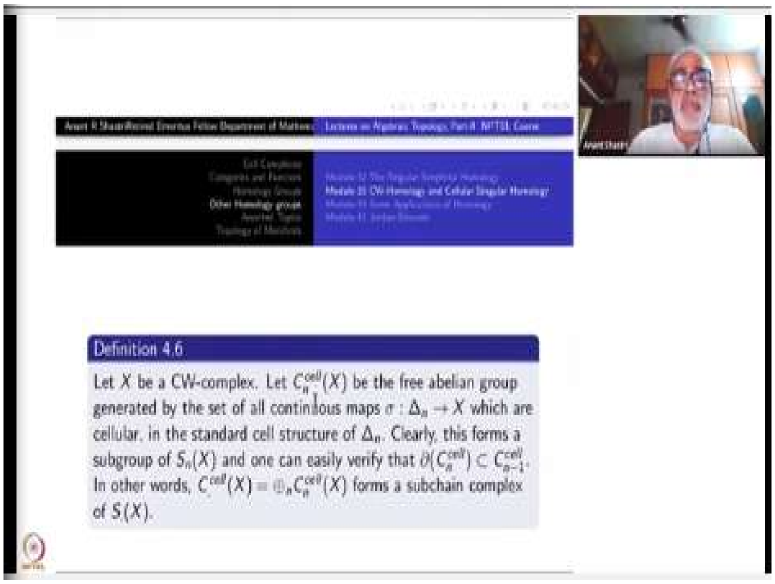

So, take a CW complex X. I am going to define a subgroup of $S_n(X)$. What is this subgroup? I am using the notation $C_n^{cell}(X)$ for the free abelian group generated by the set of all continuous maps sigma from Δ_n to X, which are cellular. Here, Δ_n are taken with their standard cell structure. If you take all continuous maps as the generating set, you get the whole group $S_n(X)$, ok? So, obviously this free abelian group is a subgroup of $S_n(X)$ and the boundary operator in $S.(X)$ takes $C_n^{cell}(X)$ into $C_{n-1}^{cell}(X)$, because boundary of a cellular map is a cellular chain. That is why you can take the restriction of the boundary operator to make $C_.^{cell}(X)$ into a subchain complex.

Now look C_n^{CW}? It is defined as a free abelian group on the set of characteristic maps of n-cells. Since the domain of each characteristic map phi of an n-cell can be taken to be Δ_n and then ϕ becomes automatically cellular, it follows that it is a subsets of the basis for $C_n^{cell}(X)$. Therefore $C_n^{CW}(X)$ is a subchain complex of $C_n^{cell}(X)$. Ok?

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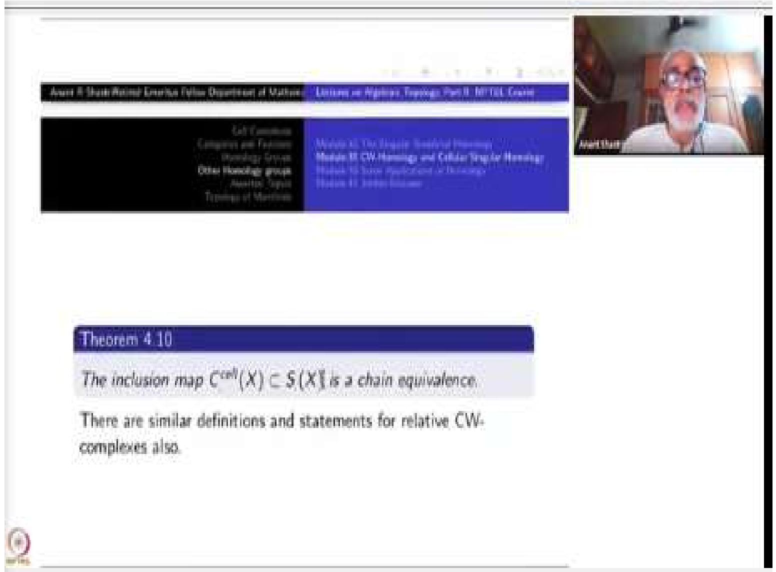

Now the statement is that the inclusion map $C_{\cdot}^{CW}$ to $C_{\cdot}^{cell}$ is chain equivalence, Ok? Therefore when you pass to the homology, it will be induce an isomorphism. Ok?

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As usual we will postpone the proof of this one. Ok?

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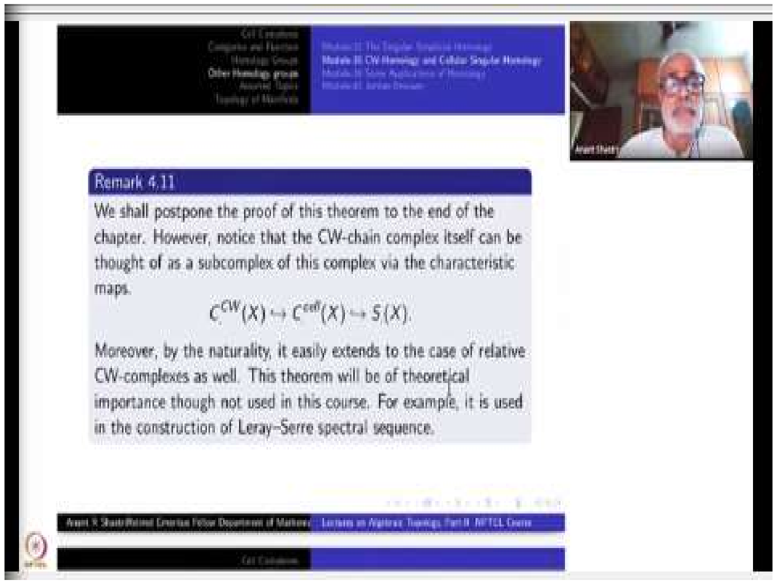

Also, you can always take relative versions of these things for CW pairs. We are not going to use this one so much here but understanding this one will be a must when you want to to study certain concepts in higher algebraic topology, such as Serre-spectral sequence etc. The idea of spectral sequence actually has roots in this kind of observations with CW complexes Ok? So, let us stop here and next time we will start doing applications of homology, thank you.